
Efficient Discrepancy Testing for Learning with Distribution Shift

Gautam Chandrasekaran*
UT Austin

Adam R. Klivans[†]
UT Austin

Vasilis Kotonis[‡]
UT Austin

Konstantinos Stavropoulos[§]
UT Austin

Arsen Vasilyan[¶]
UC Berkeley

Abstract

A fundamental notion of distance between train and test distributions from the field of domain adaptation is discrepancy distance. While in general hard to compute, here we provide the first set of provably efficient algorithms for testing *localized* discrepancy distance, where discrepancy is computed with respect to a fixed output classifier. These results imply a broad set of new, efficient learning algorithms in the recently introduced model of Testable Learning with Distribution Shift (TDS learning) due to Klivans et al. (2023).

Our approach generalizes and improves all prior work on TDS learning: (1) we obtain *universal* learners that succeed simultaneously for large classes of test distributions, (2) achieve near-optimal error rates, and (3) give exponential improvements for constant depth circuits. Our methods further extend to semi-parametric settings and imply the first positive results for low-dimensional convex sets. Additionally, we separate learning and testing phases and obtain algorithms that run in fully polynomial time at test time.

1 Introduction

Distribution shift remains a central challenge in machine learning. While practitioners may exert some level of control over a model’s training distribution, they have far less insight into future, potentially adversarial, test distributions. Developing algorithms that can predict whether a trained classifier will perform well on an unseen test set is therefore critical to the widescale deployment of modern foundation models.

A heavily-studied framework for modeling distribution shift is domain adaptation, where a learner has access to labeled examples from some training distribution, unlabeled examples from some test distribution and is asked to output a hypothesis with low error on the test distribution. Over

*gautamc@cs.utexas.edu. Supported by the NSF AI Institute for Foundations of Machine Learning (IFML).

[†]klivans@cs.utexas.edu. Supported by NSF award AF-1909204 and the NSF AI Institute for Foundations of Machine Learning (IFML).

[‡]vasilis@cs.utexas.edu. Supported by the NSF AI Institute for Foundations of Machine Learning (IFML).

[§]kstavrop@cs.utexas.edu. Supported by the NSF AI Institute for Foundations of Machine Learning (IFML) and by scholarships from Bodossaki Foundation and Leventis Foundation.

[¶]arsenvasilyan@gmail.com. Supported in part by NSF awards CCF-2006664, DMS-2022448, CCF-1565235, CCF-1955217, CCF-2310818, Big George Fellowship and Fintech@CSAIL. Work done in part while visiting UT Austin. Part of this work was conducted while the author was visiting the Simons Institute for the Theory of Computing.

the last twenty years, researchers in domain adaptation and related fields [BDBCP06, BCK⁺07, MMR09, BDBC⁺10, RMH⁺20, ZLWJ20, KM21b, HKM23, KZZ24] have established bounds for out-of-distribution generalization in terms of some type of distance between train and test distributions. By far the most commonly studied notion is discrepancy distance:

$$\text{disc}_{\mathcal{C}}(\mathcal{D}, \mathcal{D}') = \sup_{f_1, f_2 \in \mathcal{C}} \left| \mathbb{P}_{\mathbf{x} \sim \mathcal{D}} [f_1(\mathbf{x}) \neq f_2(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim \mathcal{D}'} [f_1(\mathbf{x}) \neq f_2(\mathbf{x})] \right|$$

Estimating or even testing discrepancy distance, however, seems difficult, as its definition involves an enumeration over all classifiers from some underlying function class (in [Appendix F](#) we give the first hardness result for computing discrepancy distance in general). As such, obtaining provably efficient algorithms for domain adaptation has seen little progress (none of the above works give polynomial-time guarantees).

In search of efficient algorithms for learning with distribution shift with certifiable error guarantees, recent work by [KSV24b] defined the *Testable Learning with Distribution Shift* (TDS learning) framework. In this model (similar to domain adaptation), a learner receives labeled examples from train distribution \mathcal{D} , *unlabeled* examples from test distribution \mathcal{D}' , and then runs a test. If the (efficiently computable) test accepts, the learner outputs h that is guaranteed to have low test error with respect to \mathcal{D}' . No guarantees are given if the test rejects, but it must accept (with high probability) if the marginals of \mathcal{D} and \mathcal{D}' are equal. This framework has led to the first provably efficient algorithms for learning with distribution shift for certain concept classes (for example, halfspaces) [KSV24b, KSV24a].

It is straightforward to see that if algorithm \mathcal{A} learns concept class \mathcal{C} in the (ordinary) PAC/agnostic model, and we have an efficient *localized* discrepancy tester for \mathcal{C} , then \mathcal{C} is learnable in the TDS framework: simply apply the discrepancy tester to the output of \mathcal{A} and accept if this quantity is small. A dream scenario would be to augment all known PAC/agnostic learning algorithms with associated localized discrepancy testers. This is nontrivial in part because we cannot make any assumptions on the test distribution \mathcal{D}' (our test has to always accept or reject correctly). Nevertheless, our main contribution is a suite of new discrepancy testers for well-studied function class/training distribution pairs that unifies and greatly expands all prior work on TDS learning.

1.1 Our Contributions

Optimal Error Guarantees via \mathcal{L}_1 Sandwiching. The work of [KSV24b] followed a moment-matching approach to show that the existence of \mathcal{L}_2 sandwiching polynomial approximators implies TDS learning up to a constant factor of the optimum error. Although their result implies TDS learning for several fundamental concept classes, the \mathcal{L}_2 sandwiching requirement seems restrictive for classes such as constant-depth circuits or polynomial threshold functions. In [Theorem 3.1](#), we provide TDS learning results in terms of the much more well-understood notion of \mathcal{L}_1 sandwiching, resolving one of the main questions left open in [KSV24b]. As such, we obtain exponential improvements for TDS learning constant depth circuits (AC^0), and the first results for degree-2 polynomial threshold functions (see [Table 1](#)). Our result also bridges a gap between TDS learning and testable agnostic learning [RV23], since the latter has been known to be implied by \mathcal{L}_1 sandwiching [GKK23]. Additionally, in the agnostic setting, the error guarantees we achieve are essentially optimal (as opposed to the constant-factor approximation by [KSV24b]).

Universal TDS Learners. A natural and important goal in TDS learning is to design algorithms that accept and make trustworthy predictions whenever the distribution shift is benign. In [Theorems 4.2](#) and [5.1](#), we give the first TDS learners that are guaranteed to accept whenever the test marginal falls in a wide class of distributions that are not necessarily close to the training distribution (in say statistical distance) but, instead, share some mild structural properties. In the literature of testable agnostic learning, testers with relaxed completeness criteria are called universal [GKSV23]. Our universal TDS learners accept all distributions that are sufficiently concentrated and anti-concentrated and work for convex sets with low intrinsic dimension ([Theorem 4.2](#)) and halfspace intersections ([Theorem 5.1](#)). Surprisingly, our algorithms can handle distributions that are heavy-tailed and multimodal, for which efficient (ordinary) agnostic learning algorithms are not known to exist. Our algorithms exploit localization guarantees from the training phase (e.g., subspace or boundary recovery) to relax the requirements of the testing phase.

Fully Polynomial-Time Testing. All of the TDS learners we provide consist of two decoupled phases. In the training phase, the algorithm uses labeled training examples to output a candidate hypothesis h . The testing phase receives the candidate h and uses unlabeled test examples to decide whether to reject or accept and output h . Separation of the two phases is an important feature of our approach, as it may be desirable for these tasks to be performed by distinct parties who have different amounts of available (computing) resources. Efficient implementations of the testing phase are of utmost importance, especially for potential users of large pre-trained models who need to certify that the candidate model at hand is safe to deploy. In [Theorem 5.1](#), we give *the first TDS learner for intersections of halfspaces that runs in fully polynomial test time*, and additionally improves the overall runtime of the previous state-of-the-art TDS learner for intersection of halfspaces by [\[KSV24a\]](#). In fact, our TDS learner’s overall runtime is polynomial in the dimension d , while the time complexity of the TDS learner given by [\[KSV24a\]](#) involved a factor of $d^{O(\log(1/\epsilon))}$, where ϵ is the error parameter.

1.2 Our Techniques

Our approach for designing TDS learners focuses on efficient algorithms for testing a new notion of localized discrepancy distance:

Definition 1.1 (Localized Discrepancy). Let \mathcal{D} be a distribution over $\mathcal{X} \subseteq \mathbb{R}^d$ and let $\mathcal{H}, \mathcal{C} \subseteq \{\pm 1\}^{\mathcal{X}}$ be hypothesis and concept classes respectively. Define neighborhood \mathbf{N} to be a function $\mathbf{N} : \mathcal{H} \rightarrow 2^{\mathcal{C}}$. For $\hat{f} \in \mathcal{H}$, the (\hat{f}, \mathbf{N}) -localized discrepancy from \mathcal{D} to \mathcal{D}' is defined as:

$$\text{disc}_{\hat{f}, \mathbf{N}}(\mathcal{D}, \mathcal{D}') = \sup_{f \in \mathbf{N}(\hat{f})} \left(\mathbb{P}_{\mathbf{x} \sim \mathcal{D}'}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim \mathcal{D}}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] \right)$$

Testing localized discrepancy is clearly easier than testing the traditional (global) discrepancy distance, since global discrepancy is defined with respect to a supremum over all pairs of concepts within some given class, while localized discrepancy only depends on a small neighborhood of concepts around some given reference classifier \hat{f} .

Assume for a moment that we have fixed a neighborhood function \mathbf{N} and have obtained a learner that always outputs a classifier close to the ground truth function f^* (i.e., $f^* \in \mathbf{N}(\hat{f})$). In this case, if we can test localized discrepancy, then we obtain a TDS learner as follows: output \hat{f} if the corresponding localized discrepancy is small and reject otherwise (recall \hat{f} is close to the ground truth for both training and test distributions).

The algorithmic challenge is finding a definition of neighborhood that admits both an efficient learner (for outputting a classifier close to the ground truth) and an efficient localized discrepancy tester. Smaller neighborhoods make the learning problem more difficult while larger neighborhoods make discrepancy testing more challenging.

Ultimately, the appropriate localized discrepancy relaxation of the testing phase depends on the guarantees one can ensure during training, which, in turn, depends on the properties of the concept class \mathcal{C} and the training distribution. For our main applications below we briefly describe the choice of neighborhood and the corresponding discrepancy tester. Note that we give a different discrepancy tester for each of the following cases.

Classes with Low-Degree Sandwiching Approximators. We show that the existence of degree- ℓ \mathcal{L}_1 -sandwiching approximators for a class \mathcal{C} over $\mathcal{X} \subseteq \mathbb{R}^d$ turns out to be sufficient to design a localized discrepancy tester that runs in time $d^{O(\ell)}$ where the notion of neighborhood is widest possible, i.e., $\mathbf{N}(\hat{f}) = \mathcal{C}$.⁶ In this case, the requirement for the training algorithm is minimal, as the ground truth f^* lies within \mathcal{C} , which coincides with $\mathbf{N}(\hat{f})$. The proposed tester is based on estimating the chow parameters of the reference hypothesis \hat{f} under the test marginal and checking whether they closely match the chow parameters of \hat{f} under the training marginal. For more details, see [Section 3](#).

Convex Sets with Low Intrinsic Dimension. For convex sets with few relevant dimensions, there are algorithms from standard PAC learning that guarantee approximate recovery of the relevant subspace. This guarantee allows one to choose a much stronger notion of neighborhood while

⁶The discrepancy is still localized, since it is defined with respect to a reference hypothesis \hat{f} .

still ensuring that $f^* \in \mathcal{N}(\hat{f})$. The appropriate notion of neighborhood contains low-dimensional concepts whose relevant subspace is geometrically close to the subspace of the reference hypothesis. The corresponding tester exhaustively checks that the marginal \mathcal{D}' is well-behaved on the relevant subspace. For more details, see [Section 4](#).

Intersections of Halfspaces. For intersections of halfspaces, we prove a structural result stating that finding a hypothesis with low Gaussian disagreement with the ground truth f^* implies approximate pointwise recovery of the boundary of f^* . It is therefore sufficient to check whether the marginal of the test distribution assigns unreasonably large mass near the boundary of the training output hypothesis \hat{f} , which can be done in fully polynomial time. Any proper algorithm for learning halfspace intersections under Gaussian training marginals is then sufficient for our purposes. For more details, see [Section 5](#).

1.3 Related Work

Domain Adaptation. In the past two decades, there has been a long line of research on generalization bounds for domain adaptation. The work of [\[MMR09\]](#) introduced the notion of discrepancy distance, following work by [\[BDBC06, BDBC⁺10\]](#), which used similar notions of distance between distributions. Other important notions of distribution similarity include bounded density ratios [\[SSK12\]](#) and related notions [\[KM21b, KZZ24\]](#). A type of localized discrepancy distance was defined by [\[ZLWJ20\]](#) and used to provide improved sample complexity bounds for domain adaptation. None of the above works give efficient (polynomial-time) algorithms. Here, we give a more general notion of localization and use it to obtain efficient and universal algorithms for TDS learning.

TDS Learning and Related Models. The framework of TDS learning was defined by [\[KSV24b\]](#), where it was shown that any class that admits degree- ℓ \mathcal{L}_2 -sandwiching approximators can be TDS learned in time $d^{O(\ell)}$ up to error $O(\lambda)$, where λ is the standard (and necessary) benchmark for the error in domain adaptation when the training and test distributions are allowed to be arbitrary. Here, we show that the relaxed notion of \mathcal{L}_1 -sandwiching approximators suffices for TDS learning and we improve the error guarantee to nearly-match the information-theoretically optimal λ (see [Section 3](#)). For intersections of halfspaces under Gaussian training marginals, [\[KSV24a\]](#) gave TDS learners with improved guarantees compared to those given by [\[KSV24b\]](#) through \mathcal{L}_2 sandwiching. Our TDS learners for halfspace intersections are superior to the ones from [\[KSV24a\]](#) in terms of overall runtime, universality and test-time efficiency (see [Section 5](#)).

Another related framework for learning with distribution shift is *PQ learning*, which was defined by [\[GKKM20\]](#). In PQ learning, the learner may reject regions of the domain where it is not confident to make predictions, but the total mass of these regions under the training distribution must be small. In fact, PQ learning is known to imply TDS learning (see [\[KSV24b\]](#)). However, the only known algorithms for PQ learning, which were given by [\[GKKM20, KK21\]](#), require access to oracles for learning primitives that are known to be hard even for simple classes (see [\[KK21\]](#)).

The framework of TDS learning is also related to testable agnostic learning, where the goal of the tester is to certify a near-optimal error guarantee. Testable agnostic learning was defined by [\[RV23\]](#) and there are several subsequent works in this framework [\[GKK23, GKS24, GKS23, DKK⁺23\]](#). There are many important differences between TDS learning and testable agnostic learning, including the fact that, in testable agnostic learning, there is no distribution shift and that in TDS learning, the learner does not have access to labels from the distribution on which it is evaluated. In particular, testable agnostic learning is only defined in the presence of noise in the labels, while TDS learning is meaningful even when the labels are generated noise-free (i.e., realizable learning).

PAC Learning. In the standard framework of PAC learning, there is an abundance of algorithmic ideas and techniques that aim to achieve efficient learning, under various assumptions (see e.g., [\[LW94, BK97, KOS04, KLT09, KOS08a, Vem10b, Vem10a, GKM12, KKM13, DKS18a, DTK22\]](#)). In this work, we make use of polynomial regression [\[KKMS08\]](#), dimension reduction techniques [\[Vem10a\]](#), as well as techniques for robustly learning geometric concepts [\[DKS18b\]](#), in order to obtain efficient TDS learners. In fact, our approach of designing TDS learning algorithms through localized discrepancy testing sheds a light on what kinds of guarantees from the training algorithms are desirable for learning in the presence of distribution shift. For example, we show that if approximate subspace recovery is guaranteed after training, then the discrepancy testing problem can be relaxed to

an easier, localized version. Moreover, our results on TDS learning halfspace intersections emphasize the importance of proper learners in the context of learning with distribution shift.

2 Preliminaries

We use standard big-O notation (and \tilde{O} to hide poly-logarithmic factors), \mathbb{R}^d is the d -dimensional euclidean space and \mathcal{N}_d the standard Gaussian over \mathbb{R}^d , $\{\pm 1\}^d$ is the d -dimensional hypercube and $\text{Unif}(\{\pm 1\}^d)$ the uniform distribution over $\{\pm 1\}^d$, \mathbb{N} is the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbf{x} \in \mathbb{R}^d$ denotes a vector with $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_d)$ and inner products $\mathbf{x} \cdot \mathbf{v}$. See also [Appendix A](#).

Localized Discrepancy Testing. Testing localized discrepancy ([Definition 1.1](#)) is defined as follows.

Definition 2.1 (Testing Localized Discrepancy). For a set \mathbb{D} of distributions and \mathcal{D} over \mathcal{X} and $\epsilon > 0$, we say that \mathcal{T} is a (\mathbb{N}, ϵ) -tester for localized discrepancy from \mathcal{D} with respect to \mathbb{D} , if, \mathcal{T} , upon receiving $\hat{f} \in \mathcal{H}$ and a set X of $m_{\mathcal{T}}$ i.i.d. examples from some distribution \mathcal{D}' over \mathcal{X} satisfies:

- (a) (Soundness.) With probability at least $3/4$: If \mathcal{T} accepts, then $\text{disc}_{\hat{f}, \mathbb{N}}(\mathcal{D}, \mathcal{D}') \leq \epsilon$.
- (b) (Completeness.) If $\mathcal{D}' \in \mathbb{D}$, then \mathcal{T} accepts with probability at least $3/4$.

For a concept class \mathcal{C} , a distribution \mathcal{D} over \mathcal{X} , $\epsilon \in (0, 1)$, we say that \mathcal{C} has ϵ - \mathcal{L}_1 **sandwiching degree** ℓ with respect to \mathcal{D} if for any $f \in \mathcal{C}$, there exist polynomials $p_{\text{up}}, p_{\text{down}}$ over \mathcal{X} with degree at most ℓ such that (1) $p_{\text{down}}(\mathbf{x}) \leq f(\mathbf{x}) \leq p_{\text{up}}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$ and (2) $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq \epsilon$.

Learning Setting. For $\mathcal{X} \subseteq \mathbb{R}^d$, the learner is given labeled samples from a training distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ over $\mathcal{X} \times \{\pm 1\}$ with \mathcal{X} -marginal $\mathcal{D}_{\mathcal{X}}^{\text{train}} = \mathcal{D}$ and unlabeled examples from the marginal $\mathcal{D}_{\mathcal{X}}^{\text{test}}$ of a test distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}$ over $\mathcal{X} \times \{\pm 1\}$. For a concept class $\mathcal{C} \subseteq \{\mathcal{X} \rightarrow \{\pm 1\}\}$, in the **realizable setting**, there is $f^* \in \mathcal{C}$ that generates the labels for both $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ and $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}$. In the **agnostic setting**, the standard goal in domain adaptation is to achieve an error guarantee that is competitive with the information-theoretically optimal joint error $\lambda = \min_{f \in \mathcal{C}} (\text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}))$, achieved by some $f^* \in \mathcal{C}$, where $\text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) = \mathbb{P}_{(\mathbf{x}, y) \sim \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}}[y \neq f(\mathbf{x})]$ (and similarly for $\text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}})$).

Definition 2.2 (Universal TDS Learning). Let \mathcal{C} be a concept class over $\mathcal{X} \subseteq \mathbb{R}^d$, \mathcal{D} a distribution over \mathcal{X} and \mathbb{D} some class of distributions over \mathcal{X} . The algorithm \mathcal{A} is said to \mathbb{D} -universally TDS learn \mathcal{C} with respect to \mathcal{D} up to error ψ and probability of failure δ if, upon receiving m_{train} labeled samples from a training distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ with \mathcal{X} -marginal \mathcal{D} and m_{test} unlabeled samples from a test distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}$, w.p. at least $1 - \delta$, algorithm \mathcal{A} either rejects, or accepts and outputs a hypothesis $h : \mathcal{X} \rightarrow \{\pm 1\}$ such that:

- (a) (Soundness.) If \mathcal{A} accepts, then the output h satisfies $\text{err}(h; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}) \leq \psi$.
- (b) (Completeness.) If $\mathcal{D}_{\mathcal{X}}^{\text{test}} \in \mathbb{D}$ then \mathcal{A} accepts.

In the agnostic setting, parameter ψ may depend on $\lambda = \lambda(\mathcal{C}; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}, \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}})$, whereas in the realizable setting, $\psi = \epsilon \in (0, 1)$. If $\mathbb{D} = \{\mathcal{D}\}$, then we simply say that \mathcal{A} ψ -TDS learns \mathcal{C} w.r.t. \mathcal{D} .

Note that the success probability for TDS learning can be amplified through repetition [[KSV24b](#)] and we will consider $\delta = 0.1$ unless specified otherwise.

3 Classes with Low Sandwiching Degree

Prior work on TDS learning by [[KSV24b](#)] showed that the existence of degree- ℓ \mathcal{L}_2 -sandwiching approximators implies TDS learning in time $d^{O(\ell)}$. A major question left open was whether the more traditional notion of \mathcal{L}_1 sandwiching (see [Definition C.1](#)) suffices for TDS learning. We answer this question in the affirmative, and as a consequence we obtain exponential improvements in the runtime of TDS learning for constant depth circuits (AC^0) and the first TDS learning results for degree-2 polynomial threshold functions (see [Table 1](#)). For more details, see [Appendix C](#).

Theorem 3.1 (\mathcal{L}_1 -sandwiching implies TDS learning). *Let $\epsilon, \delta \in (0, 1)$ and let $\mathcal{C} \subseteq \{\mathcal{X} \rightarrow \{\pm 1\}\}$ be a concept class such that the ϵ -approximate \mathcal{L}_1 -sandwiching degree of \mathcal{C} under \mathcal{D} is $\ell(\epsilon) \in \mathbb{N}$.*

Then, there exists a TDS learning algorithm for \mathcal{C} with respect to \mathcal{D} up to error $\lambda + \text{opt}_{\text{train}} + O(\epsilon)$ and fails with probability at most δ with time and sample complexity $\text{poly}(d^{\ell(\epsilon)}, \frac{1}{\epsilon}) \log(1/\delta)$.

Note that prior work [KSV24b] had only obtained a bound of $O(\lambda)$ in the above error guarantee. Our techniques allow us to achieve the optimal dependence of simply λ .

	Concept class	Training Marginal	Time	Prior Work
1	Degree-2 PTFs	\mathcal{N}_d or $\text{Unif}(\{\pm 1\}^d)$	$d^{\tilde{O}(1/\epsilon^9)}$	None
2	Circuits of size s , depth t	$\text{Unif}(\{\pm 1\}^d)$	$d^{O(\log(s/\epsilon))^{O(t)}}$	$d^{\sqrt{s} \cdot O(\log(s/\epsilon))^{O(t)}}$ only for formulas

Table 1: New results for TDS learning through \mathcal{L}_1 sandwiching. For constant-depth formulas, we achieve an exponential improvement compared to [KSV24b] (which used \mathcal{L}_2 -sandwiching), and our results work for circuits as well.

For Gaussian and uniform halfspaces, intersections and functions of halfspaces, as well as for decision trees over the uniform distribution, the \mathcal{L}_2 -sandwiching approach of [KSV24b] provided TDS learning algorithms with similar runtime as the one obtained here, but their error guarantee was $O(\lambda) + \epsilon$ instead of $\lambda + \text{opt}_{\text{train}} + \epsilon$ (where $\text{opt}_{\text{train}} = \min_{f \in \mathcal{C}} \text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}})$), which is the best known upper bound on the error, even information theoretically (see [BDBC⁺10, DLLP10]).

Localized discrepancy testing via Chow matching. The improvements we obtain here are based on the idea of substituting the moment-matching tester of [KSV24b] with a more localized test, depending on a candidate output hypothesis \hat{f} provided by a training algorithm run on samples from the training distribution. In particular, we estimate the Chow parameters [OS08] $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}}[\hat{f}(\mathbf{x})\mathbf{x}^\alpha]$ for all low-degree monomials $\mathbf{x}^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$ and reject if they do not match the corresponding quantities $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[\hat{f}(\mathbf{x})\mathbf{x}^\alpha]$ under the training marginal. We obtain the following result.

Proposition 3.2 (Informal, see [Theorem C.3](#)). *For any class \mathcal{C} with low sandwiching degree under \mathcal{D} , the low-degree chow matching tester is a tester for localized discrepancy for the neighborhood $\mathcal{N}(\hat{f}) = \mathcal{C}$, i.e., it certifies that $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] \leq \mathbb{P}_{\mathbf{x} \sim \mathcal{D}}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] + \epsilon$ for all $f \in \mathcal{C}$.*

Proof Outline. The main observation for obtaining the localized discrepancy testing result is that the disagreement between two functions is a linear function of their correlation, i.e., $2\mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] = 1 - \mathbb{E}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}}[\hat{f}(\mathbf{x})f(\mathbf{x})]$, and, because $f \in \mathcal{C}$, it is sandwiched by two polynomials $p_{\text{up}}, p_{\text{down}}$, which implies $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}}[\hat{f}(\mathbf{x})f(\mathbf{x})] \geq \mathbb{E}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}}[\hat{f}(\mathbf{x})p_{\text{up}}(\mathbf{x})] - \mathbb{E}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}}[p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})]$. The latter quantity can be certified to be close to the corresponding quantity under the training marginal \mathcal{D} by Chow (and moment) matching.

Although the notion of neighborhood we require here is quite generic, it is sufficient to provide significant improvements over prior work. The discrepancy tester is localized in the sense that it certifies properties of the tested marginal distribution that are related to a particular candidate hypothesis \hat{f} , but actually considers the whole concept class \mathcal{C} to be inside the neighborhood of \hat{f} . Since the concept f^* that achieves $\lambda = \min_{f \in \mathcal{C}} (\text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}))$ lies within \mathcal{C} by definition, the total test error of \hat{f} is directly related to the error achieved by the training algorithm, whenever the Chow matching tester accepts.

4 Non-Parametric Low-Dimensional Classes

For non-parametric classes like convex sets over \mathbb{R}^d , dimension-efficient TDS learning is impossible, even from an information-theoretic perspective [KSV24b] and $2^{\Omega(d)}$ time is required even in the realizable setting. However, the best known upper bound on the \mathcal{L}_1 sandwiching degree for convex sets is given indirectly by known results in approximation of convex sets by intersections of halfspaces (see, e.g., [DNS23] and references therein) and implies a TDS learning algorithm that runs in time doubly exponential in d . Improving on the doubly exponential bound based on \mathcal{L}_1 -sandwiching,

we provide a realizable TDS learner with singly exponential (in $\text{poly}(d)$) runtime for convex sets that are ϵ -balanced, meaning that the Gaussian mass of both the interior and the exterior of the convex set is at least ϵ . For convex sets with only a few relevant dimensions, our results actually give dimension-efficient TDS learners. For more details, see [Appendix D](#).

Theorem 4.1 (TDS Learning of Convex Subspace Juntas). *For $\epsilon \in (0, 1/2)$, $d, k \in \mathbb{N}$, let \mathcal{C} be the class of ϵ -balanced convex sets over \mathbb{R}^d with k relevant dimensions. There is an $O(\epsilon)$ -TDS learner for \mathcal{C} with respect to \mathcal{N}_d in the realizable setting, which, for the training phase, uses $\text{poly}(d)2^{\text{poly}(k/\epsilon)}$ samples and time and, for the testing phase, uses $\text{poly}(d)(k/\epsilon)^{O(k)}$ samples and time.*

We note that the balancing assumption is mild, since it can be tested by using examples from the training distribution and has been used in prior work on realizable TDS learning of intersections of halfspaces with respect to the Gaussian distribution [[KSV24a](#)].

Universal TDS Learners. Importantly, the TDS learner of [Theorem 4.1](#) can be made universal with respect to a wide class of distributions that enjoy some mild concentration and anti-concentration properties. The cost is an exponential deterioration of the runtime of the training phase. In other words, finding a hypothesis with better performance on the training distribution suffices to give error guarantees for a wide range of test distributions, including, for example, multi-modal and heavy-tailed distributions. We believe that this result is interesting even from an information-theoretic perspective. In [Table 2](#) in the appendix, we give a more precise trade-off between universality and training runtime.

Let \mathbb{D}_k be the class of distributions \mathcal{D} over \mathbb{R}^d such that $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[(\mathbf{v} \cdot \mathbf{x})^4] \leq C$ for any $\mathbf{v} \in \mathbb{S}^{d-1}$ and for any subspace $W \subseteq \mathbb{R}^d$ of dimension at most k , the marginal density of \mathcal{D} on W is upper bounded by C^{k^2} , where C is some positive universal constant. Then the following is true.

Theorem 4.2 (Universal TDS Learning of Convex Subspace Juntas). *There is a \mathbb{D}_k -universal $O(\epsilon)$ -TDS learner for k -dimensional ϵ -balanced convex sets over \mathbb{R}^d with respect to \mathcal{N}_d in the realizable setting, which, for the training phase, uses $\text{poly}(d) \exp(2^{O(k^2/\epsilon)})$ samples and time and, for the testing phase, uses $\text{poly}(d)k^{O(k^3/\epsilon^2)}$ samples and time.*

We remark that the testing time for the universal TDS learner of [Theorem 4.2](#) is still singly exponential in $\text{poly}(k)$, although the dependence on ϵ is exponentially worse. Having lower testing runtime is a desirable feature because the potential users of large machine learning models might have limited resources compared to those available during training. We provide a more thorough discussion about this feature in the following section.

Cylindrical grids tester for localized discrepancy. To obtain our TDS learning results of [Theorems 4.1](#) and [4.2](#), we once more make use of the localized discrepancy testing framework. In particular, we identify low-dimensionality ([Definition D.1](#)) and boundary smoothness ([Definition D.4](#)) of the underlying concept class as sufficient conditions for efficient testing of localized discrepancy when the notion of localization is defined with respect to the subspace neighborhood ([Theorem D.7](#)). The subspace neighborhood $\mathbb{N}_s(\hat{f})$ contains low-dimensional concepts f whose relevant subspace is geometrically close to the relevant subspace for \hat{f} (see [Definition D.2](#)). For TDS learning, we combine such testers with known learning algorithms for subspace recovery of low-dimensional convex sets (see, e.g., [[Vem10a](#), [KSV24a](#)] and [Theorem D.13](#)) to ensure that the training phase will output some hypothesis \hat{f} such that the ground truth f^* lies within $\mathbb{N}_s(\hat{f})$.

In other words, we exploit the existence of training algorithms with stronger guarantees (i.e., approximate subspace recovery) than merely training error bounds, to relax the discrepancy testing problem to a low-dimensional localized version, while still providing end-to-end results for TDS learning. This relaxation not only improves the testing runtime, but also enables universality, since the localized discrepancy between two distributions can be much smaller than the global discrepancy between them (see also [[ZLWJ20](#)] and references therein).

The idea behind the localized discrepancy tester for the subspace neighborhood is to split the disagreement between \hat{f} and an arbitrary concept $f \in \mathbb{N}_s(\hat{f})$ under the test distribution in two parts: (1) the disagreement between \hat{f} and a rotated version \tilde{f} of f where the input \mathbf{x} is projected on the relevant subspace of the given hypothesis \hat{f} instead of the actual, unknown relevant subspace of f and (2) the disagreement between \tilde{f} and f . For part (2), we use the fact that the relevant subspace of f is geometrically close to the relevant subspace for \hat{f} (since $f \in \mathbb{N}_s(\hat{f})$). We conclude that f

and \tilde{f} can only disagree far from the origin and, hence, testing that the test marginal is appropriately concentrated suffices to give the desired bound.

Low-dimensional disagreement between concepts with smooth boundaries. For part (1), we use the fact that the k -dimensional relevant subspace V for \hat{f} is known. We construct a grid on V and run tests to certify that the probability (under the test marginal) of falling inside each of the cells is not unreasonably large. In order to bound the size of the grid, we also test that the probability of falling far from the origin on the subspace V is appropriately bounded. We then argue that the disagreement region can be approximated reasonably well by discretizing with respect to an appropriately refined grid. To ensure that the discretization of the near-boundary region does not introduce a significant error blow-up, it is important that \hat{f} and \tilde{f} have smooth boundaries (see [Figure 2](#) in the appendix).

5 Fully Polynomial-Time Testers

Algorithms for TDS learning that are efficient in testing time, can be useful to check whether a pre-trained model can be applied to a particular population, without the need for overly expensive resources. Here, we focus on the class of balanced intersections of halfspaces (see [Definition E.9](#)) and provide the first TDS learner for this class that runs in fully polynomial time during test time. Moreover, the proposed tester is universal with respect to a wide class of distributions that satisfy some concentration and anticoncentration properties.

Let \mathbb{D}_1 be the class of distributions \mathcal{D} over \mathbb{R}^d such that for any $\mathbf{v} \in \mathbb{S}^{d-1}$ we have $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[(\mathbf{v} \cdot \mathbf{x})^4] \leq C$ and, also, that the one-dimensional density of the projection $\mathbf{v} \cdot \mathbf{x}$ where $\mathbf{x} \sim \mathcal{D}$ is upper bounded by C , where C is some positive universal constant. Then the following is true (see also [Theorem E.10](#)).

Theorem 5.1 (Universal TDS Learning of Balanced Intersections). *For $\epsilon \in (0, 1/2)$, $d, k \in \mathbb{N}$, there is a \mathbb{D}_1 -universal $O(\epsilon)$ -TDS learner for the class of ϵ -balanced intersections of k halfspaces over \mathbb{R}^d w.r.t. \mathcal{N}_d in the realizable setting, which, for the training phase, uses $\text{poly}(d) \exp(O(k^5/\epsilon))$ samples and time and, for the testing phase, uses $\text{poly}(d, k, 1/\epsilon)$ samples and time.*

For comparison, the previous state-of-the-art TDS learning algorithm for halfspace intersections by [[KSV24a](#)] had overall runtime $d^{O(\log(k/\epsilon))} + \text{poly}(d) \exp(O(k^6/\epsilon^8))$ and testing runtime $d^{O(\log(k/\epsilon))} + \text{poly}(d)(k/\epsilon)^{O(k^2)}$ (although training and testing were not explicitly separated). Hence, the overall runtime of the algorithm of [Theorem 5.1](#) is better than the previous state-of-the-art, but also enjoys two additional properties: (1) the testing time is fully polynomial and (2) the tester is universal with respect to a wide class (of multimodal and even heavy-tailed distributions).

We note that it is not by chance that these two properties are satisfied simultaneously: they both relate to the fact that it suffices to solve a simple discrepancy testing problem. Since the tested property is relaxed, more distributions should satisfy it and testing the property can be made efficient. For comparison, as well as to provide a TDS learner with better overall runtime in some regimes, we may trade-off universality and test-time efficiency to obtain the following result (see [Theorem E.10](#)).

Theorem 5.2 (TDS Learning of Balanced Intersections). *For $\epsilon \in (0, 1/2)$, $d, k \in \mathbb{N}$, there is an $O(\epsilon)$ -TDS learner for the class of ϵ -balanced intersections of k halfspaces over \mathbb{R}^d w.r.t. \mathcal{N}_d in the realizable setting, which, for the training phase, uses $\text{poly}(d)(k/\epsilon)^{O(k^3)}$ samples and time and, for the testing phase, uses $(dk)^{O(\log(1/\epsilon))}$ samples and time.*

Remark 5.3. The algorithms of [Theorems 5.1](#) and [5.2](#) can both tolerate some amount of noise, i.e., provide an $O(\epsilon)$ error guarantee even when $\lambda = \min_{f \in \mathcal{C}} (\text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}))$ is non-zero (but sufficiently small). For [Theorem 5.1](#), the amount of noise that can be tolerated is $\lambda = \exp(-\tilde{O}(k/\epsilon))$, while for [Theorem 5.2](#), the tolerated amount is $\lambda = (k/\epsilon)^{-O(k)}$ (see [Table 3](#)). The amount of noise tolerated by the non-universal tester is more, because the test is more expensive and, therefore, does a better job in translating the guarantees of the training phase to guarantees for the test error. For comparison, the Chow matching tester of [Theorem 3.1](#) runs much more expensive tests and can, therefore, tolerate much more noise, i.e., $\lambda = O(\epsilon)$.

Discrepancy testing through boundary proximity. We once more use the framework of localized discrepancy testing, in order to obtain TDS learners with strong guarantees. In order to achieve fully polynomial-time performance, we aim to use a tester that is as simple as possible. In particular, for a given halfspace intersection \hat{f} , we test whether the probability that an example drawn from the test

marginal falls close to the boundary of \hat{f} , i.e., close to at least one of the defining halfspaces (see [Lemma E.13](#) and [Definition E.3](#)). We also test concentration of the test distribution marginal.

Interestingly, we show that these two tests are sufficient for certifying low localized discrepancy from the Gaussian distribution with respect to the notion of disagreement neighborhood \mathbf{N}_e , i.e., $f \in \mathbf{N}_e(\hat{f})$ if the Gaussian disagreement $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_d}[f(\mathbf{x}) \neq \hat{f}(\mathbf{x})]$ between f and \hat{f} is small enough (see [Definition E.2](#)). In particular, we show that if f is a balanced intersection and $f \in \mathbf{N}_e(\hat{f})$, then f and \hat{f} can only differ either (1) far from the origin or (2) close to the boundary of \hat{f} (see [Proposition E.4](#) and [Lemma E.12](#)). Importantly, this property is point-wise: for any $\mathbf{x} \in \mathbb{R}^d$ such that $f(\mathbf{x}) \neq \hat{f}(\mathbf{x})$, \mathbf{x} will either satisfy (1) or (2) and, hence, no distribution over \mathbb{R}^d can fool our tester.

In the heart of our proof is a geometric lemma which demonstrates that any balanced convex set is locally balanced as well ([Lemma E.12](#)), meaning that for any point $\mathbf{x} \in \mathbb{R}^d$, there is a large number of points near \mathbf{x} with the same label as \mathbf{x} . Therefore (unless the norm of \mathbf{x} is large), any hypothesis \hat{f} with low Gaussian disagreement from the ground truth f^* , must encode all of the local structure (or boundary) of f^* that is not very far from the origin. To show this, we use a geometric argument about convex sets (see [Figure 1](#) for the case when the label of \mathbf{x} is 1. The other case is simpler and follows by the existence of a separating hyperplane between a convex set and any point outside it).

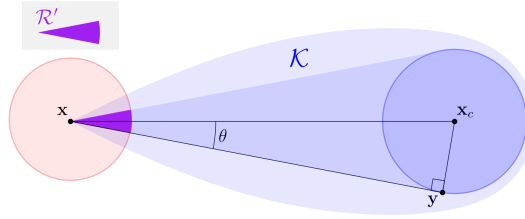


Figure 1: If \mathbf{x} lies within a balanced convex set \mathcal{K} , then many points close to \mathbf{x} lie within \mathcal{K} as well, i.e., there is a cone \mathcal{R}' with $\mathcal{R}' \subseteq \mathbb{B}(\mathbf{x}, \rho) \cap \mathcal{K}$, where $\mathbb{B}(\mathbf{x}, \rho)$ is a ball around \mathbf{x} . The ball centered at \mathbf{x}_c exists due to the fact that \mathcal{K} is balanced: any balanced convex set contains some ball with non-negligible radius. The convex hull of \mathbf{x} and the ball at \mathbf{x}_c lies within \mathcal{K} . (See also [Fig. 3](#))

Since we have a localized discrepancy tester with respect to the disagreement neighborhood, all we need from the training phase is to output some intersection of halfspaces \hat{f} with low training error (so that the ground truth f^* lies within $\mathbf{N}_e(\hat{f})$). Hence, we may use any proper PAC learning algorithm for intersections of halfspaces under the Gaussian distribution. We use the algorithm by [\[DKS18b\]](#) (see also [Theorem E.11](#)).

Remark 5.4. We note that the three important properties we used to apply the method of boundary proximity are that (1) the hypothesis \hat{f} returned by the learning algorithm admits an efficient boundary proximity tester and (2) the ground truth f^* is locally balanced and (3) that \hat{f} and f^* are both low-dimensional. For more details, see [Appendix E](#).

6 Limitations, Future Work and Broader Impacts

TDS learning beyond discrepancy testing. We show that all of the known results in TDS learning can be achieved (and improved) by decoupling the training and testing phases. While separating training and testing phases is appealing and well-motivated by real-world scenarios, it is an interesting open question whether using the examples from the test marginal during training time could lead to improved TDS learning algorithms.

Characterizations of discrepancy testing complexity. We provide several positive results for localized discrepancy testing which imply new results in TDS learning. Moreover, on the lower bounds side, in [Appendix F](#), we show that global discrepancy testing is NP-hard even for simple classes under no further assumptions. It is an interesting open question to explore tight characterizations for dimension-efficient, universal or fully polynomial-time localized discrepancy testing.

Lifting the balancing assumption. For our universal TDS learners (and universal discrepancy testers), we require that the underlying concept class only contains concepts that are not too biased

towards one of the two possible labels under the training distribution (so that the training examples include enough information for localization). This condition is mild and can be easily tested by using training examples. However, better understanding of the importance of this condition for (universal) TDS learning could potentially lead to (or rule out) improved and/or universal algorithms for broader concept classes, e.g., polynomial threshold functions.

Relaxing assumptions on training marginal. Our main results in this work hold under the assumption that the marginal of the training distribution is either the Gaussian distribution or the uniform distribution over the hypercube. Such assumptions are standard in learning theory, as they serve as a concrete theoretical testbed for simplifying the analysis and presentation of the proposed algorithms and ideas. Relaxing those assumptions is an important and obvious goal for future work and parts of our analysis hint towards such relaxations (see, e.g., [Remarks D.10](#) and [E.5](#)).

Broader Impacts. We do not identify any direct potential negative societal impacts. In fact, although our results are of theoretical nature, our algorithms might, in principle, help mitigate potentially unfair outcomes of applying certain pre-trained models on populations that are misrepresented in training data. Our discrepancy testers will either certify low prediction error on the deployment population or signal that the model at hand might not be applicable to the deployment population and another model should be considered.

References

- [AGM03] Noga Alon, Oded Goldreich, and Yishay Mansour. Almost k -wise independence versus k -wise independence. *Information Processing Letters*, 88(3):107–110, 2003.
- [Bal93] Keith Ball. The reverse isoperimetric problem for gaussian measure. *Discrete Comput. Geom.*, 10(4):411–420, dec 1993.
- [Baz09] Louay MJ Bazzi. Polylogarithmic independence can fool dnf formulas. *SIAM Journal on Computing*, 38(6):2220–2272, 2009.
- [BCK⁺07] John Blitzer, Koby Crammer, Alex Kulesza, Fernando Pereira, and Jennifer Wortman. Learning bounds for domain adaptation. *Advances in neural information processing systems*, 20, 2007.
- [BDBC⁺10] Shai Ben-David, John Blitzer, Koby Crammer, Alex Kulesza, Fernando Pereira, and Jennifer Wortman Vaughan. A theory of learning from different domains. *Machine learning*, 79:151–175, 2010.
- [BDBCP06] Shai Ben-David, John Blitzer, Koby Crammer, and Fernando Pereira. Analysis of representations for domain adaptation. *Advances in neural information processing systems*, 19, 2006.
- [BGS18] Arnab Bhattacharyya, Suprovat Ghoshal, and Rishi Saket. Hardness of learning noisy halfspaces using polynomial thresholds. In Sébastien Bubeck, Vianney Perchet, and Philippe Rigollet, editors, *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pages 876–917. PMLR, 06–09 Jul 2018.
- [BK97] Avrim Blum and Ravindran Kannan. Learning an intersection of a constant number of halfspaces over a uniform distribution. *J. Comput. Syst. Sci.*, 54(2):371–380, 1997.
- [Bog98] Vladimir Igorevich Bogachev. *Gaussian measures*. Number 62. American Mathematical Soc., 1998.
- [Bra10] Mark Braverman. Polylogarithmic independence fools AC^0 circuits. *Journal of the ACM (JACM)*, 57(5):1–10, 2010.
- [DKK⁺23] Ilias Diakonikolas, Daniel Kane, Vasilis Kontonis, Sihan Liu, and Nikos Zarifis. Efficient testable learning of halfspaces with adversarial label noise. *Advances in Neural Information Processing Systems*, 36, 2023.

- [DKN10] Ilias Diakonikolas, Daniel M Kane, and Jelani Nelson. Bounded independence fools degree-2 threshold functions. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science*, pages 11–20. IEEE, 2010.
- [DKS18a] Ilias Diakonikolas, Daniel M. Kane, and Alistair Stewart. Learning geometric concepts with nasty noise. In Ilias Diakonikolas, David Kempe, and Monika Henzinger, editors, *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018*, pages 1061–1073. ACM, 2018.
- [DKS18b] Ilias Diakonikolas, Daniel M Kane, and Alistair Stewart. Learning geometric concepts with nasty noise. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1061–1073, 2018.
- [DLLP10] Shai Ben David, Tyler Lu, Teresa Luu, and Dávid Pál. Impossibility theorems for domain adaptation. In *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics*, pages 129–136. JMLR Workshop and Conference Proceedings, 2010.
- [DNS23] Anindya De, Shivam Nadimpalli, and Rocco A Servedio. Gaussian approximation of convex sets by intersections of halfspaces. *arXiv preprint arXiv:2311.08575*, 2023.
- [DTK22] Ilias Diakonikolas, Christos Tzamos, and Daniel M Kane. A strongly polynomial algorithm for approximate forster transforms and its application to halfspace learning. *arXiv preprint arXiv:2212.03008*, 2022.
- [GKK23] Aravind Gollakota, Adam R Klivans, and Pravesh K Kothari. A moment-matching approach to testable learning and a new characterization of rademacher complexity. *Proceedings of the fifty-fifth annual ACM Symposium on Theory of Computing*, 2023.
- [GKKM20] Shafi Goldwasser, Adam Tauman Kalai, Yael Kalai, and Omar Montasser. Beyond perturbations: Learning guarantees with arbitrary adversarial test examples. *Advances in Neural Information Processing Systems*, 33:15859–15870, 2020.
- [GKM12] Parikshit Gopalan, Adam R. Klivans, and Raghu Meka. Learning functions of halfspaces using prefix covers. In Shie Mannor, Nathan Srebro, and Robert C. Williamson, editors, *COLT 2012 - The 25th Annual Conference on Learning Theory, June 25-27, 2012, Edinburgh, Scotland*, volume 23 of *JMLR Proceedings*, pages 15.1–15.10. JMLR.org, 2012.
- [GKSV23] Aravind Gollakota, Adam Klivans, Konstantinos Stavropoulos, and Arsen Vasilyan. Tester-learners for halfspaces: Universal algorithms. *Advances in Neural Information Processing Systems*, 36, 2023.
- [GKSV24] Aravind Gollakota, Adam R Klivans, Konstantinos Stavropoulos, and Arsen Vasilyan. An efficient tester-learner for halfspaces. *The Twelfth International Conference on Learning Representations*, 2024.
- [HKM23] Steve Hanneke, Samory Kpotufe, and Yasaman Mahdaviyeh. Limits of model selection under transfer learning. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 5781–5812. PMLR, 2023.
- [HS19] Prahladh Harsha and Srikanth Srinivasan. On polynomial approximations to AC^0 . *Random Structures & Algorithms*, 54(2):289–303, 2019.
- [KK21] Adam Tauman Kalai and Varun Kanade. Efficient learning with arbitrary covariate shift. In *Algorithmic Learning Theory*, pages 850–864. PMLR, 2021.
- [KKM13] Daniel M. Kane, Adam R. Klivans, and Raghu Meka. Learning halfspaces under log-concave densities: Polynomial approximations and moment matching. In Shai Shalev-Shwartz and Ingo Steinwart, editors, *COLT 2013 - The 26th Annual Conference on Learning Theory, June 12-14, 2013, Princeton University, NJ, USA*, volume 30 of *JMLR Workshop and Conference Proceedings*, pages 522–545. JMLR.org, 2013.

- [KKMS08] Adam Tauman Kalai, Adam R Klivans, Yishay Mansour, and Rocco A Servedio. Agnostically learning halfspaces. *SIAM Journal on Computing*, 37(6):1777–1805, 2008.
- [KLT09] Adam Klivans, Philip Long, and Alex Tang. Baum’s algorithm learns intersections of halfspaces with respect to log-concave distributions. pages 588–600, 01 2009.
- [KM21a] Zander Kelley and Raghu Meka. Random restrictions and prgs for pfts in gaussian space. *arXiv preprint arXiv:2103.14134*, 2021.
- [KM21b] Samory Kpotufe and Guillaume Martinet. Marginal singularity and the benefits of labels in covariate-shift. *The Annals of Statistics*, 49(6):3299–3323, 2021.
- [KOS04] Adam R. Klivans, Ryan O’Donnell, and Rocco A. Servedio. Learning intersections and thresholds of halfspaces. *J. Comput. Syst. Sci.*, 68(4):808–840, 2004.
- [KOS08a] Adam R. Klivans, Ryan O’Donnell, and Rocco A. Servedio. Learning geometric concepts via gaussian surface area. In *49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2008, October 25-28, 2008, Philadelphia, PA, USA*, pages 541–550. IEEE Computer Society, 2008.
- [KOS08b] Adam R Klivans, Ryan O’Donnell, and Rocco A Servedio. Learning geometric concepts via gaussian surface area. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 541–550. IEEE, 2008.
- [KSV24a] Adam Klivans, Konstantinos Stavropoulos, and Arsen Vasilyan. Learning intersections of halfspaces with distribution shift: Improved algorithms and sq lower bounds. In Shipra Agrawal and Aaron Roth, editors, *Proceedings of Thirty Seventh Conference on Learning Theory*, volume 247 of *Proceedings of Machine Learning Research*, pages 2944–2978. PMLR, 30 Jun–03 Jul 2024.
- [KSV24b] Adam Klivans, Konstantinos Stavropoulos, and Arsen Vasilyan. Testable learning with distribution shift. In Shipra Agrawal and Aaron Roth, editors, *Proceedings of Thirty Seventh Conference on Learning Theory*, volume 247 of *Proceedings of Machine Learning Research*, pages 2887–2943. PMLR, 30 Jun–03 Jul 2024.
- [KZZ24] Alkis Kalavasis, Ilias Zadik, and Manolis Zampetakis. Transfer learning beyond bounded density ratios. *arXiv preprint arXiv:2403.11963*, 2024.
- [LV07] László Lovász and Santosh Vempala. The geometry of logconcave functions and sampling algorithms. *Random Structures & Algorithms*, 30(3):307–358, 2007.
- [LW94] Philip M. Long and Manfred K. Warmuth. Composite geometric concepts and polynomial predictability. *Inf. Comput.*, 113(2):230–252, 1994.
- [MMR09] Yishay Mansour, Mehryar Mohri, and Afshin Rostamizadeh. Domain adaptation: Learning bounds and algorithms. In *Proceedings of The 22nd Annual Conference on Learning Theory (COLT 2009)*, Montréal, Canada, 2009.
- [Nel73] Edward Nelson. The free markoff field. *Journal of Functional Analysis*, 12(2):211–227, 1973.
- [O’D14] Ryan O’Donnell. *Analysis of boolean functions*. Cambridge University Press, 2014.
- [OS08] Ryan O’Donnell and Rocco A Servedio. The chow parameters problem. In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pages 517–526, 2008.
- [RMH⁺20] Ievgen Redko, Emilie Morvant, Amaury Habrard, Marc Sebban, and Younès Bennani. A survey on domain adaptation theory: learning bounds and theoretical guarantees. *arXiv preprint arXiv:2004.11829*, 2020.
- [RV23] Ronitt Rubinfeld and Arsen Vasilyan. Testing distributional assumptions of learning algorithms. *Proceedings of the fifty-fifth annual ACM Symposium on Theory of Computing*, 2023.

- [SSK12] Masashi Sugiyama, Taiji Suzuki, and Takafumi Kanamori. *Density ratio estimation in machine learning*. Cambridge University Press, 2012.
- [Tal17] Avishay Tal. Tight bounds on the Fourier spectrum of AC^0 . In *32nd Computational Complexity Conference (CCC 2017)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.
- [Vem10a] Santosh S Vempala. Learning convex concepts from gaussian distributions with pca. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science*, pages 124–130. IEEE, 2010.
- [Vem10b] Santosh S Vempala. A random-sampling-based algorithm for learning intersections of halfspaces. *Journal of the ACM (JACM)*, 57(6):1–14, 2010.
- [Ver18] Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press, 2018.
- [ZLWJ20] Yuchen Zhang, Mingsheng Long, Jianmin Wang, and Michael I Jordan. On localized discrepancy for domain adaptation. *arXiv preprint arXiv:2008.06242*, 2020.

A Extended Preliminaries

We use standard big-O notation (and \tilde{O} to hide poly-logarithmic factors), \mathbb{R}^d is the d -dimensional euclidean space and \mathcal{N}_d the standard Gaussian over \mathbb{R}^d , $\{\pm 1\}^d$ is the d -dimensional hypercube and $\text{Unif}(\{\pm 1\}^d)$ the uniform distribution over $\{\pm 1\}^d$, \mathbb{N} is the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbf{x} \in \mathbb{R}^d$ denotes a vector with $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_d)$ and inner products $\mathbf{x} \cdot \mathbf{v}$. For $\alpha \in \mathbb{N}^d$, we denote with \mathbf{x}^α the product $\prod_{i \in [d]} \mathbf{x}_i^{\alpha_i}$, $M_\alpha = \mathbb{E}[\mathbf{x}^\alpha]$ and $\|\alpha\|_1 = \sum_{i \in [d]} \alpha_i$. For a polynomial⁷ p over \mathbb{R}^d and $\alpha \in \mathbb{N}^d$, we denote with p_α the coefficient of p corresponding to \mathbf{x}^α , i.e., we have $p(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^d} p_\alpha \mathbf{x}^\alpha$. If p is a polynomial over $\{\pm 1\}^d$, then we express it in its multilinear form, using only coefficients p_α with $\alpha \in \{0, 1\}^d$, i.e., $p(\mathbf{x}) = \sum_{\alpha \in \{0, 1\}^d} p_\alpha \mathbf{x}^\alpha$. We define the degree of p and denote $\deg(p)$ the maximum degree of a monomial whose coefficient in p is non-zero. We use standard notations for norms $\|\mathbf{x}\|_1 = \sum_{i \in [d]} |\mathbf{x}_i|$, $\|\mathbf{x}\|_2 = (\sum_{i \in [d]} \mathbf{x}_i^2)^{1/2}$ and $\|\mathbf{x}\|_\infty = \max_{i \in [d]} |\mathbf{x}_i|$. We denote with \mathbb{S}^{d-1} the $d-1$ dimensional sphere on \mathbb{R}^d and, for $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, $\mathbb{B}_k(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^k : \|\mathbf{x} - \mathbf{y}\|_2 \leq r\}$.

For any $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$, we denote with $\mathbf{v}_1 \cdot \mathbf{v}_2$ the inner product between \mathbf{v}_1 and \mathbf{v}_2 and we let $\angle(\mathbf{v}_1, \mathbf{v}_2)$ be the angle between the two vectors, i.e., the quantity $\theta \in [0, \pi]$ such that $\|\mathbf{v}_1\|_2 \|\mathbf{v}_2\|_2 \cos(\theta) = \mathbf{v}_1 \cdot \mathbf{v}_2$. For $\mathbf{v} \in \mathbb{R}^d, \tau \in \mathbb{R}$, we call a function of the form $\mathbf{x} \mapsto \text{sign}(\mathbf{v} \cdot \mathbf{x})$ an origin-centered (or homogeneous) halfspace and a function of the form $\mathbf{x} \mapsto \text{sign}(\mathbf{v} \cdot \mathbf{x} + \tau)$ a general halfspace over \mathbb{R}^d .

We let $\mathcal{X} \subseteq \mathbb{R}^d$ be either the d -dimensional hypercube $\{\pm 1\}^d$ or \mathbb{R}^d . For a distribution \mathcal{D} over \mathcal{X} , we use $\mathbb{E}_{\mathcal{D}}$ (or $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}$) to refer to the expectation over distribution \mathcal{D} and for a given set X , we use \mathbb{E}_X (or $\mathbb{E}_{\mathbf{x} \sim X}$) to refer to the expectation over the uniform distribution on X (i.e., $\mathbb{E}_{\mathbf{x} \sim X}[g(\mathbf{x})] = \frac{1}{|X|} \sum_{\mathbf{x} \in X} g(\mathbf{x})$, counting possible duplicates separately). We let $\mathbb{R}_+ = (0, \infty)$.

We define the notion of balance as follows.

Definition A.1 (Balanced Concepts). For $\beta \in (0, 1)$, we say that a function $f : \mathbb{R}^d \rightarrow \{\pm 1\}$ is (globally) β -balanced if for any $\mathbf{x} \in \mathbb{R}^d$ we have $\mathbb{P}_{\mathbf{z} \sim \mathcal{N}}[f(\mathbf{z}) = f(\mathbf{x})] > \beta$.

B Additional Tools

B.1 Boundary Smoothness of Structured Concepts

In this section, we prove that low dimensional polynomial threshold functions and convex sets have smooth boundary, i.e., a non-asymptotic anticoncentration bounds that scales linearly with the distance from the boundary. We first prove that PTFs have smooth boundary.

Lemma B.1 (Smooth Boundary for PTFs). *Let p be a polynomial of degree ℓ over \mathbb{R}^k . Let $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ be the function defined as $F(\mathbf{x}) = \text{sign}(p(\mathbf{x}))$. Then, F has a $C\ell^3 k$ -smooth boundary with respect to \mathcal{N}_k for a large universal constant C .*

Proof. Let C be a large universal constant that we fix later. Let $\delta = 3C\ell^3 \gamma k$. Define the set $S := \{\mathbf{x} \mid \exists i \in [\ell], \|\nabla^i p(\mathbf{x})\|_2 > (C\ell^3/\delta) \cdot \|\nabla^{i-1} p(\mathbf{x})\|_2\}$. Observe that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in \partial_\gamma F] \leq \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S] + \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in \partial_\gamma f \mid \mathbf{x} \notin S]$. We bound these two terms separately. To bound the first term, we use the following theorem from [KM21a].

Lemma B.2 (Lemma 1.6 from [KM21a]). *Let C be a large universal constant. For any polynomial $p : \mathbb{R}^k \rightarrow \mathbb{R}$ of degree ℓ and $\mathbf{x} \sim \mathcal{N}_k$, the following event occurs with probability at least $1 - \delta$:*

$$\|\nabla^i p(\mathbf{x})\|_2 \leq (C\ell^3/\delta) \|\nabla^{i-1} p(\mathbf{x})\|_2, \text{ for all } 1 \leq i \leq \ell.$$

Thus, we have that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S] \leq \delta$. Now consider a point $\mathbf{x} \notin S$. From a multivariate Taylor expansion, we have that $p(\mathbf{x} + \mathbf{z}) = p(\mathbf{x}) + \sum_{\alpha \in \mathbb{N}^k, 1 \leq |\alpha| \leq \ell} \frac{\partial^\alpha p(\mathbf{x})}{\alpha!} \cdot \mathbf{z}^\alpha$. Thus, for $\mathbf{z} \in \mathbb{R}^k$ with

⁷In Appendices D and E, we use the notation p to denote natural numbers and use q for polynomials instead.

$\|\mathbf{z}\|_2 \leq \gamma$, we obtain that

$$\begin{aligned} |p(\mathbf{x}) - p(\mathbf{x} + \mathbf{z})| &\leq \sum_{1 \leq |\alpha| \leq \ell} |\partial^\alpha p(\mathbf{x})| \cdot \|\mathbf{z}\|_\infty^{|\alpha|} \leq \sum_{i \in [\ell]} \|\mathbf{z}\|_2^i \cdot \|\nabla^i p(\mathbf{x})\|_1 \\ &\leq \sum_{i \in [\ell]} \gamma^i k^i \|\nabla^i p(\mathbf{x})\|_2 \leq \sum_{i \in [\ell]} \gamma^i k^i (C\ell^3/\delta)^i |p(\mathbf{x})| \leq |p(\mathbf{x})|/2. \end{aligned}$$

The first inequality follows from the multivariate Taylor expansion. The third inequality follows from the fact that $\|\mathbf{z}\|_2 \leq \gamma$ and the bound on the number of monomials of size i by k^{2i} . The penultimate inequality follows from the definition of the set S and the last inequality is true by our choice of δ .

Since $|p(\mathbf{x}) - p(\mathbf{x} + \mathbf{z})| \leq |p(\mathbf{x})|/2$, we have that $F(\mathbf{x}) = F(\mathbf{x} + \mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^k$ with $\|\mathbf{z}\|_2 \leq \gamma$. Thus, we have that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in \partial_\gamma F \mid \mathbf{x} \notin S] = 0$. Thus, we have that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in \partial_\gamma F] \leq 3C\ell^3\gamma k$. \square

We now move on to proving that low dimensional convex sets. To prove this, we will crucially use the notion of Gaussian surface area (an asymptotic anticoncentration bound) that we will now define.

Definition B.3 (Gaussian Surface Area). Let f be a boolean function. The Gaussian surface area $\Gamma(f)$ is defined as

$$\Gamma(f) = \liminf_{\delta \rightarrow 0} \frac{1}{\delta} \mathbb{P}_{\mathbf{z} \sim \mathcal{N}(0, I_k)}[\mathbf{z} \in A_f^\delta \setminus A_f],$$

where $A_f = \mathbb{1}\{\mathbf{x} \mid f(\mathbf{x}) = 1\}$, $A_f^\delta = \{\mathbf{u} : \min_{\mathbf{v} \in A_f} \|\mathbf{u} - \mathbf{v}\|_2 \leq \delta\}$.

We prove that convex sets have smooth boundary in two steps. We first prove that the set of points inside the set that are close to its boundary have small mass. To do this, we use a noise sensitivity argument ([Lemma B.5](#)). Then, we prove that points outside it that are close to the boundary ([Lemma B.7](#)). This will follow from an argument uses the definition of Gaussian Surface area and a bound on this quantity for convex sets due to [[Bal93](#)]. Together, these two lemmas imply that convex sets have smooth boundary.

The following lemma will be useful in proving the smooth boundary of the interior of the set.

Lemma B.4. Let $\lambda \in (0, 1/2)$. Let S be a convex set on \mathbb{R}^k and let $f(\mathbf{x}) = \mathbb{1}\{\mathbf{x} \in S\}$ be the indicator function of S . Then, we have that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[f(\mathbf{x}) \neq f(\mathbf{x}/\sqrt{1-\lambda})] \leq k \log k \sqrt{\lambda}$.

Proof. For any vector $\mathbf{w} \in \mathbb{R}^k$ with $\|\mathbf{w}\|_2 = 1$, let $f_{\mathbf{w}} : \mathbb{R}^+ \rightarrow \mathbb{R}$ be the function defined as $f_{\mathbf{w}}(r) = f(r \cdot \mathbf{w})$. Also, note that $f_{\mathbf{w}}$ is the indicator function of a one dimensional convex set. Observe that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[f(\mathbf{x}) \neq f(\mathbf{x}/\sqrt{1-\lambda})] \leq \sup_{\|\mathbf{w}\|_2=1} \mathbb{P}_{r \sim \chi^2(k)}[f_{\mathbf{w}}(\sqrt{r}) \neq f_{\mathbf{w}}(\sqrt{r}/\sqrt{1-\lambda})]$ from the fact that the k dimensional Gaussian conditioned on pointing in direction \mathbf{w} is distributed as $\sqrt{r}\mathbf{w}$ where $r \sim \chi^2(k)$. Here, $\chi^2(k)$ is the one dimensional Chi-squared distribution with mean k .

We have thus reduced the problem to one dimension. Consider a function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(x) = \mathbb{1}\{x \in [\sqrt{a/(1-\lambda)}, \sqrt{b/(1-\lambda)}]\}$ where a, b are from $\mathbb{R}^+ \cup \{+\infty\}$. All one dimensional indicators of convex sets are of this form. We will now prove that $\mathbb{P}_{r \sim \chi^2(k)}[g(\sqrt{r}) \neq g(\sqrt{r}/\sqrt{1-\lambda})] \leq k\lambda \log(k/\lambda)$.

Observe that $\mathbb{P}_{r \sim \chi^2(k)}[g(\sqrt{r}) \neq g(\sqrt{r}/\sqrt{1-\lambda})] \leq \mathbb{P}_{r \sim \chi^2(k)}[r \in [a, a/(1-\lambda)] \cup [b, b/(1-\lambda)]]$. It suffices to bound $\mathbb{P}_{r \sim \chi^2(k)}[r \in [a, a/(1-\lambda)]]$ for $a \in \mathbb{R}^+$ as the claim then follows from a union bound. We bound this by splitting into two cases.

Case 1: $a \geq 2k \log(k/\lambda)$. Since $\chi^2(k)$ is the distribution of the sum of squares of k independent $\mathcal{N}(0, 1)$ Gaussian random variables, we have that $\mathbb{P}_{r \sim \chi^2(k)}[r \geq a] \leq k \mathbb{P}_{x \sim \mathcal{N}(0,1)}[|x|^2 \geq a/k] \leq k e^{-a/(2k)}$. Thus, when $a \geq 2k \log(k/\lambda)$, we have that $\mathbb{P}_{r \sim \chi^2(k)}[r \in [a, a/(1-\lambda)]] \leq \mathbb{P}_{r \sim \chi^2(k)}[r \geq a] \leq \lambda$.

Case 2: $a < 2k \log(k/\lambda)$. Let ψ be the density function for $\chi^2(k)$. It is a standard fact from probability that $\psi(x) = \frac{x^{k/2-1}}{2^{k/2}\Gamma(k/2)} e^{-x/2}$. For $k = 1$, it is a fact that $\psi(x) \leq 1$. For $k \geq 2$, by taking

a derivative, we can see that this density function is maximized at $x = k - 2$. We obtain that

$$\psi(x) = \frac{(k-2)^{k/2-1}}{2^{k/2}\Gamma(k/2)} e^{-k/2+1} \leq \frac{((k-2) \cdot e)^{k/2-1}}{2^{k/2}(k/2)^{k/2-1}} e^{-k/2+1} \leq \frac{1}{2}$$

where the second inequality follows from the fact that $\Gamma(t) \geq (\frac{t}{e})^{t-1}$ for all $t \geq 2$ and $\Gamma(1) = 1$. We have that

$$\mathbb{P}_{r \sim \chi^2(k)} [r \in [a, a/(1-\lambda)]] \leq \|\psi\|_\infty \cdot a(1/(1-\lambda) - 1) \leq 4k\lambda \log(k/\lambda) \leq 4k \log k \sqrt{\lambda}.$$

We get the first inequality from the upper bound on the density. The second follows from the fact that $1/(1-\lambda) \leq 1 + 2\lambda$ when $\lambda < 1/2$. The third inequality follows from the assumption on a . The final inequality follows from the fact that $x \log(1/x) \leq \sqrt{x}$. \square

We are now ready to prove the set of points inside the convex set that are close to its boundary have small mass under the Gaussian.

Lemma B.5. *Let S be a convex set on \mathbb{R}^k . Let $\varrho \in (0, 1)$. Then, we have that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S \cap \partial_\varrho S] \leq Ck \log k \varrho$ where C is a large universal constant.*

Proof. Define the function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ as $f(\mathbf{x}) = \mathbb{1}\{\mathbf{x} \in S\}$. We now use a restatement of Corollary 12 from [KOS08b].

Lemma B.6. *Let g be a boolean function on \mathbb{R}^k . For any $\lambda \in (0, 1)$, it holds that*

$$\mathbb{P}_{\mathbf{x}, \mathbf{y} \sim \mathcal{N}_k} \left[g(\mathbf{x}) \neq g\left(\sqrt{1-\lambda}\mathbf{x} + \sqrt{\lambda}\mathbf{y}\right) \right] \leq C\sqrt{\lambda}\Gamma(g)$$

for large universal constant C .

Let g be the function $g(\mathbf{x}) = f(\mathbf{x}/\sqrt{1-\lambda})$. Observe that g is also an indicator of a convex set. From [Bal93] we have that $\Gamma(g) \leq 4k^{1/4}$. Thus, applying Lemma B.6 to g , we obtain that for any $\lambda \in (0, 1)$

$$\mathbb{P}_{\mathbf{x}, \mathbf{y} \sim \mathcal{N}_k} \left[f(\mathbf{x}/\sqrt{1-\lambda}) \neq f\left(\mathbf{x} + \sqrt{\frac{\lambda}{1-\lambda}}\mathbf{y}\right) \right] \leq C\sqrt{\lambda}k^{1/4}$$

where C is a large constant. Combining the above expression with Lemma B.4, we obtain that for any $\lambda \in (0, 1/2)$,

$$\mathbb{P}_{\mathbf{x}, \mathbf{y} \sim \mathcal{N}_k} \left[f(\mathbf{x}) \neq f\left(\mathbf{x} + \sqrt{\frac{\lambda}{1-\lambda}}\mathbf{y}\right) \right] \leq C\sqrt{\lambda}k^{1/4} + 2k \log k \sqrt{\lambda}. \quad (\text{B.1})$$

Now, consider any point \mathbf{p} in $S \cap \partial_\varrho S$. Since S is convex, there exists a hyperplane $h(\mathbf{x}) = \mathbb{1}\{\mathbf{w} \cdot \mathbf{x} + b \geq 0\}$ for $\mathbf{w} \in \mathbb{R}^k$ with $\|\mathbf{w}\|_2 = 1$ and $b \in \mathbb{R}$ such that $h(\mathbf{y}) = 1$ for all $\mathbf{y} \in S$ and $\mathbf{w} \cdot \mathbf{p} + b \leq \varrho$. This hyperplane corresponds to the tangential plane whose normal vector is the line joining \mathbf{p} and the point closest to it in ∂S . We have that for any $\gamma > 0$, $\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[\mathbf{w} \cdot \gamma \mathbf{z} \leq -\varrho] \geq \frac{1}{2} - \frac{\varrho}{2\gamma}$ as the Gaussian density is upper bounded by 1 pointwise. Thus, for any $\gamma > 0$, $\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[f(\mathbf{p} + \gamma \mathbf{z}) \neq f(\mathbf{p})] \geq \frac{1}{2} - \frac{\varrho}{2\gamma}$. Combining this with Equation (B.1), we obtain that

$$\left(\frac{1}{2} - \frac{\varrho}{2} \cdot \sqrt{\frac{1-\lambda}{\lambda}}\right) \cdot \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S \cap \partial_\varrho S] \leq C\sqrt{\lambda}k^{1/4} + 2k \log k \sqrt{\lambda}.$$

Setting $\lambda = 4\varrho^2$ and rearranging terms, we obtain that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S \cap \partial_\varrho S] \leq C'k \log k \varrho$ where C' is a sufficiently large universal constant. \square

We now prove the smoothness result for points outside the set.

Lemma B.7. *Let S be a convex set on \mathbb{R}^k . Let $\varrho \in (0, 1)$. Then, we have that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S^c \cap \partial_\varrho S] \leq Ck^{1/4}\varrho$ where C is a sufficiently large universal constant.*

Proof. For $t > 0$, define the set S_t as $S_t = \{\mathbf{x} \in \mathbb{R}^k \mid \inf_{\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|_2 \leq t\}$. We have that

$$\begin{aligned} \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S^c \cap \partial_\varrho] &= \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in S_\varrho \setminus S] \\ &= \int_{t=0}^\varrho \int_{\mathbf{x} \in \partial S_t} \mathcal{N}(\mathbf{x}; 0, I_k) d\mathbf{x} dt \leq \int_{t=0}^\varrho Ck^{1/4} dt \leq Ck^{1/4}\varrho \end{aligned}$$

where C is a large universal constant. We obtained the penultimate inequality using the definition of Gaussian surface area. \square

We now state our final result on the smooth boundary of convex sets.

Lemma B.8 (Smooth Boundary for Convex sets). *Let S be a convex set. Let $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ be the function defined as $F(\mathbf{x}) = \mathbb{1}\{\mathbf{x} \in S\}$. Then, F has a $Ck \log k$ -smooth boundary with respect to \mathcal{N}_k for a sufficiently large universal constant C .*

Proof. The proof is immediate from [Lemma B.5](#) and [Lemma B.7](#). \square

B.2 Sandwiching Polynomials

In this section, we present known results from pseudorandomness literature on the existence of sandwiching polynomials for various function classes with respect to $\text{Unif}\{\pm 1\}^d$ and \mathcal{N}_d . Although previously known, these results are mostly not stated in the manner in which we need them. In particular, the coefficient bounds are not explicitly stated in previous work. We state these results in terms of existence of sandwiching polynomials with coefficient bounds for completeness.

We now introduce the important notion of (δ, ℓ) -independent distributions.

Definition B.9 ((δ, ℓ) -independent distribution). Let $\mathcal{D}, \mathcal{D}'$ be distributions on \mathbb{R}^d . For $\delta > 0$ and $\ell \in \mathbb{N}$, we say that the distribution \mathcal{D}' is (δ, ℓ) -independent with respect to \mathcal{D} if $|\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[\mathbf{x}^\alpha] - \mathbb{E}_{\mathbf{x} \sim \mathcal{D}'}[\mathbf{x}^\alpha]| \leq \delta$ for all $\alpha \in \mathbb{N}^d$.

We drop the "with respect to \mathcal{D} " when the distribution is clear from context. Let $\mathcal{D}, \mathcal{D}'$ be distributions on $\mathcal{X} \subseteq \mathbb{R}^d$ and $f : \mathcal{X} \rightarrow \{\pm 1\}$. For $\epsilon > 0$, we say that \mathcal{D}' ϵ -fools f with respect to \mathcal{D} if $|\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[f(\mathbf{x})] - \mathbb{E}_{\mathbf{x} \sim \mathcal{D}'}[f(\mathbf{x})]| \leq \epsilon$ (again, we drop the "with respect to" when the target distribution is clear from context). For a concept class \mathcal{C} , we say that \mathcal{D}' ϵ -fools \mathcal{C} with respect to \mathcal{D} if \mathcal{D}' ϵ -fools f with respect to \mathcal{D} for all functions $f \in \mathcal{C}$.

We will use the following result from [\[GKK23\]](#) which is a generalization of a result from [\[Baz09\]](#). We will only need one direction of the result which we state below.

Lemma B.10. *[Theorem 3.2 from [\[GKK23\]](#)] Let \mathcal{D} be a distribution on $\mathcal{X} \subseteq \mathbb{R}^d$. Let $\delta, \epsilon > 0$ and $\ell \in \mathbb{N}$. Let $f : \mathcal{X} \rightarrow \mathbb{R}^d$ be a function that satisfies the following property: given any distribution \mathcal{D}' that is (δ, ℓ) -independent with respect to \mathcal{D} , we have that $|\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[f(\mathbf{x})] - \mathbb{E}_{\mathbf{x} \sim \mathcal{D}'}[f(\mathbf{x})]| \leq \epsilon$. Then, there exists degree ℓ polynomials $p_{\text{down}}, p_{\text{up}}$ such that $p_{\text{down}} \leq f \leq p_{\text{up}}$ and $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] + \delta(|p_{\text{up}}| + |p_{\text{down}}|) \leq \epsilon$.*

B.2.1 Sandwiching Polynomials: Boolean

In this section, the target distribution is $\text{Unif}\{\pm 1\}^d$. We will find the following lemma useful.

Lemma B.11. *Let $\epsilon > 0$ and $\ell \in \mathbb{N}$. Let $f : \{\pm 1\}^d \rightarrow \{\pm 1\}$ be a function such that all $(0, \ell)$ -independent distributions ϵ -fool f . Then, there exists polynomials $p_{\text{up}}, p_{\text{down}}$ of degree ℓ and coefficients bounded by $O(d^\ell)$ such that $p_{\text{down}} \leq f \leq p_{\text{up}}$ and $\mathbb{E}_{\mathbf{x} \sim \text{Unif}\{\pm 1\}^d}[p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq O(\epsilon)$.*

Proof. We use the following theorem from [\[AGM03\]](#) that states that for any (δ, ℓ) -distribution, there exists a $(0, \ell)$ distribution that is ϵ -close to it in TV distance.

Lemma B.12 (Theorem 2.1 from [\[AGM03\]](#)). *For $\delta > 0$ and $\ell \in \mathbb{N}$, let \mathcal{D} be a (δ, ℓ) -independent distribution on $\{\pm 1\}^d$. Then, there exists a distribution \mathcal{D}' that is $(0, \ell)$ -independent such that the TV distance between \mathcal{D} and \mathcal{D}' is at most δd^ℓ .*

From the above claim, we have that any $(\epsilon/d^\ell, \ell)$ -independent distribution 2ϵ -fools f . Thus, from [Lemma B.10](#), there exists polynomials $p_{\text{up}}, p_{\text{down}}$ of degree ℓ with coefficients bounded by $O(d^\ell)$ such that $\mathbb{E}_{\mathbf{x} \sim \text{Unif}\{\pm 1\}^d} [p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq 2\epsilon$. This proves the claim. \square

Lemma B.13 (Sandwiching polynomials for degree 2 PTFs). *Let \mathcal{C} be the class of degree 2 PTFs. For $\epsilon > 0$, the $O(\epsilon)$ -approximate \mathcal{L}_1 sandwiching degree of \mathcal{C} under $\text{Unif}\{\pm 1\}^d$ is at most $\ell = \tilde{O}(1/\epsilon^9)$ with coefficient bound $O(d^\ell)$.*

Proof. From [\[DKN10\]](#), we have that $(0, \ell)$ -independent distributions ϵ -fools \mathcal{C} when $\ell = \tilde{O}(1/\epsilon^9)$. Now, we apply [Lemma B.11](#) to finish the proof. \square

Lemma B.14 (Sandwiching polynomials for depth- t AC_0). *Let \mathcal{C} be the class of depth- t AC_0 circuits of size s on $\{\pm 1\}^d$. For $\epsilon > 0$, the $O(\epsilon)$ -approximate \mathcal{L}_1 sandwiching degree of \mathcal{C} under $\text{Unif}\{\pm 1\}^d$ is at most $\ell = (\log s)^{O(t)} \log(1/\epsilon)$ with coefficient bound $O(d^\ell)$.*

Proof. From [\[Bra10, Tal17, HS19\]](#), we have that $(0, \ell)$ -independent distributions ϵ -fools f when $\ell = (\log s)^{O(t)} \log(1/\epsilon)$. Now, we apply [Lemma B.11](#) to finish the proof. \square

B.2.2 Sandwiching Polynomials: Gaussian

Lemma B.15. *Let $\epsilon > 0$ and $\ell \in \mathbb{N}$. Let $f : \mathbb{R}^d \rightarrow \{\pm 1\}$ be a function such that all $(0, \ell)$ -independent distributions ϵ -fool f . Then, there exists polynomials $p_{\text{up}}, p_{\text{down}}$ of degree ℓ and coefficients bounded by $O(d^\ell)$ such that $p_{\text{down}} \leq f \leq p_{\text{up}}$ and $\mathbb{E}_{\mathbf{x} \sim \mathcal{N}_d} [p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq O(\epsilon)$.*

Proof. From [Lemma B.10](#), we have that there exists $p_{\text{up}}, p_{\text{down}}$ of degree ℓ such that $\mathbb{E}_{\mathbf{x} \sim \mathcal{N}_d} [p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq 2\epsilon$ and $p_{\text{down}} \leq f \leq p_{\text{up}}$. The claim now follows from the following lemma (proof is included in the end of this section) that states that any sandwiching polynomial with respect to \mathcal{N}_d must have bounded coefficients.

Lemma B.16. *Let $f : \mathbb{R}^d \rightarrow \{\pm 1\}$ be a function, and let p_{up} and p_{down} be degree- ℓ polynomials satisfying the following (i) for every $\mathbf{x} \in \mathbb{R}^d$ we have $p_{\text{up}}(\mathbf{x}) \geq f(\mathbf{x}) \geq p_{\text{down}}(\mathbf{x})$. (ii) $\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq 1$. Then, the polynomials p_{up} and p_{down} both have coefficients bounded by $2 \cdot (10d)^\ell$ in absolute value.*

\square

Lemma B.17 (Sandwiching polynomials for degree 2 PTFs). *Let \mathcal{C} be the class of degree 2 PTFs. For $\epsilon > 0$, the $O(\epsilon)$ -approximate \mathcal{L}_1 sandwiching degree of \mathcal{C} under \mathcal{N}_d is at most $\ell = \tilde{O}(1/\epsilon^8)$ with coefficient bound $O(d^\ell)$.*

Proof. From [\[DKN10\]](#), we have that $(0, \ell)$ -independent distributions ϵ -fools \mathcal{C} when $\ell = \tilde{O}(1/\epsilon^8)$. Now, we apply [Lemma B.15](#) to finish the proof. \square

In the remainder of this section, we prove [Lemma B.16](#). We will use the notion of Hermite polynomials. Recall that for $i = 0, 1, 2, \dots$ Hermite polynomials $\{H_i\}$ are the unique collection of polynomials over \mathbb{R} that are orthogonal with respect to Gaussian distribution. In other words $\mathbb{E}_{x \in \mathcal{N}(0, 1)} [H_i(x)H_j(x)] = 0$ whenever $i \neq j$. In this work, we normalize the Hermite polynomials to further satisfy $\mathbb{E}_{x \in \mathcal{N}(0, 1)} [H_i(x)H_i(x)] = 1$. It is a standard fact from theory of orthogonal polynomials that $H_0(x) = 1$, $H_1(x) = x$ and for $i \geq 2$ Hermite polynomials satisfy the following recursive identity:

$$H_{i+1}(x) \cdot \sqrt{(i+1)!} = xH_i(x) \cdot \sqrt{i!} - i \cdot H_{i-1}(x) \cdot \sqrt{(i-1)!}$$

Proposition B.18. *Each coefficient of H_i is bounded by 2^i in absolute value.*

Proof. This follows immediately from the recursion relation. \square

Proposition B.19. *All coefficients of multi-dimensional polynomial $H_{i_1}(\mathbf{x}_1)H_{i_2}(\mathbf{x}_2) \cdots H_{i_d}(\mathbf{x}_d)$ are bounded by $2^{i_1+i_2+\dots+i_d}$.*

Proof. Each monomial of $H_{i_1}(\mathbf{x}_1)H_{i_2}(\mathbf{x}_2)\cdots H_{i_d}(\mathbf{x}_d)$ can be expressed as $\prod_j m_j(\mathbf{x}_j)$ where each $m_j(\mathbf{x}_j)$ is a monomial of $H_{i_j}(\mathbf{x}_j)$. But we know that the coefficient of m_j is bounded by 2^{i_j} in absolute value. Thus, each coefficient of $H_{i_1}(\mathbf{x}_1)H_{i_2}(\mathbf{x}_2)\cdots H_{i_d}(\mathbf{x}_d)$ is at most $2^{i_1+i_2+\cdots+i_d}$. \square

Proposition B.20. *Let p be a polynomial over \mathbb{R}^d of degree ℓ . Suppose that p satisfies*

$$\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [(p(\mathbf{x}))^2] \leq 1,$$

then every monomial of p has a coefficient of at most $(2d)^\ell$ in absolute value.

Proof. For an element $\mathbf{x} \in \mathbb{R}^d$ we let $(\mathbf{x}_1, \dots, \mathbf{x}_d)$ be its coordinates. We expand $p(\mathbf{x})$ as a sum of multidimensional Hermite polynomials⁸:

$$p(\mathbf{x}) = \sum_{\substack{i_1, i_2, \dots, i_d \geq 0 \\ i_1 + i_2 + \dots + i_d \leq \ell}} \alpha_{i_1, i_2, \dots, i_d} H_{i_1}(\mathbf{x}_1) H_{i_2}(\mathbf{x}_2) \cdots H_{i_d}(\mathbf{x}_d) \quad (\text{B.2})$$

Due to orthogonality of Hermite polynomials, we have:

$$\sum_{\substack{i_1, i_2, \dots, i_d \geq 0 \\ i_1 + i_2 + \dots + i_d \leq \ell}} \alpha_{i_1, i_2, \dots, i_d}^2 = \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [(p(\mathbf{x}))^2] \leq 1$$

In particular, this implies that each coefficient $\alpha_{i_1, i_2, \dots, i_d}$ is bounded by 1 in absolute value. Combining this with Equation B.2, Proposition B.19 and the fact that there are at most d^ℓ ways to choose $i_1, i_2, \dots, i_d \geq 0$ satisfying $\sum_j i_j \leq \ell$, we see that each coefficient of p is bounded by $(2d)^\ell$ in absolute value. \square

Finally, we need the following standard fact.

Fact B.21 (Gaussian Hypercontractivity [Bog98],[Nel73]). *If $p : \mathbb{R}^d \rightarrow \mathbb{R}$ is a polynomial of degree at most ℓ , for every $t \geq 2$,*

$$\mathbb{E}_{\mathbf{x} \sim \mathcal{N}(0, I_d)} [|p(\mathbf{x})|^t]^{\frac{1}{t}} \leq (t-1)^{\ell/2} \sqrt{\mathbb{E}_{\mathbf{x} \sim \mathcal{N}_d} [p^2(\mathbf{x})]}.$$

The following is a standard corollary:

Proposition B.22. *If $p : \mathbb{R}^d \rightarrow \mathbb{R}$ is a polynomial of degree ℓ , then*

$$\sqrt{\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [(p(\mathbf{x}))^2]} \leq e^\ell \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [|p(\mathbf{x})|]$$

Proof. The proof is standard, and is included here for completeness (a completely analogous proof for the Boolean case can be found in Theorem 9.22 from [O'D14]). Let $\lambda > 0$ be a parameter and let $\theta = \frac{1}{2} \frac{\lambda}{1+\lambda}$. Using Generalized Holder's inequality and Gaussian Hypercontractivity, we have

$$\begin{aligned} \sqrt{\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [(p(\mathbf{x}))^2]} &\leq \left(\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [|p(\mathbf{x})|] \right)^\theta \left(\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [(p(\mathbf{x}))^{2+\lambda}] \right)^{\frac{1-\theta}{2+\lambda}} \leq \\ &\leq \left(\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [|p(\mathbf{x})|] \right)^\theta \left((1+\lambda)^{\ell/2} \sqrt{\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [(p(\mathbf{x}))^2]} \right)^{1-\theta} \end{aligned}$$

Overall,

$$\left(\sqrt{\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [(p(\mathbf{x}))^2]} \right)^\theta \leq (1+\lambda)^{(1-\theta)\ell/2} \left(\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0,I)} [|p(\mathbf{x})|] \right)^\theta$$

⁸Note that the expansion below is always possible for a degree ℓ polynomials because polynomials of the form $H_{i_1}(\mathbf{x}_1)H_{i_2}(\mathbf{x}_2)\cdots H_{i_d}(\mathbf{x}_d)$ are polynomials of degree at most ℓ that are linearly independent, because they are orthonormal with respect to the standard d -dimensional Gaussian.

Taking power $1/\theta$ of both sides and recalling that $\theta = \frac{1}{2} \frac{\lambda}{1+\lambda}$ we get:

$$\sqrt{\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [(p(\mathbf{x}))^2]} \leq (1 + \lambda)^{\frac{(1-\theta)\ell/2}{\theta}} \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [|p(\mathbf{x})|] = (1 + \lambda)^{\left(\frac{1}{\lambda} - \frac{1}{2}\right)\ell} \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [|p(\mathbf{x})|].$$

Finally, taking $\lambda \rightarrow 0$ proves the proposition. \square

Finally, we are ready to prove Theorem B.16.

Proof of Theorem B.16. Without loss of generality⁹, we bound the coefficients of $p_{\text{up}}(\mathbf{x})$. We have

$$\begin{aligned} \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [|p_{\text{up}}(\mathbf{x})|] &\leq \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [|f(\mathbf{x})|] + \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [|p_{\text{up}}(\mathbf{x}) - f(\mathbf{x})|] \leq \\ &\leq \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [|f(\mathbf{x})|] + \mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq 2. \end{aligned}$$

Note that in the last inequality the value of $\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [|f(\mathbf{x})|]$ is bounded by 1 because f is $\{\pm 1\}$ -valued, and $\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})]$ was bounded by 1 by the premise of the theorem. Combining the equation above with Proposition B.22, we get

$$\sqrt{\mathbb{E}_{\mathbf{x} \in \mathcal{N}(0, I)} [(p_{\text{up}}(\mathbf{x}))^2]} \leq 2 \cdot e^\ell.$$

Finally, together with Proposition B.20 implies that each coefficient of $\frac{p_{\text{up}}}{2 \cdot e^\ell}$ is bounded by $(2d)^\ell$ in absolute value. This allows us to conclude that each coefficient of p_{up} is bounded by $2 \cdot (10d)^\ell$ in absolute value. \square

C Chow Matching Tester

We now focus on functions that have low-degree sandwiching polynomials approximators under the training distribution.

Definition C.1 (\mathcal{L}_1 -sandwiching polynomials). Consider $\mathcal{X} \subseteq \mathbb{R}^d$ and a distribution \mathcal{D} over \mathcal{X} . For $\epsilon > 0$ and $f : \mathcal{X} \rightarrow \{\pm 1\}$, we say that the polynomials $p_{\text{up}}, p_{\text{down}} : \mathcal{X} \rightarrow \mathbb{R}$ are ϵ -approximate \mathcal{L}_1 -sandwiching polynomials for f under \mathcal{D} if the following are true.

1. $p_{\text{down}}(\mathbf{x}) \leq f(\mathbf{x}) \leq p_{\text{up}}(\mathbf{x})$, for all $\mathbf{x} \in \mathcal{X}$.
2. $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}} [p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq \epsilon$

We say that the ϵ -approximate \mathcal{L}_1 -sandwiching degree of \mathcal{C} under \mathcal{D} is at most ℓ and with (coefficient) bound B if for any $f \in \mathcal{C}$ there are ϵ -approximate \mathcal{L}_1 -sandwiching polynomials $p_{\text{up}}, p_{\text{down}}$ for f such that $\deg(p_{\text{up}}), \deg(p_{\text{down}}) \leq \ell$ and each of the coefficients of $p_{\text{up}}, p_{\text{down}}$ are absolutely bounded by B .

It turns out that given a function class \mathcal{C} with low degree sandwiching approximators, we can test localized discrepancy of a hypothesis \hat{f} with respect to a very global notion of neighbourhood: the entire concept class \mathcal{C} ! We state the definition here.

Definition C.2 (Global Neighborhood). The global $(\mathcal{H}, \mathcal{C})$ neighborhood is defined as $\mathbf{N}(\hat{f}) = \mathcal{C}$ for all $\hat{f} \in \mathcal{H}$. We denote this by $\mathbf{N}_{\mathcal{C}}$.

C.1 Discrepancy Testing Result

We now present our discrepancy tester for concept classes with bounded ϵ -approximate \mathcal{L}_1 sandwiching degree. The primary advantage of this tester is it's global nature: given a hypothesis \hat{f} , it certifies low localized discrepancy with respect to every function in the concept class.

⁹This is indeed without loss of generality, because the function $-f$ is bounded from above by $-p_{\text{down}}$ and from below by $-p_{\text{up}}$.

Theorem C.3 (Chow Matching Tester). *Let \mathcal{D} be a distribution over a set $\mathcal{X} \subseteq \mathbb{R}^d$. Let $\mathcal{C} \subseteq \{\mathcal{X} \rightarrow \{\pm 1\}\}$ be a concept class. Let $\epsilon > 0$, $m_{\text{conc}} \in \mathbb{N}$. Let $\mathcal{H} = \{\pm 1\}^{\mathcal{X}}$. Assume that the following are true.*

1. (\mathcal{L}_1 -sandwiching) *The $\frac{\epsilon}{3}$ -approximate \mathcal{L}_1 -sandwiching degree of \mathcal{C} w.r.t. \mathcal{D} is ℓ with bound B .*
2. (Chow-concentration) *For any function $\hat{f} \in \mathcal{H}$, if $X \sim \mathcal{D}^{\otimes m}$ with $m \geq m_{\text{conc}}$, then with probability at least $9/10$, we have that for all $\alpha \in \mathbb{N}^d$ with $\|\alpha\|_1 \leq \ell$, $|\mathbb{E}_{\mathcal{D}}[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha] - \mathbb{E}_X[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha]| \leq \frac{\epsilon}{Bd^{2\ell}}$.*

Then, there exists a $(\mathbf{N}_{\mathcal{C}}, \epsilon)$ -tester \mathcal{T} for localized discrepancy from \mathcal{D} with respect to $\{\mathcal{D}\}$ that uses $m_{\text{conc}} + O(\frac{1}{\epsilon^2})$ samples and runs in time $\text{poly}(m_{\text{conc}}, d^\ell, \frac{1}{\epsilon})$.

Proof. For an input distribution \mathcal{D}' and function $\hat{f} \in \mathcal{H}$, the tester runs **Algorithm 1** with m_{conc} samples X from \mathcal{D}' and function \hat{f} as input. We now prove it's correctness.

Soundness We first consider the case where \mathcal{T} accepts \mathcal{D}' . Let $f^* = \arg \max_{f \in \mathcal{C}} (\mathbb{P}_{\mathbf{x} \sim \mathcal{D}'}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim \mathcal{D}}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})])$. Since $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}'}[\hat{f}(\mathbf{x}) \neq f^*(\mathbf{x})] = (1 - \mathbb{E}_{\mathcal{D}'}[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})])/2$, it is sufficient to prove a lower bound on the second term. From a Chernoff bound, we have that $\mathbb{E}_{\mathcal{D}'}[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] \geq \mathbb{E}_X[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] - \epsilon$ with probability at least $3/4$ when $|X| \geq C/\epsilon^2$ for some universal constant $C \geq 1$. We now bound $\mathbb{E}_X[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})]$. Let $p_{\text{up}}, p_{\text{down}}$ be ϵ -approximate \mathcal{L}_1 -sandwiching polynomials for f^* under \mathcal{D} . We have that

$$\begin{aligned} \mathbb{E}_X[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] &= \mathbb{E}_X[(f^*(\mathbf{x}) - p_{\text{up}}(\mathbf{x})) \cdot \hat{f}(\mathbf{x})] + \mathbb{E}_X[p_{\text{up}}(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] \\ &\geq \mathbb{E}_X[p_{\text{down}}(\mathbf{x}) - p_{\text{up}}(\mathbf{x})] + \mathbb{E}_X[p_{\text{up}}(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] \geq \mathbb{E}_{\mathcal{D}}[p_{\text{down}}(\mathbf{x}) - p_{\text{up}}(\mathbf{x})] + \mathbb{E}_{\mathcal{D}}[p_{\text{up}}(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] - 3\epsilon \\ &\geq \mathbb{E}_{\mathcal{D}}[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] + \mathbb{E}_{\mathcal{D}}[(p_{\text{up}}(\mathbf{x}) - f^*(\mathbf{x})) \cdot \hat{f}(\mathbf{x})] - 4\epsilon \geq \mathbb{E}_{\mathcal{D}}[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] - 5\epsilon. \end{aligned}$$

The first inequality follows from the fact that $p_{\text{down}}(\mathbf{x}) \leq f^*(\mathbf{x}) \leq p_{\text{up}}(\mathbf{x})$. To obtain the second inequality, we use the fact that the tester accepts if and only if $|\mathbb{E}_X[\mathbf{x}^\alpha] - \mathbb{E}_{\mathcal{D}}[\mathbf{x}^\alpha]| < \Delta$ and $|\mathbb{E}_X[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha] - \mathbb{E}_{\mathcal{D}}[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha]| < \Delta$ for $\Delta = \frac{\epsilon}{Bd^{2\ell}}$ and all $\alpha \in \mathbb{N}$ such that $\|\alpha\|_1 \leq \ell$. Since the coefficients of $p_{\text{up}}, p_{\text{down}}$ are bounded by B and each have at most $d^{2\ell}$ monomials, we obtain the second inequality. The last two inequalities use the fact that $\mathbb{E}_{\mathcal{D}}[p_{\text{up}}(\mathbf{x}) - p_{\text{down}}(\mathbf{x})] \leq \epsilon$.

Thus, we obtain that $\mathbb{E}_{\mathcal{D}'}[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] \geq \mathbb{E}_{\mathcal{D}}[f^*(\mathbf{x}) \cdot \hat{f}(\mathbf{x})] - 6\epsilon$ with probability at least $3/4$. This implies that $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}'}[f^*(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \leq \mathbb{P}_{\mathbf{x} \sim \mathcal{D}}[f^*(\mathbf{x}) \neq \hat{f}(\mathbf{x})] + 3\epsilon$. From the definition of f^* , we therefore have that $\text{disc}_{\hat{f}, \mathbf{N}_{\mathcal{C}}}(\mathcal{D}, \mathcal{D}') \leq 3\epsilon$ with probability at least $3/4$ when the tester accepts.

Completeness In this case, we have that $\mathcal{D}' = \mathcal{D}$. Clearly, from our assumption on Chow concentration, we have that with probability at least $4/5$, $|\mathbb{E}_X[\mathbf{x}^\alpha] - \mathbb{E}_{\mathcal{D}}[\mathbf{x}^\alpha]| < \Delta$ and $|\mathbb{E}_X[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha] - \mathbb{E}_{\mathcal{D}}[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha]| < \Delta$ for $\Delta = \frac{\epsilon}{Bd^{2\ell}}$ and all $\alpha \in \mathbb{N}$ such that $\|\alpha\|_1 \leq \ell$. Thus, with probability at least $4/5$, the tester will accept. \square

Algorithm 1: Chow Matching Tester

Input: Set X from \mathcal{D}' , function $\hat{f} : \mathcal{X} \rightarrow \{\pm 1\}$, parameters $\epsilon > 0$, $\ell \in \mathbb{N}$, $B > 0$

Set $\Delta = \frac{\epsilon}{Bd^{2\ell}}$

For each $\alpha \in \mathbb{N}^d$ with $\|\alpha\|_1 \leq \ell$, compute the quantity $\hat{M}_\alpha = \mathbb{E}_X[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha]$.

Accept if $|\hat{M}_\alpha - \mathbb{E}_{\mathcal{D}}[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha]| < \Delta$ and $|\mathbb{E}_X[\mathbf{x}^\alpha] - \mathbb{E}_{\mathcal{D}}[\mathbf{x}^\alpha]| < \Delta$ for all α with $\|\alpha\|_1 \leq \ell$.

Reject otherwise.

C.2 Applications to TDS Learning

In this section we prove that any concept class with \mathcal{L}_1 sandwiching polynomials can be TDS learned. This improves on the results of Klivans et al. 2023 which proved that \mathcal{L}_2 sandwiching implies TDS learning. In particular, our result implies a new TDS learning algorithm for the class of all constant depth circuits(AC0) which was unknown in prior work. We also achieve tight dependence on the parameter λ as compared to prior work which was off by constant factors.

Algorithm 2: TDS learning through Chow matching

Input: Sets S_{train} from $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$, X_{test} from $\mathcal{D}_{\mathcal{X}}^{\text{test}}$, Training Algorithm \mathcal{A} , $\epsilon \in (0, 1)$, $\ell \in \mathbb{N}$, $B > 0$

Let \hat{f} be the output of \mathcal{A} when run on input S_{train}

Run the Chow matching tester(Algorithm 1) with inputs X_{test} , \hat{f} , ϵ , ℓ and B with source distribution $\mathcal{D}_{\mathcal{X}}^{\text{train}}$.

Accept and output \hat{f} if the Chow matching tester accepts.

Reject otherwise.

We now state our general theorem about the connection between \mathcal{L}_1 sandwiching and TDS learning. In contrast to prior work, we completely decouple the training and testing phase of the TDS learner.

Theorem C.4 (\mathcal{L}_1 -sandwiching implies TDS learning). *Let \mathcal{D} be a distribution over a set $\mathcal{X} \subseteq \mathbb{R}^d$. Let $\mathcal{C} \subseteq \{\mathcal{X} \rightarrow \{\pm 1\}\}$ be a concept class. Let $\epsilon, \delta \in (0, 1)$. Let $\mathcal{H} = \{\pm 1\}^{\mathcal{X}}$. Assume that the following are true.*

1. (\mathcal{L}_1 -sandwiching) *The ϵ -approximate \mathcal{L}_1 sandwiching degree of \mathcal{C} under \mathcal{D} is at most ℓ with bound B .*
2. (Chow-concentration) *For any function $\hat{f} \in \mathcal{H}$, if $X \sim \mathcal{D}^{\otimes m}$ with $m \geq m_{\text{conc}}$, then with probability at least $9/10$, we have that for all $\alpha \in \mathbb{N}^d$ with $\|\alpha\|_1 \leq \ell$, $|\mathbb{E}_{\mathcal{D}}[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha] - \mathbb{E}_X[\hat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha]| \leq \frac{\epsilon}{Bd^{2\ell}}$.*
3. (Agnostic Learning Algorithm) *There exists an algorithm \mathcal{A} that takes m_{train} samples from $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$, runs in time T_{train} , and outputs w.p. at least $1 - \frac{\delta}{2}$ a hypothesis \hat{f} such that $\mathbb{P}_{(x,y) \sim \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}} [y \neq \hat{f}(\mathbf{x})] \leq \text{err}_{\mathcal{A}}$.*

Then, there exists an algorithm that takes m_{train} labelled samples from the training distribution, $O((m_{\text{conc}} + 1/\epsilon^2) \log(1/\delta))$ unlabelled test samples, runs in time $T_{\text{train}} + \text{poly}(m_{\text{conc}}, d^\ell, \frac{1}{\epsilon}, \log(1/\delta))$ and TDS learns \mathcal{C} with respect to \mathcal{D} up to error $\lambda + \text{err}_{\mathcal{A}} + \epsilon$ and fails with probability at most δ .

Proof. Let $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ be the training distribution with marginal $\mathcal{D}_{\mathcal{X}}^{\text{train}} = \mathcal{D}$ and let $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}$ be the test distribution with marginal equal. Let S_{train} be a set of m_{train} samples from $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ and let X_{test} be a set of $m_{\text{conc}} + 1/\epsilon^2$ samples from $\mathcal{D}_{\mathcal{X}}^{\text{test}}$. Run Algorithm 2 with inputs S_{train} , X_{test} , \mathcal{A} , ϵ , ℓ and B . We now prove it's correctness.

Soundness We first consider the case when the input distribution is accepted. This happens when $\mathcal{D}_{\mathcal{X}}^{\text{test}}$ is accepted by the Chow Matching tester from Algorithm 1. From Theorem C.3, we have that with probability at least $3/4$, $\text{disc}_{\hat{f}, \mathcal{N}_{\mathcal{C}}}(\mathcal{D}_{\mathcal{X}}^{\text{train}}, \mathcal{D}_{\mathcal{X}}^{\text{test}}) \leq \epsilon$. This probability can be boosted to $1 - \delta/2$ by repeating the Chow matching tester $O(\log(1/\delta))$ times with independent samples and accepting if and only if a majority of the tests accept. Let $f^* = \arg \min_{f \in \mathcal{C}} \{\text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}})\}$. That is, $\lambda = \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}})$. From Definition C.2 and the fact that $\text{disc}_{\hat{f}, \mathcal{N}_{\mathcal{C}}}(\mathcal{D}_{\mathcal{X}}^{\text{train}}, \mathcal{D}_{\mathcal{X}}^{\text{test}}) \leq \epsilon$, we have that

$$\mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}} [f^*(\mathbf{x}) \neq \hat{f}(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{train}}} [f^*(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \leq \epsilon \quad (\text{C.1})$$

We also have that $\text{err}(\widehat{f}; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) \leq \text{err}_{\mathcal{A}}$ with probability at least $1 - \delta/2$ from the error guarantee of \mathcal{A} . We are now ready to bound $\text{err}(\widehat{f}; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}})$. We have that

$$\begin{aligned} \text{err}(\widehat{f}; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}) &\leq \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}) + \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}} [f^*(\mathbf{x}) \neq \widehat{f}(\mathbf{x})] \\ &\leq \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}) + \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{train}}} [f^*(\mathbf{x}) \neq \widehat{f}(\mathbf{x})] + \epsilon \\ &\leq \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}) + \mathbb{P}_{(\mathbf{x}, y) \sim \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}} [f^*(\mathbf{x}) \neq y] + \mathbb{P}_{(\mathbf{x}, y) \sim \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}} [\widehat{f}(\mathbf{x}) \neq y] \\ &\leq \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}) + \text{err}_{\mathcal{A}} \leq \lambda + \text{err}_{\mathcal{A}} + \epsilon. \end{aligned}$$

The first and third inequalities follow from the triangle inequality. The second inequality follows from [Equation \(C.1\)](#). The penultimate inequality follows from the error guarantee of \mathcal{A} . The last inequality follows from the definition of λ .

Completeness This follows immediately from the completeness guarantee of [Theorem C.3](#). As seen before, the success probability can be boosted to $1 - \delta/2$. Thus, the tester accepts when $\mathcal{D}_{\mathcal{X}}^{\text{test}} = \mathcal{D}_{\mathcal{X}}^{\text{train}}$ with probability at least $1 - \delta/2$. \square

Remark C.5. The above theorem completely decouples training and testing. This is in contrast to the Klivans et al. 2023 which don't make this distinction. In particular, this forces their output hypothesis to be polynomial threshold function. In our theorem, the hypothesis can be any function output by the training algorithm \mathcal{A} that achieves low error. This is also in contrast with the other TDS learning algorithms in this paper that require additional structure from the hypothesis output by the training algorithm.

In fact, we can drop Assumption 3 from [Theorem C.4](#) entirely, if we restrict our training algorithm. In particular, we use the following theorem from [\[KKMS08\]](#).

Theorem C.6 (Theorem 5 from [\[KKMS08\]](#)). *Let \mathcal{D} be a distribution on $\mathcal{X} \times \{\pm 1\}$ for $\mathcal{X} \subseteq \mathbb{R}^d$ with marginal $\mathcal{D}_{\mathcal{X}}$. Let $\epsilon, \delta \in (0, 1)$. Let \mathcal{C} be a class of functions such that for all $f \in \mathcal{C}$, there exists polynomials p of degree ℓ such that $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}} [|f(\mathbf{x}) - p(\mathbf{x})|] \leq \epsilon$. Then there exists an agnostic learning algorithm \mathcal{A} that has run time and sample complexity at most $\text{poly}(d^\ell, 1/\epsilon, \log(1/\delta))$ that outputs a hypothesis \widehat{f} such that with probability at least $1 - \delta$, we have that*

$$\mathbb{P}_{(\mathbf{x}, y) \sim \mathcal{D}} [y \neq \widehat{f}(\mathbf{x})] \leq \inf_{f \in \mathcal{C}} \mathbb{P}_{(\mathbf{x}, y) \sim \mathcal{D}} [f(\mathbf{x}) \neq y]$$

Armed with this, we give our end to end result that \mathcal{L}_1 sandwiching implies TDS learning.

Theorem C.7 (\mathcal{L}_1 -sandwiching implies TDS learning). *Let \mathcal{D} be a distribution over a set $\mathcal{X} \subseteq \mathbb{R}^d$. Let $\mathcal{C} \subseteq \{\mathcal{X} \rightarrow \{\pm 1\}\}$ be a concept class. Let $\epsilon, \delta \in (0, 1)$. Let $\mathcal{H} = \{\pm 1\}^{\mathcal{X}}$. Assume that the following are true.*

1. (\mathcal{L}_1 -sandwiching) *The ϵ -approximate \mathcal{L}_1 sandwiching degree of \mathcal{C} under \mathcal{D} is at most ℓ with bound B .*
2. (Chow-concentration) *For any function $\widehat{f} \in \mathcal{H}$, if $X \sim \mathcal{D}^{\otimes m}$ with $m \geq m_{\text{conc}}$, then with probability at least $9/10$, we have that for all $\alpha \in \mathbb{N}^d$ with $\|\alpha\|_1 \leq \ell$, $|\mathbb{E}_{\mathcal{D}}[\widehat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha] - \mathbb{E}_X[\widehat{f}(\mathbf{x}) \cdot \mathbf{x}^\alpha]| \leq \frac{\epsilon}{B d^{2\ell}}$.*

Then, there exists an algorithm that takes $\text{poly}(d^\ell, 1/\epsilon)$ labelled samples from the training distribution, $O((m_{\text{conc}} + 1/\epsilon^2) \cdot \log(1/\delta))$ unlabelled test samples, runs in time $\text{poly}(m_{\text{conc}}, d^\ell, \frac{1}{\epsilon}, \log(1/\delta))$ and TDS learns \mathcal{C} with respect to \mathcal{D} up to error $\lambda + \text{opt}_{\text{train}} + \epsilon$ and fails with probability at most δ .

Proof. Observe that \mathcal{L}_1 sandwiching polynomials are also \mathcal{L}_1 approximating polynomials. Thus, \mathcal{C} satisfies the requirements of [Theorem C.6](#). Thus, we can run [Algorithm 2](#) with \mathcal{A} instantiated to be the algorithm from [Theorem C.6](#). The proof of correctness follows from [Algorithm 2](#). \square

We now argue that when $\mathcal{D}_{\mathcal{X}}^{\text{train}} \in \{\text{Unif}\{\pm 1\}^d, \mathcal{N}_d\}$, then we have that Assumption 2 of [Theorem C.4](#) is always true with $m_{\text{conc}} \leq \text{poly}(d^\ell B/\epsilon)$.

Lemma C.8. Let $\mathcal{D} \in \{\text{Unif}\{\pm 1\}^d, \mathcal{N}_d\}$. Let f be a function taking values in $\{\pm 1\}$. Let $\ell \in \mathbb{N}$. Let $X \sim \mathcal{D}^{\otimes m_{\text{conc}}}$ for $m_{\text{conc}} \geq \text{poly}(d^\ell/\epsilon)$. Then, with probability at least $9/10$ over S , we have that for all $\alpha \in \mathbb{N}^d$ with $\|\alpha\|_1 \leq \ell$,

$$\left| \mathbb{E}_{\mathcal{D}}[f(\mathbf{x}) \cdot \mathbf{x}^\alpha] - \mathbb{E}_X[f(\mathbf{x}) \cdot \mathbf{x}^\alpha] \right| \leq \epsilon.$$

Proof. For $\alpha \in \mathbb{N}^d$, let $\widehat{Z} = \mathbb{E}_X[f(\mathbf{x}) \cdot \mathbf{x}^\alpha]$ be the empirical mean over the samples. Let $Z = \mathbb{E}_{\mathcal{D}}[f(\mathbf{x}) \cdot \mathbf{x}^\alpha]$ be the true mean. Clearly, $\mathbb{E}_X[\widehat{Z}] = Z$. Thus, we have that $\mathbb{P}_X[|\widehat{Z} - Z| \geq \epsilon] \leq \frac{\text{Var}_X[\widehat{Z}]}{\epsilon^2}$. We have that $\text{Var}_X[\widehat{Z}] \leq \frac{1}{m_{\text{conc}}} \text{Var}[f(\mathbf{x}) \cdot \mathbf{x}^\alpha]$. We have that $\text{Var}[f(\mathbf{x}) \cdot \mathbf{x}^\alpha] \leq \mathbb{E}_{\mathcal{D}}[\mathbf{x}^{2\alpha}]$ from the fact that f takes values in $\{\pm 1\}$. When $\mathcal{D} = \text{Unif}\{\pm 1\}^d$, $\mathbf{x}^{2\alpha} = 1$. When $\mathcal{D} = \mathcal{N}_d$, we have that $\mathbb{E}_{\mathcal{D}}[\mathbf{x}^{2\alpha}] \leq \text{poly}(d^\ell)$ (see Proposition 2.5.2 [Ver18]). Thus, choosing $m_{\text{conc}} = \text{poly}(d^\ell/\epsilon)$, we have that $\mathbb{P}_X[|\widehat{Z} - Z| \geq \epsilon] \leq \frac{\epsilon}{d^{\Omega(\ell)}}$. Taking a union bound over all $\alpha \in \mathbb{N}^d$ completes the proof. \square

Applying [Theorem C.7](#), [Lemma C.8](#) and the bounds on the sandwiching degrees ([Lemmas B.13](#), [B.14](#) and [B.17](#)) from [Appendix B.2](#), we immediately get the following results on TDS learning as corollaries.

Corollary C.9 (TDS learning for degree 2 PTFs with respect to $\text{Unif}\{\pm 1\}^d$ or \mathcal{N}_d). *Let \mathcal{C} be the class of degree-2 PTFs. Let $\epsilon > 0$ and $\ell = \tilde{O}(1/\epsilon^9)$. Then, there exists an algorithm that runs in time $d^{O(\ell)}$ and TDS learning \mathcal{C} with respect to $\text{Unif}\{\pm 1\}^d$ or \mathcal{N}_d with error at most $\text{opt}_{\text{train}} + \lambda + \epsilon$.*

Corollary C.10 (TDS learning for depth- t AC_0). *Let \mathcal{C} be the class of depth- t AC_0 circuits of size s on $\{\pm 1\}^d$. Let $\epsilon > 0$ and $\ell = (\log s)^{O(t)} \log(1/\epsilon)$. Then, there exists an algorithm that runs in time $d^{O(\ell)}$ and TDS learning \mathcal{C} with respect to $\text{Unif}\{\pm 1\}^d$ with error at most $\text{opt}_{\text{train}} + \lambda + \epsilon$.*

D Cylindrical Grids Tester

We focus on functions whose values only depend on the projection of the input on some low-dimensional subspace, i.e., we focus on the class of subspace juntas, which is formally defined as follows.

Definition D.1 (Subspace Junta). We say that a function $f : \mathbb{R}^d \rightarrow \{\pm 1\}$ is a k -subspace junta if there exists $W \in \mathbb{R}^{k \times d}$ with $\|W\|_2 = 1$ and $WW^\top = I_k$ as well as a function $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ such that

$$f(\mathbf{x}) = f_W(\mathbf{x}) = F(W\mathbf{x}) \text{ for any } \mathbf{x} \in \mathbb{R}^d$$

Since such functions only depend on a low-dimensional subspace, one might hope to exploit this property to obtain more efficient discrepancy testers. However, the relevant subspaces of different subspace juntas can be completely different and the low dimensional structure of a class of subspace juntas does not seem enough to provide significant improvements for global discrepancy testing. Nevertheless, it turns out that testing the localized discrepancy with respect to a notion of subspace neighborhood can be benefited by the low-dimensional structure. In particular, we define the notion of subspace neighborhood as follows.

Definition D.2 (Subspace Neighborhood). Let \mathcal{H} be the class of k -subspace juntas (see [Definition D.1](#)) and \mathcal{C} be some concept class. We define the (γ_s, γ_e) -subspace neighborhood $\mathbf{N}_s : \mathcal{H} \rightarrow \text{Pow}(\mathcal{C})$ as follows for any $\widehat{f} = \widehat{f}_V \in \mathcal{H}$.

$$\mathbf{N}_s(\widehat{f}_V) = \{f_W \in \mathcal{C} \mid \|W - V\|_2 \leq \gamma_s \text{ and } \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[f(\mathbf{x}) \neq \widehat{f}(\mathbf{x})] \leq \gamma_e\}$$

To design efficient testers for localized discrepancy in terms of the subspace neighborhood, we also use the notion of boundary of concepts and we require the boundaries to be smooth, meaning that the measure of the region close to the boundaries scales proportionally to its thickness. Formally, we provide the following definitions.

Definition D.3 (Boundary of Concept). Let $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ some concept. For $\varrho \geq 0$, we denote $\partial_\varrho F$ the ϱ -boundary of F , i.e., the region $\{\mathbf{x} \in \mathbb{R}^k : \exists \mathbf{z} \in \mathbb{R}^k \text{ with } \|\mathbf{z}\|_2 \leq \varrho \text{ and } F(\mathbf{x} + \mathbf{z}) \neq F(\mathbf{x})\}$.

Definition D.4 (Smooth Boundary). Let $F : \mathbb{R}^k \rightarrow \{\pm 1\}$. For $\sigma \geq 1$, we say that F has σ -smooth boundary with respect to \mathcal{N}_k if for any $\varrho \geq 0$

$$\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [\mathbf{x} \in \partial_\varrho F] := \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [\exists \mathbf{z} : \|\mathbf{z}\|_2 \leq \varrho, F(\mathbf{x} + \mathbf{z}) \neq F(\mathbf{x})] \leq \sigma \varrho$$

As we will show shortly, the choice of the subspace neighborhood not only enables obtaining faster localized discrepancy testers, but also testers that are guaranteed to accept much wider classes of distributions. This is because the properties of the test marginal that need to be tested in order to ensure low localized discrepancy are much simpler, compared to the properties required for global discrepancy. Such properties are not only easy to test, but are also satisfied by more distributions. The structural properties we will require for the completeness criteria of our algorithms are concentration in every direction and anti-concentration of low-dimensional marginals. More formally, we consider structured distributions to be as follows.

Definition D.5 (Structured Distributions). For $\mu_c : \mathbb{N} \rightarrow \mathbb{R}_+$, $\mu_{ac} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $k, d \in \mathbb{N}$ with $k \leq d$, we say that the distribution \mathcal{D} over \mathbb{R}^d is (μ_c, μ_{ac}) -structured on k -dimensions (w.r.t. \mathcal{N}_k), if the following are true.

1. (Concentration) For any $\mathbf{v} \in \mathbb{S}^{d-1}$ and $p \in \mathbb{N}$, we have $\mathbb{E}_{\mathbf{x} \sim \mathcal{D}} [(\mathbf{v} \cdot \mathbf{x})^{2p}] \leq \mu_c(p)$.
2. (Anti-concentration) For any subspace \mathcal{U} of dimension k , if \mathcal{Q} is the density of the marginal of \mathcal{D} on \mathcal{U} we have $\frac{\mathcal{Q}(\mathbf{x})}{\mathcal{N}_k(\mathbf{x})} \leq \mu_{ac}(R)$ for any $\mathbf{x} \in \mathbb{R}^k$ with $\|\mathbf{x}\|_2 \leq R$.

Moreover, if $k = d$, we simply say that \mathcal{D} is (μ_c, μ_{ac}) -structured.

Remark D.6. We note that the two conditions of **Definition D.5** are not always independent. For example, if $\mu_{ac}(R) = O(1)$, then the distribution \mathcal{Q} of condition 2 is subgaussian, which implies a bound on $\mu_c(p)$ for all $p \in \mathbb{N}$ (i.e., implies some version of condition 1). However, the anti-concentration condition does not always imply the concentration condition (e.g., if $\mu_{ac}(R) = \Theta(e^{R^2/2})$) and both conditions are important.

For example, isotropic log-concave distributions are structured on k -dimensions with $\mu_c(p) \leq (O(p))^{2p}$ and $\mu_{ac}(R) = (O(k))^k \exp(\frac{R^2}{2})$.

D.1 Discrepancy Testing Result

We now provide our main localized discrepancy testing result for subspace juntas with smooth boundaries, where we use some free parameters R, p that can be chosen according to how structured the target accepted class of distribution is.

Theorem D.7 (Discrepancy Testing through Cylindrical Grids). *Let $\mu_c : \mathbb{N} \rightarrow \mathbb{R}_{\geq 1}$, $\mu_{ac} : \mathbb{R}_+ \rightarrow \mathbb{R}_{\geq 1}$, $p \in \mathbb{N}$, $R, \sigma, \hat{\sigma} \geq 1$ and $\gamma_s, \gamma_e \in (0, 1)$. Let also \mathcal{H} (resp. \mathcal{C}) be a class whose elements are k -subspace juntas over \mathbb{R}^d with $\hat{\sigma}$ -smooth (resp. σ -smooth) boundaries. Consider \mathbb{D} to be the class of distributions over \mathbb{R}^d that are (μ_c, μ_{ac}) -structured on k -dimensions and $\mathbf{N}_s : \mathcal{H} \rightarrow \text{Pow}(\mathcal{C})$ the (γ_s, γ_e) -subspace neighborhood. For any $\epsilon \in (0, 1)$, there is a $(\mathbf{N}_s, \psi + \epsilon)$ -tester ([Algorithm 3](#)) for localized discrepancy from \mathcal{N}_d with respect to \mathbb{D} with sample complexity $m = \frac{10\mu_c(2)}{(\mu_c(1))^2} d^4 + \frac{12R^{2p}}{k\mu_c(p)} + \frac{14k(\sqrt{2\pi} \exp(R^2))^k}{\mu_{ac}(R\sqrt{k})\eta^k} \ln(\frac{3R}{\eta}) + O(\frac{1}{\epsilon^2})$ and time complexity $O(md^3 + mdk(2\lceil \frac{R}{\eta} \rceil)^k)$, where $\eta = \frac{\gamma_s R^p}{2\hat{\sigma}\sqrt{k}} \sqrt{\mu_c(1)/\mu_c(p)}$ and the error parameter ψ is*

$$\psi = \frac{14k\mu_c(p)}{R^{2p}} + 12 \left(\frac{2kR^{2p}\mu_c(1) \ln \mu_{ac}(R\sqrt{k})}{\mu_c(p)} \right)^{\frac{1}{2}} \mu_{ac}(R\sqrt{k})\sigma\gamma_s + 2\mu_{ac}(R\sqrt{k})\gamma_e$$

For different target distribution classes we obtain different results, that reveal a trade-off between universality and the size of the subspace neighborhood tested. To accept wider classes of distributions, we restrict to testing localized discrepancy with respect to narrower neighborhoods, which is parameterized by γ_s and γ_e in the following corollary. Eventually, for applications in TDS learning, this will result into requiring the training algorithm to provide stronger error guarantees by using more training examples and time.

Algorithm 3: Cylindrical Grids Tester

Input: Set X of points in \mathbb{R}^d , matrix $V \in \mathbb{R}^{k \times d}$, parameters $p \in \mathbb{N}$, $R \geq 1$, $\eta > 0$
Compute the matrix $M = \mathbb{E}_{\mathbf{x} \sim X}[\mathbf{x}\mathbf{x}^\top]$ and **reject** if the largest eigenvalue is larger than $2\mu_c(1)$.
Compute the quantity $\mathbb{P}_{\mathbf{x} \sim X}[\|\mathbf{V}\mathbf{x}\|_\infty > R]$ and **reject** if the value is larger than $\frac{2k\mu_c(p)}{R^{2p}}$.
Let $\mathcal{I} = \{-\lceil \frac{R}{\eta} \rceil, \dots, -1, 0, \dots, \lceil \frac{R}{\eta} \rceil - 1\}$ and consider the grid
 $\mathcal{G}_{\eta, R} = \{[i_1\eta, (i_1 + 1)\eta] \times \dots \times [i_k\eta, (i_k + 1)\eta] : i_1, \dots, i_k \in \mathcal{I}\}$
for each grid cell $G \in \mathcal{G}_{\eta, R}$ **do**
 Compute the quantity $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{V}\mathbf{x} \in G]$ and **reject** if the value is larger than
 $2\mu_{ac}(R\sqrt{k}) \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[\mathbf{V}\mathbf{x} \in G]$.
end
Otherwise, **accept**.

Corollary D.8. Let $\epsilon \in (0, 1)$, let $\mathcal{H}, \mathcal{C}, \sigma, \hat{\sigma}$ be as in [Theorem D.7](#) and let $\mathbf{N}_s : \mathcal{H} \rightarrow \text{Pow}(\mathcal{C})$ be the (γ_s, γ_e) -subspace neighborhood (on k dimensions). For a class of distributions \mathbb{D} over \mathbb{R}^d , there is a (\mathbf{N}_s, ϵ) -tester for localized discrepancy from \mathcal{N}_d with respect to \mathbb{D} in each of the following cases for appropriately large universal constants $C_1, C_2 \geq 1$.

1. $\mathbb{D} = \{\mathcal{N}_d\}$, $\sigma\gamma_s \leq \gamma_e \leq (\frac{\epsilon}{C_1 k})^{C_2}$. The tester has time and sample complexity $\text{poly}(d)(\frac{k}{\epsilon})^{O(k)}(\sigma\hat{\sigma})^k$.
2. \mathbb{D} is the class of C -subgaussian and isotropic log-concave measures over \mathbb{R}^d for some $C = O(1)$ and $\sigma\gamma_s \leq \gamma_e \leq (\frac{\epsilon}{C_1})^{C_2 k}$. The tester has time and sample complexity $\text{poly}(d)(\frac{k}{\epsilon})^{O(k^2)}(\sigma\hat{\sigma})^k$.
3. \mathbb{D} is the class of isotropic log-concave measures over \mathbb{R}^d and also $\sigma\gamma_s \leq \gamma_e \leq (\frac{1}{C_1})^{-C_2 k^2 \log^2(1/\epsilon)}$. The tester has time and sample complexity $\text{poly}(d)k^{O(k^3 \log^2(1/\epsilon))}(\sigma\hat{\sigma})^k$.
4. \mathbb{D} is the class of distributions over \mathbb{R}^d that are (μ_c, μ_{ac}) -structured on k -dimensions, with $\mu_c(2) \leq C$ and $\mu_{ac}(R) \leq C^{k^2} e^{kR^2/2}$ for some $C = O(1)$ and $\sigma\gamma_s \leq \gamma_e \leq (\frac{1}{C_1})^{-C_2 k^2/\epsilon}$. The tester has time and sample complexity $\text{poly}(d)k^{O(k^3/\epsilon^2)}(\sigma\hat{\sigma})^k$.

Proof. To apply [Theorem D.7](#) in each case, it suffices to show bounds for $\mu_c(p)$ and $\mu_{ac}(R\sqrt{k})$ for each of the choices for \mathbb{D} . We then pick $p = \log(1/\epsilon)$ in Cases [1,2](#) and $p = 1$ in Case [4](#) and R sufficiently small to achieve error guarantee ϵ . For Case [1](#), $\mu_c(p) \leq (Cp)^p$ and $\mu_{ac}(R\sqrt{k}) \leq 1$. For case [2](#), $\mu_c(p) \leq (2Cp)^p$ and $\mu_{ac}(R\sqrt{k}) \leq (Ck)^k e^{kR^2/2}$. Finally, for Case [3](#), $\mu_c(p) \leq (Cp)^{2p}$ and $\mu_{ac}(R\sqrt{k}) \leq (Ck)^k e^{kR^2/2}$. These bounds follow from properties of log-concave and subgaussian distributions (see, e.g., [[LV07](#), [Ver18](#)]). \square

In order to prove [Theorem D.7](#), we first provide a tester which can certify that the mass assigned by the tested distribution to the region near the boundary of any function with smooth boundary is bounded. Structured distributions ([Definition D.5](#)) indeed have this property and the proposed tester can certify it universally over the class of such distributions.

This can be done by considering a cover the low-dimensional space by a grid of bounded size and checking whether the probability of falling within each of the grid cells is appropriately bounded. To account for grid cells that are far from the origin, it suffices to check that the tested distribution is sufficiently concentrated. If these tests pass, then we have a certificate that the mass of the tested distribution close to the boundary of any smooth function is appropriately bounded, because such regions can be covered by the union of a relatively small number of grid cells (see [Figure 2](#)).

Lemma D.9 (Grids Tester). Let $\mu_c : \mathbb{N} \rightarrow \mathbb{R}_+$, $\mu_{ac} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $p \in \mathbb{N}$, $R, \sigma \geq 1$ and $\varrho \in (0, 1)$. There is a tester \mathcal{T} which, upon receiving a set X of vectors in \mathbb{R}^k , and in time $|X| \cdot (O(\frac{R\sqrt{k}}{\varrho}))^k$, either accepts or rejects and satisfies the following.

(a) (Soundness.) If \mathcal{T} accepts, then for any $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ with σ -smooth boundary we have

$$\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in \partial_\varrho F] \leq \frac{2k\mu_c(p)}{R^{2p}} + 4\sigma\varrho\mu_{ac}(R\sqrt{k})$$

(b) (Completeness.) If X consists of at least $\frac{12R^{2p}}{k\mu_c(p)} + \frac{14k(3\sqrt{2\pi k} \exp(R^2))^k}{\mu_{ac}(R\sqrt{k})\varrho^k} \ln\left(\frac{9Rk}{\varrho}\right)$ i.i.d. examples from some (μ_c, μ_{ac}) -structured distribution over \mathbb{R}^k , then \mathcal{T} accepts with probability at least 99%.

Proof. Let $\eta = \frac{\varrho}{3\sqrt{k}}$ be some parameter, $\mathcal{I} = \{-\lceil \frac{R}{\eta} \rceil, \dots, -1, 0, \dots, \lceil \frac{R}{\eta} \rceil - 1\}$ be a set of indices and $\mathcal{G}_{\eta, R} = \{[i_1\eta, (i_1 + 1)\eta] \times \dots \times [i_k\eta, (i_k + 1)\eta] : i_1, \dots, i_k \in \mathcal{I}\}$ the corresponding finite grid with cell length η (each cell corresponds to a hypercube in \mathbb{R}^k , the cartesian product of k intervals each of length η). The tester does the following.

1. Computes the quantity $\mathbb{P}_{\mathbf{x} \sim X}[\|\mathbf{x}\|_\infty > R]$ and rejects if the computed value is larger than $\frac{2k\mu_c(p)}{R^{2p}}$.
2. For each cell G in the grid $\mathcal{G}_{\eta, R}$, computes the quantity $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in G]$ and rejects if the computed value is $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in G] > 2\mu_{ac}(R\sqrt{k})\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in G]$.
3. Otherwise, the tester accepts.

Soundness. Suppose that the tester \mathcal{T} has accepted. This means that the quantities $\mathbb{P}_{\mathbf{x} \sim X}[\|\mathbf{x}\|_\infty > R]$ and $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in G]$ are appropriately bounded (for any $G \in \mathcal{G}_{\eta, R}$). Let F be any function with σ -smooth boundary with respect to \mathcal{N}_k .

Consider $\tilde{\mathcal{G}} \subseteq \mathcal{G}_{\eta, R}$ to be the set of grid cells that have non-empty intersection with the set $\partial_\varrho F$ (see [Definition D.3](#)), i.e., $\tilde{\mathcal{G}} := \{G \in \mathcal{G}_{\eta, R} : G \cap \partial_\varrho F \neq \emptyset\}$. Observe that if $\mathbf{x} \in \partial_\varrho F$ then either $\|\mathbf{x}\|_\infty > R$, or $\mathbf{x} \in G$ for some $G \in \tilde{\mathcal{G}}$, because the grid covers the set $\{\mathbf{x} : \|\mathbf{x}\|_\infty \leq R\}$. Moreover, if $\mathbf{x} \in \tilde{\mathcal{G}}$, then there is a point $\mathbf{y} \in \tilde{\mathcal{G}} \cap \partial_\varrho F$ that falls in the same cell as \mathbf{x} and, therefore, $\|\mathbf{x} - \mathbf{y}\|_2 \leq \eta\sqrt{k}$, because each cell has length η . This implies that $\mathbf{x} \in \partial_{\varrho + \eta\sqrt{k}} F$. We overall have the following (see also [Figure 2](#)).

$$\partial_\varrho F \setminus \{\mathbf{x} : \|\mathbf{x}\|_\infty > R\} \subseteq \bigcup_{G \in \tilde{\mathcal{G}}} G \subseteq \partial_{\tilde{\varrho}} F, \text{ where } \tilde{\varrho} := \varrho + \eta\sqrt{k} \quad (\text{D.1})$$

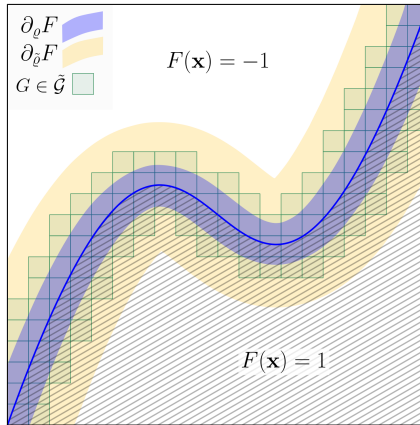


Figure 2: Discretization of smooth boundary

Combining the first inclusion in expression (D.1) with the fact that the tester has accepted, the quantity $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in \partial_\varrho F]$ is bounded as follows.

$$\begin{aligned} \mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in \partial_\varrho F] &\leq \mathbb{P}_{\mathbf{x} \sim X}[\|\mathbf{x}\|_\infty > R] + \sum_{G \in \tilde{\mathcal{G}}} \mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in G] \\ &\leq \frac{2k\mu_c(p)}{R^{2p}} + 2\mu_{ac}(R\sqrt{k}) \sum_{G \in \tilde{\mathcal{G}}} \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in G] \end{aligned}$$

For any $G, G' \in \tilde{\mathcal{G}}$ with $G \neq G'$, the events that $\mathbf{x} \in G$ and that $\mathbf{x} \in G'$ are mutually exclusive. Therefore $\sum_{G \in \tilde{\mathcal{G}}} \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in G] = \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in \cup_{G \in \tilde{\mathcal{G}}} G] \leq \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in \partial_{\tilde{\varrho}} F]$, where the final inequality follows from the second inclusion in expression (D.1). Since F has σ -smooth boundary, we have $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in \partial_{\tilde{\varrho}} F] \leq \sigma \tilde{\varrho}$. Overall, we have

$$\begin{aligned} \mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in \partial_\varrho F] &\leq \frac{2k\mu_c(p)}{R^{2p}} + 2\sigma(\varrho + \eta\sqrt{k})\mu_{ac}(R\sqrt{k}) \\ &\leq \frac{2k\mu_c(p)}{R^{2p}} + 4\sigma\varrho\mu_{ac}(R\sqrt{k}), \text{ as desired.} \end{aligned}$$

Completeness. Suppose, now, that the examples X are drawn independently from a (μ_c, μ_{ac}) -structured distribution \mathcal{Q} . We first show that, with probability at least $1 - \frac{1}{200}$, we have $\mathbb{P}_{\mathbf{x} \sim X}[\|\mathbf{x}\|_\infty > R] \leq \frac{2k\mu_c(p)}{R^{2p}}$.

We first bound the quantity $\mathbb{P}_{\mathbf{x} \sim \mathcal{Q}}[\|\mathbf{x}\|_\infty > R]$, by using Markov's inequality as follows.

$$\begin{aligned} \mathbb{P}_{\mathbf{x} \sim \mathcal{Q}}[\|\mathbf{x}\|_\infty > R] &\leq k \sup_{\mathbf{v} \in \mathbb{S}^{k-1}} \mathbb{P}_{\mathbf{x} \sim \mathcal{Q}}[|\mathbf{v} \cdot \mathbf{x}| > R] \\ &\leq k \frac{\sup_{\mathbf{v} \in \mathbb{S}^{k-1}} \mathbb{E}_{\mathbf{x} \sim \mathcal{Q}}[(\mathbf{v} \cdot \mathbf{x})^{2p}]}{R^{2p}} \\ &\leq \frac{k\mu_c(p)}{R^{2p}}, \text{ since } \mathcal{Q} \text{ is structured.} \end{aligned}$$

By the multiplicative Chernoff bound¹⁰, we have that $\mathbb{P}_{\mathbf{x} \sim X}[\|\mathbf{x}\|_\infty > R] \leq 2\mathbb{P}_{\mathbf{x} \sim \mathcal{Q}}[\|\mathbf{x}\|_\infty > R]$ with probability at least $1 - \exp(-|X| \frac{k\mu_c(p)}{2R^{2p}}) \geq 1 - \frac{1}{200}$, since $|X| \geq \frac{12R^{2p}}{k\mu_c(p)}$.

We will show that for each $G \in \mathcal{G}_{\eta, R}$, $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in G] \leq 2\mu_{ac}(R\sqrt{k})\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k}[\mathbf{x} \in G]$, with probability at least $1 - \exp(-\frac{|X|}{2}\mu_{ac}(R\sqrt{k})\eta^k/(\sqrt{2\pi}e^{R^2})^k)$. The desired result then follows by a union bound over $\mathcal{G}_{\eta, R}$ (where $|\mathcal{G}_{\eta, R}| \leq (3R/\eta)^k$) and the fact that $|X| \geq \frac{14k(\sqrt{2\pi}\exp(R^2))^k}{\mu_{ac}(R\sqrt{k})\eta^k} \ln(\frac{3R}{\eta})$.

We first bound $\mathbb{P}_{\mathbf{x} \sim \mathcal{Q}}[\mathbf{x} \in G]$ as follows by using the fact that \mathcal{Q} is structured and $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_\infty \sqrt{k} \leq R\sqrt{k}$ for all $\mathbf{x} \in G$ (because $G \in \mathcal{G}_{\eta, R}$).

$$\mathbb{P}_{\mathbf{x} \sim \mathcal{Q}}[\mathbf{x} \in G] = \int_{\mathbf{x} \in G} \mathcal{Q}(\mathbf{x}) d\mathbf{x} \leq \mu_{ac}(R\sqrt{k}) \int_{\mathbf{x} \in G} \mathcal{N}(\mathbf{x}) d\mathbf{x} = \mu_{ac}(R\sqrt{k}) \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[\mathbf{x} \in G]$$

By the multiplicative Chernoff bound, we once more have that $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in G] \leq 2\mathbb{P}_{\mathbf{x} \sim \mathcal{Q}}[\mathbf{x} \in G]$ with probability at least $1 - \exp(-\frac{|X|}{2}\mu_{ac}(R\sqrt{k})\mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[\mathbf{x} \in G])$ and conclude the proof by observing that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[\mathbf{x} \in G] \geq (\frac{\eta}{\sqrt{2\pi}\exp(R^2)})^k$. \square

Remark D.10. We note that **Lemma D.9** is not specialized to the Gaussian distribution. The only requirement is that the distribution of the completeness criterion is structured with respect to the same distribution for which the functions F of the soundness criterion have smooth boundary. In particular, in **Definition D.5**, the anti-concentration condition 2 is defined with respect to the Gaussian, but it could also be defined with respect to some other distribution. The concentration condition 1 is always the same.

¹⁰We use the version of the Chernoff bound that uses an upper bound on the expectation rather than the exact value, through a standard coupling argument.

We are now ready to prove [Theorem D.7](#). The idea is that if a function f lies within the subspace neighborhood of another function \hat{f} , then the disagreement region between the two functions is bounded by the union of: (1) their disagreement after projecting on the relevant subspace for \hat{f} (since the subspace is known, it can be tested exhaustively, similarly to [Lemma D.9](#)) and (2) the region far from the origin (for which testing concentration suffices).

Proof of [Theorem D.7](#). Let \mathcal{D}' be the unknown distribution and X a set of m i.i.d. samples from \mathcal{D}' and let $\eta = \frac{\gamma_s R^p}{2\hat{\sigma}\sqrt{k}} \sqrt{\frac{\mu_c(1)}{\mu_c(p)}}$. Let (\hat{f}_V, X) be an instance of the localized discrepancy testing problem (see [Definition 1.1](#)). We run [Algorithm 3](#) with input (X, V, p, R, η) and accept (or reject) accordingly.

Soundness. Suppose that the algorithm accepts. We will show that $\mathbb{P}_{\mathbf{x} \sim X}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] \leq \psi$ for any $f \in \mathbf{N}_s(\hat{f})$. Since the event that $\hat{f}(\mathbf{x}) \neq f(\mathbf{x})$ is independent for each $\mathbf{x} \in X$, we may apply the Hoeffding bound to show that $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}'}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] \leq \psi + \epsilon$ with probability at least $3/4$ whenever $|X| \geq \frac{3}{\epsilon^2}$. To bound the empirical quantity, we have the following, for $R_s = R^p(\mu_c(1)/\mu_c(p))^{1/2}$ and $\varrho = \frac{\gamma_s R_s}{\hat{\sigma}}$.

$$\mathbb{P}_{\mathbf{x} \sim X}[F(W\mathbf{x}) \neq \hat{F}(V\mathbf{x})] \leq \underbrace{\mathbb{P}_{\mathbf{x} \sim X}[F(W\mathbf{x}) \neq F(V\mathbf{x})]}_{P_1} + \underbrace{\mathbb{P}_{\mathbf{x} \sim X}[F(V\mathbf{x}) \neq \hat{F}(V\mathbf{x})]}_{P_2}$$

For the term P_1 , we observe that $F(W\mathbf{x}) = F((W - V)\mathbf{x} + V\mathbf{x})$ and therefore

$$\begin{aligned} P_1 &\leq \mathbb{P}_{\mathbf{x} \sim X}[\|(W - V)\mathbf{x}\|_2 \geq \gamma_s R_s] + \mathbb{P}_{\mathbf{x} \sim X}[\exists \mathbf{z} \in \mathbb{R}^k : \|\mathbf{z}\|_2 \leq \gamma_s R_s, F(V\mathbf{x} + \mathbf{z}) \neq F(V\mathbf{x})] \\ &= \mathbb{P}_{\mathbf{x} \sim X}[\|(W - V)\mathbf{x}\|_2 \geq \gamma_s R_s] + \mathbb{P}_{\mathbf{x} \sim X}[V\mathbf{x} \in \partial_{\gamma_s R_s} F] \end{aligned}$$

By applying Chebyshev's inequality for the first term in the above expression and [Lemma D.9](#) for the second term (note that we have chosen $\eta \leq \frac{\gamma_s R_s}{3\sqrt{k}}$ and [Algorithm 3](#) runs the tester corresponding to [Lemma D.9](#)), we obtain the following bound for P_1 (recall that $\|W - V\|_2 \leq \gamma_s$ and $\|(W - V)\mathbf{x}\|_2 \leq \|W - V\|_2 \|\text{proj}_U \mathbf{x}\|_2$, where U is the span of the columns of the matrix $W - V$).

$$\begin{aligned} P_1 &\leq \frac{k \sup_{\mathbf{v} \in \mathbb{S}^{d-1}} \mathbb{E}_{\mathbf{x} \sim X}[(\mathbf{v} \cdot \mathbf{x})^2]}{R_s^2} + \frac{2k\mu_c(p)}{R^{2p}} + 4\sigma\gamma_s R_s \mu_{ac}(R\sqrt{k}) \\ &\leq \frac{2k\mu_c(1)}{R_s^2} + \frac{2k\mu_c(p)}{R^{2p}} + 4\sigma\gamma_s R_s \mu_{ac}(R\sqrt{k}) \end{aligned}$$

The last inequality follows from the spectral bound on the empirical covariance matrix $M = \mathbb{E}_{\mathbf{x} \sim X}[\mathbf{x}\mathbf{x}^\top]$ implied by [Algorithm 3](#) upon acceptance.

For the term P_2 , consider the set of grid cells $\tilde{\mathcal{G}}$ with non-zero intersection with the disagreement region, i.e., $\tilde{\mathcal{G}} = \{G \in \mathcal{G}_{\eta, R} : \text{there is } \mathbf{x} \text{ with } V\mathbf{x} \in G \text{ and } F(V\mathbf{x}) \neq \hat{F}(V\mathbf{x})\}$. Recall that $\varrho = \eta\sqrt{k}$ and let $\tilde{\mathcal{G}}_{\text{in}}$ be the interior part of $\tilde{\mathcal{G}}$, i.e., $\tilde{\mathcal{G}}_{\text{in}} = \{G \in \tilde{\mathcal{G}} : \text{for any } \mathbf{x} \text{ with } V\mathbf{x} \in G \text{ we have } F(V\mathbf{x}) \neq \hat{F}(V\mathbf{x})\}$.

Let \mathbf{x} be such that $\|V\mathbf{x}\|_\infty \leq R$, $F(V\mathbf{x}) \neq \hat{F}(V\mathbf{x})$ and $V\mathbf{x} \notin \partial_\varrho F \cup \partial_\varrho \hat{F}$. It must be that $V\mathbf{x}$ lies within some grid cell in $\tilde{\mathcal{G}}_{\text{in}}$. To see this, note that $V\mathbf{x}$ must be in exactly one grid cell G in $\tilde{\mathcal{G}}$ (by definition of $\tilde{\mathcal{G}}$) and if this grid cell was in $\tilde{\mathcal{G}} \setminus \tilde{\mathcal{G}}_{\text{in}}$, this would imply that for some \mathbf{x}' with $V\mathbf{x}' \in G$ we would have either $F(V\mathbf{x}) \neq F(V\mathbf{x}')$ or $\hat{F}(V\mathbf{x}) \neq \hat{F}(V\mathbf{x}')$ (because F, \hat{F} disagree on $V\mathbf{x}$ but agree on $V\mathbf{x}'$). However, $\|V\mathbf{x} - V\mathbf{x}'\|_2 \leq \eta\sqrt{k} = \varrho$, because they are in the same grid cell and we conclude that $V\mathbf{x} \in \partial_\varrho F \cup \partial_\varrho \hat{F}$, which is a contradiction. Overall, we have the following.

$$P_2 \leq \underbrace{\mathbb{P}_{\mathbf{x} \sim X}[\|V\mathbf{x}\|_\infty > R]}_{P_{21}} + \underbrace{\mathbb{P}_{\mathbf{x} \sim X}[V\mathbf{x} \in \partial_\varrho F]}_{P_{22}} + \underbrace{\mathbb{P}_{\mathbf{x} \sim X}[V\mathbf{x} \in \partial_\varrho \hat{F}]}_{P_{23}} + \underbrace{\sum_{G \in \tilde{\mathcal{G}}_{\text{in}}} \mathbb{P}_{\mathbf{x} \sim X}[V\mathbf{x} \in G]}_{P_{24}}$$

For the term P_{21} , we use the bound implied by [Algorithm 3](#), for the terms P_{22}, P_{23} we apply [Lemma D.9](#) and for the term P_{24} , we use the fact that (upon acceptance) $\mathbb{P}_{\mathbf{x} \sim X}[V\mathbf{x} \in G] \leq 2\mu_{ac}(R\sqrt{k}) \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[V\mathbf{x} \in G]$ to obtain the following.

$$\begin{aligned} P_{24} &\leq 2\mu_{ac}(R\sqrt{k}) \sum_{G \in \hat{\mathcal{G}}_{\text{in}}} \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[V\mathbf{x} \in G] \\ &\leq 2\mu_{ac}(R\sqrt{k}) \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[F(V\mathbf{x}) \neq \hat{F}(V\mathbf{x})] \end{aligned}$$

We bound the quantity $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[F(V\mathbf{x}) \neq \hat{F}(V\mathbf{x})]$ as follows.

$$\begin{aligned} \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[F(V\mathbf{x}) \neq \hat{F}(V\mathbf{x})] &\leq \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[F(W\mathbf{x}) \neq \hat{F}(V\mathbf{x})] + \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[F(W\mathbf{x}) \neq F(V\mathbf{x})] \\ &\leq \gamma_e + \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[\|(W - V)\mathbf{x}\|_2 > \gamma_s R'] + \mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[V\mathbf{x} \in \partial_{\gamma_s R'} F] \\ &\leq \gamma_e + 4ke^{-\frac{R'^2}{2k}} + \sigma\gamma_s R' \end{aligned}$$

where the last inequality follows from Gaussian concentration and the fact that F has σ -smooth boundary. By choosing $R' = (2k \ln(\frac{R^{2p}\mu_{ac}(R\sqrt{k})}{\mu_c(p)}))^{1/2}$, we obtain that

$$P_{24} \leq 2\mu_{ac}(R\sqrt{k})\gamma_e + \frac{4k\mu_c(p)}{R^{2p}} + 2\sigma\gamma_s\mu_{ac}(R\sqrt{k}) \left(2k \ln\left(\frac{R^{2p}\mu_{ac}(R\sqrt{k})}{\mu_c(p)}\right)\right)^{1/2}$$

Overall, for the term P_2 we have the following bound.

$$P_2 \leq \frac{10k\mu_c(p)}{R^{2p}} + 10\sigma\gamma_s R^p \sqrt{\frac{2k\mu_c(1)}{\mu_c(p)}} \mu_{ac}(R\sqrt{k}) (\ln \mu_{ac}(R\sqrt{k}))^{1/2} + 2\gamma_e \mu_{ac}(R\sqrt{k})$$

Combining the bounds for P_1 and P_2 , we obtain the desired result.

Completeness. Suppose, now, that $\mathcal{D}' \in \mathbb{D}$. It suffices to show that all the tests will accept with probability at least $3/4$. For the quantity $\mathbb{P}_{\mathbf{x} \sim X}[\|\mathbf{V}\mathbf{x}\|_\infty > R]$ as well as the quantities $\mathbb{P}_{\mathbf{x} \sim X}[V\mathbf{x} \in G]$, we apply the Chernoff Bound as described in the proof of completeness of the grid tester (see the proof of [Lemma D.9](#)). For the quantity $M = \mathbb{E}_{\mathbf{x} \sim X}[\mathbf{x}\mathbf{x}^\top]$, we use the Chebyshev's inequality on each of the random variables $M_{ij} = \mathbb{E}_{\mathbf{x} \sim X}[\mathbf{x}_i\mathbf{x}_j]$, the fact that $\mathbb{E}[M_{ij}^2] \leq \mu_c(2)$ and a union bound over $i, j \in [d]$. \square

D.2 Application to TDS Learning

Interestingly, in learning theory, there are algorithms that are guaranteed to recover the relevant subspace for certain classes of subspace juntas that have some additional properties. This enables us to use the discrepancy tester of [Theorem D.7](#) to obtain end-to-end results for TDS learning, because the training phase can guarantee that the ground truth lies within the subspace neighborhood of the output hypothesis \hat{f} , for which we have efficient localized discrepancy testers. Here, we present a TDS learning result for balanced convex subspace juntas in the realizable setting. The class of balanced convex subspace juntas is defined as follows.

Definition D.11 (Balanced Convex Subspace Juntas). A concept $f : \mathbb{R}^d \rightarrow \{\pm 1\}$ is a β -balanced convex k -subspace junta if it is β -balanced (see [Definition A.1](#)), convex and a k -subspace junta (see [Definition D.1](#)).

We make use of known algorithms from PAC learning that are guaranteed to approximately recover the effective ground-truth subspace in terms of geometric distance, which is important since the tester of [Theorem D.7](#) works with respect to the subspace neighborhood and obtain the following theorem, which underlines a trade-off between training time and universality.

Theorem D.12 (TDS Learning of Convex Subspace Juntas). *For $\beta \in (0, 1/2)$, $d, k \in \mathbb{N}$, let \mathcal{C} be the class of β -balanced convex k -subspace juntas over \mathbb{R}^d . For any $\epsilon \in (0, 1)$, there is a (decoupled) ϵ -TDS learner for \mathcal{C} with respect to \mathcal{N}_d in the realizable setting, which, for the learning phase, uses $\text{poly}(d)(\frac{1}{\beta})^{\text{poly}(k/\epsilon)}$ samples and time and, for the testing phase, uses $\text{poly}(d)(k/\epsilon)^{O(k)}$ samples and time. Moreover, in the same setting, there is a \mathbb{D} -universal ϵ -TDS learner for \mathcal{C} for each of the cases listed in [Table 2](#).*

	Class \mathbb{D} over \mathbb{R}^d	Training Time and Samples	Testing Time and Samples
1	1-subgaussian & Isotropic Log-Concave	$\text{poly}(d)(\frac{1}{\beta})^{\text{poly}(1/\epsilon^k)}$	$\text{poly}(d)(k/\epsilon)^{O(k^2)}$
2	Isotropic Log-Concave	$\text{poly}(d)(\frac{1}{\beta})^{2^{O(k^2 \log^2(1/\epsilon))}}$	$\text{poly}(d)k^{O(k^3 \log^2(1/\epsilon))}$
3	Fourth Moments Bound: $\mathbb{E}[(\mathbf{v} \cdot \mathbf{x})^4] \leq C\ \mathbf{v}\ _2^4$ & Dimension- k Marginals Density Bound: C^{k^2}	$\text{poly}(d)(\frac{1}{\beta})^{2^{O(k^2/\epsilon)}}$	$\text{poly}(d)k^{O(k^3/\epsilon^2)}$

Table 2: Specifications for \mathbb{D} -universal (ϵ, δ) -TDS learning of β -balanced convex k -subspace juntas. The properties that define the class \mathbb{D} in line 3, hold for some given universal constant $C \geq 1$, for all members of \mathbb{D} , for all $\mathbf{v} \in \mathbb{R}^d$ and the density bound holds for any projection on some k -dimensional subspace of any member of \mathbb{D} .

In order to obtain a TDS learner for some class \mathcal{C} , one might hope to learn a hypothesis \hat{f} during the training phase, such that the subspace neighborhood of \hat{f} (see [Definition D.2](#)) contains the ground truth. Then, the test error can be bounded simply by running the localized discrepancy tester of [Theorem D.7](#), assuming that both \hat{f} and the class \mathcal{C} have smooth boundaries. In [Appendix B.1](#), we show that, indeed, convex subspace juntas have smooth boundaries. However, for the learning guarantee, prior work in standard PAC learning implicitly provides the following weaker guarantee regarding subspace retrieval for convex subspace juntas, which, as we show, is, nevertheless, still sufficient for our purposes.

Theorem D.13 (Implicit in [[Vem10a](#)], see also [[KSV24a](#)]). *For any $\gamma \in (0, 1)$, $\beta \in (0, 1/2)$, there is an algorithm that, upon receiving a number of i.i.d. examples from \mathcal{N}_d , labeled by some β -balanced convex k -subspace junta $f^*(\mathbf{x}) = F^*(W^*\mathbf{x})$, runs in time $\text{poly}(d)(\frac{1}{\beta})^{\text{poly}(k/\gamma)}$ and returns, w.p. at least 0.99, some polynomial $\hat{q} : \mathbb{R}^k \rightarrow \{\pm 1\}$ of degree at most $\text{poly}(k/\gamma)$ and some $V \in \mathbb{R}^{k \times d}$ with $VV^\top = I_k$ such that the following are true for the hypothesis $\hat{f}(\mathbf{x}) = \text{sign}(\hat{q}(V\mathbf{x}))$ and some $f(\mathbf{x}) = F^*(W\mathbf{x})$ with $WW^\top = I_k$.*

(a) $f \in \mathbf{N}_s(\hat{f})$, where \mathbf{N}_s is the k -dimensional (γ, γ) -subspace neighborhood, i.e., $\|W - V\|_2 \leq \gamma$ and $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_d}[f(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \leq \gamma$.

(b) For any $\mathbf{x} \in \mathbb{R}^d$ with $\|W^*\mathbf{x}\|_2 \leq \sqrt{k/\gamma}$, we have $f(\mathbf{x}) = f^*(\mathbf{x})$.

We are now ready to prove [Theorem D.12](#).

Proof of [Theorem D.12](#). Our plan is to combine [Theorem D.7](#) with [Theorem D.13](#). We will use an additional test, to account for the fact that [Theorem D.13](#) does not provide exact subspace recovery, but, rather, recovery of the effectively relevant subspace (see [Item \(b\)](#)).

Suppose that the training distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ has marginal $\mathcal{D}_{\mathcal{X}}^{\text{train}} = \mathcal{N}_d$ and that the labels (both in training and in test distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}$ as well) are generated by some β -balanced convex k -subspace junta $f^* : \mathbb{R}^d \rightarrow \{\pm 1\}$, where $f^*(\mathbf{x}) = F^*(W^*\mathbf{x})$ for some $W^* \in \mathbb{R}^{k \times d}$ with $W^*W^{*\top} = I_k$.

Learning Phase. The learner runs the algorithm of [Theorem D.13](#) for γ chosen so that the error parameter $\epsilon'(\gamma)$ of [Theorem D.7](#) is at most $\epsilon' \leq \epsilon/3$ using labeled examples from $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ and computes $\hat{f}(\mathbf{x}) = \text{sign}(\hat{q}(V\mathbf{x}))$ with the corresponding specifications. For the particular choice of γ , see [Corollary D.8](#), where $\sigma = \text{poly}(k)$ according to [Lemma B.8](#).

Testing Phase. The tester first computes the maximum eigenvalue of the matrix $\mathbb{E}_{\mathbf{x} \sim X_{\text{test}}}[\mathbf{x}\mathbf{x}^\top]$ using samples X_{test} drawn from $\mathcal{D}_{\mathcal{X}}^{\text{test}}$ and rejects if the quantity is larger than 2. Then, the tester runs the localized discrepancy tester of [Theorem D.7](#) and rejects or accepts accordingly.

Testing Run-Time. To bound the testing run-time we use [Corollary D.8](#), where $\sigma = \text{poly}(k)$ (because \mathcal{C} is the class of convex subspace juntas and due to [Lemma B.8](#)) and $\hat{\sigma} = \text{poly}(k/\gamma)$, because \hat{f} is a polynomial threshold function of degree $\text{poly}(k/\gamma)$ and, therefore, has $\text{poly}(k/\gamma)$ -smooth boundary according to [Lemma B.1](#).

Soundness. If the tester accepts and $|X_{\text{test}}| \geq \text{poly}(1/\epsilon)$, then we have $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}} [\|W^* \mathbf{x}\|_2 > \sqrt{k/\gamma}] \leq \mathbb{P}_{\mathbf{x} \sim X_{\text{test}}} [\|W^* \mathbf{x}\|_2 > \sqrt{k/\gamma}] + \epsilon/6$ (by the Hoeffding bound) and $\mathbb{P}_{\mathbf{x} \sim X_{\text{test}}} [\|W^* \mathbf{x}\|_2 > \sqrt{k/\gamma}] \leq 2\gamma \leq \epsilon/6$ for $\gamma \leq \epsilon/12$. Hence, overall, by combining [Theorem D.13](#) with the guarantees from the fact that the testing phase has accepted, we have

$$\begin{aligned} \text{err}(\hat{f}; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}) &= \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}} [f^*(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \\ &\leq \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}} [\|W^* \mathbf{x}\|_2 > \sqrt{k/\gamma}] + \mathbb{P}_{\mathbf{x} \sim \mathcal{D}_{\mathcal{X}}^{\text{test}}} [\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] \\ &\leq \frac{\epsilon}{3} + \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_d} [\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] + \frac{\epsilon}{3} \\ &\leq \frac{2\epsilon}{3} + \gamma \leq \epsilon, \end{aligned}$$

where we used the soundness property of the cylindrical grids tester ([Theorem D.7](#) and [Corollary D.8](#)) and the fact that f is a hypothesis with the properties specified in [Theorem D.13](#) and, in particular, lies within the subspace neighborhood of \hat{f} .

Completeness. Combine the completeness guarantee of [Theorem D.7](#) and the fact that $\mathbb{E}_{\mathbf{x} \sim X_{\text{test}}} [\mathbf{x}\mathbf{x}^\top]$ has, with probability at least 0.99, bounded maximum eigenvalue whenever $\mathcal{D}_{\mathcal{X}}^{\text{test}}$ lies within \mathbb{D} (for any \mathbb{D} in [Table 2](#)) and $|S_{\text{test}}| \geq \text{poly}(d)$. \square

E Testing Boundary Proximity

We now focus on classes of low-dimensional concepts (see [Definition D.1](#)) that are locally structured. In particular, we consider subspace juntas that are locally balanced, meaning that near any point \mathbf{x} in the domain, there are several points with the same label as \mathbf{x} . This condition is important to ensure that there are, for example, no zero measure regions over the (Gaussian) training distribution that contain significant information about the ground truth. We will show that this condition actually enables significant improvements for the testing runtime for TDS learning. More formally, we give the following definition.

Definition E.1 (Locally Balanced Concepts). For $R \geq 1$ and $r, \beta \in (0, 1)$, we say that a function $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ is (R, r) -locally β -balanced if for any $\varrho \leq r$ and $\mathbf{x} \in \mathbb{R}^k$ with $\|\mathbf{x}\|_2 \leq R$, the following is true.

$$\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k} [F(\mathbf{z}) = F(\mathbf{x}) \mid \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)] > \beta$$

For a subspace junta $f(\mathbf{x}) = F(W\mathbf{x})$, we say that f is (R, r) -locally β -balanced on the relevant subspace if F is (R, r) -locally β -balanced.

For locally balanced concepts, it is possible to obtain efficient localized discrepancy testers with respect to the disagreement neighborhood, i.e., the neighborhood of concepts that have low disagreement with the reference hypothesis \hat{f} under the Gaussian distribution (or, in general, the reference distribution at hand).

Definition E.2 (Disagreement Neighborhood). Let \mathcal{H} and \mathcal{C} be some concept classes. We define the (Gaussian) γ_e -disagreement neighborhood $\mathbf{N}_e : \mathcal{H} \rightarrow \text{Pow}(\mathcal{C})$ as follows for any $\hat{f} \in \mathcal{H}$.

$$\mathbf{N}_e(\hat{f}) = \{f \in \mathcal{C} \mid \mathbb{P}_{\mathbf{x} \sim \mathcal{N}^d} [f(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \leq \gamma_e\}$$

We also define the boundary proximity tester, which directly tests whether the probability of falling close to the boundary of some reference hypothesis \hat{f} is appropriately bounded. This testing problem can be solved efficiently, for example, for the fundamental class of halfspace intersections.

Definition E.3 (Boundary Proximity Tester). For $\hat{\sigma} \geq 1$, $\varrho \in (0, 1)$, let \mathcal{H} be some class of functions from \mathbb{R}^d to $\{\pm 1\}$ and let \mathbb{D} be some class of distributions over \mathbb{R}^d . The tester \mathcal{T} is called a $(\varrho, \hat{\sigma})$ -boundary proximity tester for \mathcal{H} with respect to \mathbb{D} if, upon receiving some $\hat{f} \in \mathcal{H}$ and a set X of points in \mathbb{R}^d , the tester either accepts or rejects and satisfies the following.

- (a) (Soundness.) If \mathcal{T} accepts, then $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in \partial_\varrho \hat{f}] \leq \hat{\sigma} \varrho$.
- (b) (Completeness.) If X consists of (at least) $m_{\mathcal{T}}$ i.i.d. examples from some distribution in \mathbb{D} , then the tester \mathcal{T} accepts with probability at least 99%.

Note that the complexity of boundary proximity testing depends on the simplicity of \hat{f} and, therefore, considering applications in TDS learning, where \hat{f} is the output of the learning algorithm, highlights the importance of proper learning algorithms that output some simple hypothesis with low error. Since the hypothesis is simple, disagreement-localized discrepancy testing is tractable and since its error is low, the ground truth is likely within the disagreement neighborhood and disagreement-localized discrepancy testing suffices to guarantee low test error.

E.1 Discrepancy Testing Result

In order to obtain a localized discrepancy tester assuming access to a boundary proximity tester, we first show a simple proposition connecting local balance condition with boundary proximity testing. In particular, if two functions have low Gaussian disagreement, but one of them is locally balanced, then all of the points of disagreement are either close to the boundary of the other function, or far from the origin.

Proposition E.4 (Localization of Disagreement from Locally Balanced Concepts). *Let $F, \hat{F} : \mathbb{R}^k \rightarrow \{\pm 1\}$, where F is (R, ϱ) -locally β -balanced and F, \hat{F} have disagreement $\gamma = \beta \inf_{\|\mathbf{x}\|_2 \leq R} \mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)]$, i.e., $\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[F(\mathbf{z}) \neq \hat{F}(\mathbf{z})] \leq \gamma$. Then, for any \mathbf{x} with $\|\mathbf{x}\|_2 \leq R$ and $F(\mathbf{x}) \neq \hat{F}(\mathbf{x})$, we have $\mathbf{x} \in \partial_\varrho \hat{F}$.*

Proof of Proposition E.4. Suppose, for contradiction, that there exists some $\mathbf{x} \in \mathbb{R}^k$ with $\|\mathbf{x}\|_2 \leq R$ and $F(\mathbf{x}) \neq \hat{F}(\mathbf{x})$, for which $\mathbf{x} \notin \partial_\varrho \hat{F}$. Then, it must be that $\hat{F}(\mathbf{z}) = \hat{F}(\mathbf{x})$ for all $\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)$ (otherwise, $\mathbf{x} \in \partial_\varrho \hat{F}$). We have that $\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[F(\mathbf{z}) \neq \hat{F}(\mathbf{z})] \geq \mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho) \text{ and } F(\mathbf{z}) \neq \hat{F}(\mathbf{z})]$ and also $F(\mathbf{z}) \neq \hat{F}(\mathbf{z})$ is equivalent to $F(\mathbf{z}) \neq \hat{F}(\mathbf{x})$ (because $\hat{F}(\mathbf{z}) = \hat{F}(\mathbf{x})$), which, in turn, is equivalent to $F(\mathbf{z}) = F(\mathbf{x})$ (because $F(\mathbf{x}) \neq \hat{F}(\mathbf{x})$). Overall, $\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[F(\mathbf{z}) \neq \hat{F}(\mathbf{z})] \geq \mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho) \text{ and } F(\mathbf{z}) = F(\mathbf{x})] > \gamma$, by assumption, and we reached contradiction. \square

Remark E.5. Note that Proposition E.4 is not specialized to the Gaussian disagreement between F and \hat{F} , but would also work for any distribution \mathcal{Q} , if the local balance (Definition E.1) was also defined w.r.t. \mathcal{Q} .

We combine the boundary proximity tester with a moment matching tester for concentration (to bound the probability of falling far from the origin) to obtain a non-universal localized discrepancy tester (Theorem E.6). If we instead use a spectral tester for concentration, we obtain a universal localized discrepancy tester (Theorem E.7).

Theorem E.6 (Discrepancy Testing through Boundary Proximity). *Let $p \in \mathbb{N}$, $R, \hat{\sigma} \geq 1$, $r, \beta \in (0, 1)$ and $0 \leq \gamma_e \leq \frac{\beta r^k}{k^{k/2}} e^{-2R^2}$. Let also \mathcal{H} and \mathcal{C} be a classes whose elements are k -subspace juntas over \mathbb{R}^d and $\mathbf{N}_e : \mathcal{H} \rightarrow \text{Pow}(\mathcal{C})$ the γ_e -disagreement neighborhood. Assume that the elements of \mathcal{C} are (R, r) -locally β -balanced on the relevant subspaces and let \mathcal{T} be a $(\varrho, \hat{\sigma})$ -boundary proximity tester for \mathcal{H} w.r.t. \mathcal{N}_d , requiring $m_{\mathcal{T}}$ samples, with $\varrho = (\frac{\gamma_e}{\beta})^{1/k} \sqrt{k} e^{2R^2/k}$. For any $\epsilon \in (0, 1)$, there is a $(\mathbf{N}_e, \psi + \epsilon)$ -tester for localized discrepancy from \mathcal{N}_d with respect to \mathcal{N}_d with sample complexity $m = m_{\mathcal{T}} + O(dk)^{4p+1} + O(\frac{1}{\epsilon^2})$, that calls \mathcal{T} once and uses additional time $O(md^{2p+1})$, where the error parameter ψ is*

$$\psi = 2 \left(\frac{4kp}{R^2} \right)^p + \hat{\sigma} \sqrt{k} \left(\frac{\gamma_e \exp(2R^2)}{\beta} \right)^{1/k}$$

Proof of Theorem E.6. Let $\varrho = (\gamma_e/\beta)^{1/k} \sqrt{k} \exp(2R^2/k)$, $\Delta = \frac{1}{(2kd)^{2p}}$ and let (\hat{f}, X) be an instance of the localized discrepancy problem (see Definition 1.1). The algorithm does the following.

1. For each $\alpha \in \mathbb{N}^d$ with $\|\alpha\|_1 \leq 2p$, compute the quantities $M_\alpha = \mathbb{E}_{\mathbf{x} \sim X}[\mathbf{x}^\alpha] = \mathbb{E}_{\mathbf{x} \sim X}[\prod_{i \in [d]} \mathbf{x}_i^{\alpha_i}]$ and **reject** if for some α as such, we have $|M_\alpha - \mathbb{E}_{\mathbf{x} \sim \mathcal{N}}[\mathbf{x}^\alpha]| > \Delta$.
2. Run the boundary proximity tester \mathcal{T} with inputs (ϱ, \hat{f}, X) and **reject** if \mathcal{T} rejects.
3. Otherwise, **accept**.

Soundness. Assume, first, that all of the tests have passed. We will show that for any $f \in \mathbb{N}_e(\hat{f})$, we have $\mathbb{P}_{\mathbf{x} \sim X}[f(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \leq \psi$. Since the event that $\hat{f}(\mathbf{x}) \neq f(\mathbf{x})$ is independent for each $\mathbf{x} \in X$, we may apply the Hoeffding bound to show that $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}'}[\hat{f}(\mathbf{x}) \neq f(\mathbf{x})] \leq \psi + \epsilon$ with probability at least $3/4$ whenever $|X| \geq \frac{3}{\epsilon^2}$. Since f and \hat{f} are k -subspace juntas, we have that $f(\mathbf{x}) = F(W\mathbf{x})$ and $\hat{f}(\mathbf{x}) = \hat{F}(V\mathbf{x})$ for $W, V \in \mathbb{R}^{k \times d}$ so that $WW^\top = VV^\top = I_k$. Let $U \in \mathbb{R}^{2k \times d}$ be a matrix such that $UU^\top = I_{2k}$ and the span of the rows of U contains the span of the rows of W and of V taken together. This, together with the fact that $WW^\top = I_k$, imply that for any $\mathbf{x} \in \mathbb{R}^d$ we have $W\mathbf{x} = WU^\top U\mathbf{x}$ and, similarly, $V\mathbf{x} = VU^\top U\mathbf{x}$ (the part of \mathbf{x} that falls within the subspace spanned by the rows of W does not change by applying the projection matrix $U^\top U$ and the remaining part is irrelevant). Moreover, we have that $\|U\|_2 = \|U^\top\|_2 = \|W\|_2 = \|V\|_2^\top = 1$. Let $F'(\mathbf{z}) = F(WU^\top \mathbf{z})$ and $\hat{F}'(\mathbf{z}) = \hat{F}(VU^\top \mathbf{z})$.

We have that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}}[F'(U\mathbf{x}) \neq \hat{F}'(U\mathbf{x})] \leq \gamma_e$, by assumption. By Proposition E.4, applied on F', \hat{F}' , and since $\gamma_e \leq \frac{\beta_1^k}{k^{k/2}} e^{-2R^2}$, we have that for any $\mathbf{x} \in \mathbb{R}^d$ such that $F'(U\mathbf{x}) \neq \hat{F}'(U\mathbf{x})$ (i.e., $F(W\mathbf{x}) \neq \hat{F}(V\mathbf{x})$) at least one of the following is true: (a) $\|U\mathbf{x}\|_2 \geq R$ or (b) $U\mathbf{x} \in \partial_\varrho \hat{F}'$. According to Proposition E.8, $U\mathbf{x} \in \partial_\varrho \hat{F}'$ implies that $VU^\top U\mathbf{x} \in \partial_\varrho \hat{F}$, which, in turn, implies that $V\mathbf{x} \in \partial_\varrho \hat{F}$, since $V\mathbf{x} = VU^\top U\mathbf{x}$ and therefore, by Proposition E.8 we also have that $\mathbf{x} \in \partial_\varrho \hat{f}$. Therefore, overall, we have

$$\mathbb{P}_{\mathbf{x} \sim X}[f(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \leq \mathbb{P}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2 \geq R] + \mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in \partial_\varrho \hat{f}]$$

In order to bound the term $\mathbb{P}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2 \geq R]$, we use the fact that the test of step 1 of the algorithm has passed. In particular, by applying Markov's inequality appropriately, we obtain that $\mathbb{P}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2 \geq R] \leq \frac{1}{R^{2p}} \mathbb{E}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2^{2p}]$. Note that the expression $\|U\mathbf{x}\|_2^{2p}$ corresponds to a polynomial of degree at most $2p$ and corresponding to coefficient vector whose absolute (ℓ_1) norm is bounded by $(4kd^2)^p$. In particular, we have that (for all $\mathbf{x} \in \mathbb{R}^d$) $\|U\mathbf{x}\|_2^{2p} = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \mathbf{x}^\alpha$ (recall that $\mathbf{x}^\alpha = \prod_{i \in [d]} \mathbf{x}_i^{\alpha_i}$), where $\sum_{\alpha \in \mathbb{N}^d} |c_\alpha| \leq (4kd^2)^p$ and $c_\alpha = 0$ whenever $\|\alpha\|_1 > 2p$. Therefore, by linearity of expectation, we have $\mathbb{E}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2^{2p}] = \sum_{\alpha} c_\alpha \mathbb{E}_{\mathbf{x} \sim X}[\mathbf{x}^\alpha] = \sum_{\alpha} c_\alpha (\mathbb{E}_{\mathbf{x} \sim \mathcal{N}}[\mathbf{x}^\alpha] + \Delta_\alpha) = \mathbb{E}_{\mathbf{x} \sim \mathcal{N}}[\|U\mathbf{x}\|_2^{2p}] + \sum_{\alpha} c_\alpha \Delta_\alpha$, where $|\Delta_\alpha| \leq \frac{1}{(2kd)^{2p}}$ for any α with $\|\alpha\|_1 \leq 2p$. Hence, overall, we have $\mathbb{E}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2^{2p}] \leq \mathbb{E}_{\mathbf{x} \sim \mathcal{N}}[\|U\mathbf{x}\|_2^{2p}] + 1 \leq 2(4kp)^p$, which implies that $\mathbb{P}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2 \geq R] \leq 2 \frac{(4kp)^p}{R^{2p}}$.

For the term $\mathbb{P}_{\mathbf{x} \sim X}[V\mathbf{x} \in \partial_\varrho \hat{F}]$, we use the fact that the tester \mathcal{T} has accepted and hence we have $\mathbb{P}_{\mathbf{x} \sim X}[\mathbf{x} \in \partial_\varrho \hat{f}] \leq \hat{\sigma} \varrho \leq \hat{\sigma} \left(\frac{\gamma_e \exp(2R^2)}{\beta k^{-k/2}} \right)^{1/k}$. We have shown that $\mathbb{P}_{\mathbf{x} \sim X}[f(\mathbf{x}) \neq \hat{f}(\mathbf{x})] \leq \psi$, as desired.

Completeness. Suppose now that X consists of i.i.d. examples from the Gaussian distribution \mathcal{N}_d . To ensure that with probability at least $9/10$, the tests of step 1 pass, we pick $|X| \geq \frac{(Cdk)}{\Delta^2}$, for some sufficiently large C . This is because the Gaussian moments concentrate (e.g., due to Chebyshev's inequality) as well as a union bound. For step 2, it suffices that $|X| \geq m_{\mathcal{T}}$. \square

We now give our universal discrepancy tester though testing boundary proximity.

Theorem E.7 (Universal Discrepancy Testing through Boundary Proximity). *In the setting of Theorem E.6, if the tester \mathcal{T} works with respect to a class \mathbb{D} of distributions over \mathbb{R}^d such that for*

some $\mu_c \geq 1$ we have $\sup_{\mathbf{v} \in \mathbb{S}^{d-1}} \mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[(\mathbf{v} \cdot \mathbf{x})^4] \leq \mu_c$ for all $\mathcal{D} \in \mathbb{D}$, then there is a $(\mathbf{N}_e, \psi + \epsilon)$ -tester for localized discrepancy from \mathcal{N}_d with respect to \mathbb{D} with sample complexity $m = m_{\mathcal{T}} + 20d^4 + \frac{3}{\epsilon^2}$, that calls \mathcal{T} once and uses additional time $O(md^2 + d^3)$, where the error parameter ψ is

$$\psi = \frac{4k\mu_c}{R^2} + \hat{\sigma}\sqrt{k} \left(\frac{\gamma_e \exp(2R^2)}{\beta} \right)^{1/k}$$

Proof of Theorem E.7. Let $\varrho = (\gamma_e/\beta)^{1/k} \sqrt{k} \exp(2R^2/k)$ and let (\hat{f}, X) be an instance of the localized discrepancy problem (see Definition 1.1). The algorithm is similar to the one used in Theorem E.6, but for the first step, instead of matching low degree moments, we compute the maximum eigenvalue of the second moment matrix.

1. Compute the maximum eigenvalue of the matrix $M = \mathbb{E}_{\mathbf{x} \sim X}[\mathbf{x}\mathbf{x}^\top]$ and **reject** if the computed value is larger than $2\mu_c$.
2. Run the boundary proximity tester \mathcal{T} with inputs (ϱ, \hat{f}, X) and **reject** if \mathcal{T} rejects.
3. Otherwise, **accept**.

Soundness. For the proof of soundness, we use a similar argument to the one for Theorem E.6, but we instead bound the term $\mathbb{E}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2^{2p}]$ for $p = 1$ and as follows

$$\mathbb{E}_{\mathbf{x} \sim X}[\|U\mathbf{x}\|_2^2] = \sum_{i=1}^{2k} \mathbb{E}_{\mathbf{x} \sim X}[(\mathbf{u}^i \cdot \mathbf{x})^2] \leq 2k \sup_{\mathbf{v} \in \mathbb{S}^{d-1}} \mathbb{E}_{\mathbf{x} \in X}[(\mathbf{v} \cdot \mathbf{x})^2] \leq 4k\mu_c,$$

where \mathbf{u}^i denotes the vector corresponding to the i -th row of U .

Completeness. The completeness for step 1 follows by an application of Chebyshev's inequality to the random variables corresponding to each of the entries of the matrix M and a union bound, to show that the Frobenius norm (and hence the operator norm) of the matrix $M - \mathbb{E}_{\mathbf{x} \sim \mathcal{D}}[\mathbf{x}\mathbf{x}^\top]$ is sufficiently small (where \mathcal{D} is some distribution in \mathbb{D} and X consists of independent draws from \mathcal{D}). \square

In the proofs of Theorems E.6 and E.7 we have used the following useful proposition.

Proposition E.8. Let $f : \mathbb{R}^d \rightarrow \{\pm 1\}$ be a k -subspace junta, i.e., $f(\mathbf{x}) = F(W\mathbf{x})$, where $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ and $W \in \mathbb{R}^{k \times d}$ with $WW^\top = I_k$. Then, we have $\mathbf{x} \in \partial_\varrho f$ if and only if $W\mathbf{x} \in \partial_\varrho F$.

Proof. Note, first that since $WW^\top = I_k$ and $k \leq d$, we have that $\|W\|_2 = 1$. Consider $\mathbf{x} \in \partial_\varrho f$. Then, by Definition D.3, we have that there exists $\mathbf{z} \in \mathbb{R}^d$ with $\|\mathbf{z}\|_2 \leq \varrho$ and $f(\mathbf{x} + \mathbf{z}) \neq f(\mathbf{x})$. Note that for the same \mathbf{x} and \mathbf{z} we have $F(W\mathbf{x} + W\mathbf{z}) \neq F(W\mathbf{x})$. Since $\|W\|_2 = 1$, we have that $\|W\mathbf{z}\|_2 \leq \|\mathbf{z}\|_2 \leq \varrho$. Let $\tilde{\mathbf{z}} = W\mathbf{z} \in \mathbb{R}^k$. We have $\|\tilde{\mathbf{z}}\|_2 \leq \varrho$ and $F(W\mathbf{x} + \tilde{\mathbf{z}}) \neq F(W\mathbf{x})$, i.e., $W\mathbf{x} \in \partial_\varrho F$.

For the other direction, suppose that $W\mathbf{x} \in \partial_\varrho F$. Then, there is $\tilde{\mathbf{z}} \in \mathbb{R}^k$ with $\|\tilde{\mathbf{z}}\|_2 \leq \varrho$ such that $F(W\mathbf{x} + \tilde{\mathbf{z}}) \neq F(W\mathbf{x})$. We have that $\tilde{\mathbf{z}} = I_k \tilde{\mathbf{z}} = WW^\top \tilde{\mathbf{z}}$. Let $\mathbf{z} = W^\top \tilde{\mathbf{z}}$. We have $\tilde{\mathbf{z}} = W\mathbf{z}$ and $\|\mathbf{z}\|_2 = \|W^\top \tilde{\mathbf{z}}\|_2 \leq \|W^\top\|_2 \|\tilde{\mathbf{z}}\|_2 = \|W\|_2 \|\tilde{\mathbf{z}}\|_2 \leq \varrho$. We have that $f(\mathbf{x} + \mathbf{z}) = F(W\mathbf{x} + W\mathbf{z}) = F(W\mathbf{x} + \tilde{\mathbf{z}}) \neq F(W\mathbf{x}) = f(\mathbf{x})$. Hence, $\mathbf{x} \in \partial_\varrho f$. \square

E.2 Application to TDS Learning

We now focus on the class of balanced intersections of halfspaces, which is formally defined as follows.

Definition E.9 (Balanced Halfspace Intersections). A concept $f : \mathbb{R}^d \rightarrow \{\pm 1\}$ is called a β -balanced intersection of k halfspaces if it is β -balanced (see Definition A.1) and there are $\mathbf{w}^1, \mathbf{w}^2, \dots, \mathbf{w}^k \in \mathbb{S}^{d-1}$ and $\tau_1, \tau_2, \dots, \tau_k \in \mathbb{R}$ such that $f(\mathbf{x}) = 2 \prod_{i=1}^k \mathbb{1}\{\mathbf{w}^i \cdot \mathbf{x} \geq \tau_i\} - 1$ for all $\mathbf{x} \in \mathbb{R}^d$.

We will now combine [Theorems E.6](#) and [E.7](#) with results from robust learning ([\[DKS18b\]](#)) to obtain the following theorem regarding TDS learning balanced intersections of halfspaces with respect to Gaussian training marginals. Our results indicate a trade-off between the training runtime and testing runtime and are robust to some amount of noise (in terms of the parameter λ).

Theorem E.10 (TDS Learning of Balanced Halfspace Intersections). *For $\beta \in (0, 1/2)$, $d, k \in \mathbb{N}$, let \mathcal{C} be the class of β -balanced intersections of k halfspaces \mathbb{R}^d . For any $\epsilon \in (0, 1)$ with $\epsilon = O(\frac{\beta}{k^2})$, there is a \mathbb{D} -universal ψ -TDS learner for \mathcal{C} w.r.t. \mathcal{N}_d in the agnostic setting for each of the cases listed in [Table 3](#).*

Class \mathbb{D} over \mathbb{R}^d	Training Time	Testing Time	Error Guarantee ψ
Gaussian \mathcal{N}_d	$\text{poly}(d)(\frac{k}{\epsilon\beta})^{O(k^3)}$	$(dk)^{O(\log(1/\epsilon))}$	$(\frac{k}{\epsilon\beta})^{O(1)}\lambda^{\frac{1}{12k}} + \epsilon$
Fourth Moments Bound: $\mathbb{E}[(\mathbf{v} \cdot \mathbf{x})^4] \leq C\ \mathbf{v}\ _2^4$ & Dimension-1 Marginal Densities Bounded by C	$\text{poly}(d)(\frac{k}{\beta})^{k^3} 2^{O(\frac{k^3}{\epsilon})}$	$\text{poly}(d, k, 1/\epsilon)$	$(\frac{k}{\beta})^{O(1)} 2^{O(\frac{1}{\epsilon})} \lambda^{\frac{1}{12k}} + \epsilon$

Table 3: Specifications for \mathbb{D} -universal ψ -TDS learning of β -balanced k -halfspace intersections. The properties that define the class \mathbb{D} in line 2, hold for some given universal constant $C \geq 1$, for all members of \mathbb{D} , for all $\mathbf{v} \in \mathbb{R}^d$ and the density bound holds for all one-dimensional projections of any member of \mathbb{D} .

For the learning phase of the algorithm of [Theorem E.10](#), we use an algorithm from [\[DKS18b\]](#) in the context of learning with nasty noise. Since the algorithm works under nasty noise, it will also work in the agnostic setting. The following result follows from [\[DKS18b, Theorem 5.1\]](#).

Theorem E.11 (Reformulation of Theorem 5.1 in [\[DKS18b\]](#)). *Let \mathcal{C} be some hypothesis class that consists of intersections of k halfspaces. For any $\gamma \in (0, 1)$, there is an algorithm that, upon receiving a number of i.i.d. examples from some labeled distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}$ whose marginal is \mathcal{N}_d , runs in time $\text{poly}(d)(\frac{k}{\gamma})^{O(k^2)}$ and returns, w.p. at least 0.99, some intersection of k halfspaces $\hat{f} : \mathbb{R}^d \rightarrow \{\pm 1\}$ such that for any distribution $\mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}$ over $\mathbb{R}^d \times \{\pm 1\}$, if $f^* \in \mathcal{C}$ is the intersection that achieves $\lambda = \min_{f \in \mathcal{C}} (\text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{test}}))$, then we have $f^* \in \mathbf{N}_e(\hat{f})$, where \mathbf{N}_e is the $(Ck\lambda^{\frac{1}{12}} + \gamma)$ -disagreement neighborhood (see [Definition E.2](#)), where C is some sufficiently large universal constant.*

Note that for the above reformulation of Theorem 5.1 in [\[DKS18b\]](#), we used the following reasoning. Their algorithm returns \hat{f} with the guarantee that $\text{err}(\hat{f}; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) \leq O(k \text{opt}_{\text{train}}^{\frac{1}{12}}) + \gamma$, where $\text{opt}_{\text{train}} = \min_{f \in \mathcal{C}} \text{err}(f; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) \leq \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) \leq \lambda$. Therefore $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_d}[\hat{f}(\mathbf{x}) \neq f^*(\mathbf{x})] \leq \text{err}(\hat{f}; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) + \text{err}(f^*; \mathcal{D}_{\mathcal{X}\mathcal{Y}}^{\text{train}}) \leq Ck\lambda^{\frac{1}{12}} + \gamma$, which implies that $f^* \in \mathbf{N}_e(\hat{f})$.

Our plan is to use the discrepancy testers of [Theorems E.6](#) and [E.7](#). To this end, we have to show that (1) balanced halfspace intersections are locally balanced and (2) there is a boundary proximity tester (see [Definition E.3](#)) for the class. It turns out that any convex set that is globally balanced (see [Definition A.1](#)), is also locally balanced (see [Definition E.1](#)), as we show in the following lemma.

Lemma E.12 (Globally Balanced Convex Sets are Locally Balanced). *For $\beta \in (0, 1)$, let $F : \mathbb{R}^k \rightarrow \{\pm 1\}$ be the indicator of a (globally) β -balanced convex set $\mathcal{K} \subseteq \mathbb{R}^k$, let $C \geq 1$ some sufficiently large universal constant and let $R \geq 1$. Then, F is $(R, \frac{\beta}{Ck \log k})$ -locally β' -balanced for $\beta' = \frac{\beta^k \exp(-\frac{1}{2}R)}{(Ck^2 R \ln(\frac{1}{\beta}))^k}$.*

Proof of Lemma E.12. Let $\varrho \leq \frac{\beta}{Ck \log k}$. We will first show that for any $\mathbf{x} \in \mathbb{R}^k$ with $\|\mathbf{x}\|_2 \leq R$ and $F(\mathbf{x}) = -1$, we have $\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k}[F(\mathbf{z}) = -1 \mid \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)] \geq \frac{1}{2}e^{-eR}$. We have that $\mathbf{x} \notin \mathcal{K}$ and, therefore, there is a separating hyperplane between \mathbf{x} and \mathcal{K} , due to the convexity of \mathcal{K} . This hyperplane does not pass through \mathbf{x} and, hence, at least half of $\mathbb{B}_k(\mathbf{x}, \varrho)$ is outside \mathcal{K} . We obtain the

following.

$$\begin{aligned}
\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k} [F(\mathbf{z}) = -1 \mid \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)] &= \frac{\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k} [F(\mathbf{z}) = -1 \text{ and } \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)]}{\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k} [\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)]} \\
&\geq \frac{\frac{1}{2} \text{vol}(\mathbb{B}_k(\mathbf{x}, \varrho))}{\text{vol}(\mathbb{B}_k(\mathbf{x}, \varrho))} \cdot \frac{\inf_{\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)} \mathcal{N}_k(\mathbf{z})}{\sup_{\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)} \mathcal{N}_k(\mathbf{z})} \\
&\geq \frac{1}{2} \cdot \frac{\exp(-\frac{1}{2}(\|\mathbf{x}\|_2 + \varrho)^2)}{\exp(-\frac{1}{2}(\|\mathbf{x}\|_2 - \varrho)^2)} \\
&\geq \frac{1}{2} e^{-\frac{1}{2} \varrho \|\mathbf{x}\|_2} \geq \frac{1}{2} e^{-\frac{1}{2} \varrho R}
\end{aligned}$$

For the case where $F(\mathbf{x}) = 1$, we first prove the following claim, which states that when a convex set is (globally) balanced, it must contain some Euclidean ball with non-negligible mass.

Claim. *Since \mathcal{K} is β -balanced and convex, there is $\mathbf{x}_c \in \mathbb{R}^k$ such that $\mathbb{B}_k(\mathbf{x}_c, r) \subseteq \mathcal{K}$, where $r = \frac{\beta}{Ck \log k}$, $\|\mathbf{x}_c\|_2 \leq R_c = (2k \ln(\frac{8k}{\beta}))^{1/2}$ and $C \geq 1$ is a sufficiently large universal constant.*

Proof. Since \mathcal{K} is balanced, we have $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{x}) = 1] > \beta$. We now use [Lemma B.8](#) to obtain that $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [\mathbf{x} \in \partial_r F] \leq \frac{C}{2} rk \log k$. We have the following.

$$\begin{aligned}
\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{z}) = 1, \forall \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, r)] &= \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{x}) = 1 \text{ and } F(\mathbf{x} + \mathbf{z}) = 1, \forall \mathbf{z} \text{ with } \|\mathbf{z}\|_2 \leq r] \\
&= \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{x}) = 1] - \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{x}) = 1 \text{ and } \exists \mathbf{z} : \|\mathbf{z}\|_2 \leq r \text{ and } F(\mathbf{x} + \mathbf{z}) \neq 1] \\
&\geq \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{x}) = 1] - \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [\exists \mathbf{z} : \|\mathbf{z}\|_2 \leq r \text{ and } F(\mathbf{x} + \mathbf{z}) \neq F(\mathbf{x})] \\
&= \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{x}) = 1] - \mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [\mathbf{x} \in \partial_r F] \\
&> \beta - \frac{C}{2} rk \log k = \frac{\beta}{2}
\end{aligned}$$

Moreover, since $\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [\|\mathbf{x}\|_2 > R_c] \leq 4ke^{-\frac{R_c^2}{2k}} = \beta/2$, we overall have that

$$\mathbb{P}_{\mathbf{x} \sim \mathcal{N}_k} [F(\mathbf{z}) = 1, \forall \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, r) \text{ and } \|\mathbf{x}\|_2 \leq R_c] > 0$$

Since the probability of such an \mathbf{x} is positive, by the probabilistic method, there is some \mathbf{x}_c as desired. \square

We have shown that for some \mathbf{x}_c with $\|\mathbf{x}_c\|_2 \leq R_c$, we have $\mathbb{B}_k(\mathbf{x}_c, r) \subseteq \mathcal{K}$. Let now $\mathbf{x} \in \mathbb{R}^k$ with $\|\mathbf{x}\|_2 \leq R$ and $F(\mathbf{x}) = 1$ ($\mathbf{x} \in \mathcal{K}$). Since \mathcal{K} is convex, if \mathcal{K}' is the convex hull of $\{\mathbf{x}\} \cup \mathbb{B}_k(\mathbf{x}_c, r)$, we have $\mathcal{K}' \subseteq \mathcal{K}$. We will show that $\mathcal{K}' \cap \mathbb{B}_k(\mathbf{x}, \varrho)$ contains some cone \mathcal{R}' with non-trivial mass (see [Figure 3](#)).

Let \mathbf{y} be any point on the surface of $\mathbb{B}_k(\mathbf{x}_c, r)$ such that the tangent hyperplane of $\mathbb{B}_k(\mathbf{x}_c, r)$ on \mathbf{y} passes from \mathbf{x} . Then, if we let θ to be the angle $\widehat{\mathbf{y}\mathbf{x}\mathbf{x}_c}$, we have $\sin \theta = \|\mathbf{y} - \mathbf{x}_c\| / \|\mathbf{x} - \mathbf{x}_c\|_2 = r / \|\mathbf{x} - \mathbf{x}_c\|_2$, because $\widehat{\mathbf{x}\mathbf{y}\mathbf{x}_c} = \pi/2$, by definition of \mathbf{y} . Note that the triangle defined by \mathbf{x} , \mathbf{y} and \mathbf{x}_c lies within \mathcal{K}' and hence within \mathcal{K} as well. Since this is true for any \mathbf{y} as defined above, we have that \mathcal{K} contains a rotational cone \mathcal{R} with vertex \mathbf{x} , angle θ and height $h \in [\|\mathbf{x} - \mathbf{x}_c\|_2 - r, \|\mathbf{x} - \mathbf{x}_c\|_2]$. Note that the volume of $\mathcal{K}' \cap \mathbb{B}_k(\mathbf{x}, \varrho)$ is decreasing in $\|\mathbf{x} - \mathbf{x}_c\|_2$, as long as $\varrho \leq r$. Therefore, we may assume that $\|\mathbf{x} - \mathbf{x}_c\|_2 = R + R_c$ (which implies that $h \geq 1 \geq \varrho \geq \varrho \cos \theta$). Let $\mathcal{R}' = \mathcal{R} \cap \mathbb{B}_k(\mathbf{x}, \varrho)$.

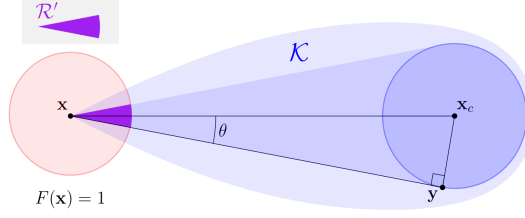


Figure 3: If $\mathbf{x} \in \mathcal{K}$, then there is a cone $\mathcal{R}' \subseteq \mathbb{B}_k(\mathbf{x}, \varrho) \cap \mathcal{K}$

By observing that \mathcal{R}' contains a cone of angle θ , height $\varrho \cos \theta$, where $\cos \theta \geq 1/2$ and $\varrho \leq R$, we overall have the following.

$$\begin{aligned}
\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k} [F(\mathbf{z}) = 1 \mid \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)] &= \frac{\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k} [F(\mathbf{z}) = 1 \text{ and } \mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)]}{\mathbb{P}_{\mathbf{z} \sim \mathcal{N}_k} [\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)]} \\
&\geq \frac{\text{vol}(\mathcal{R}')}{\text{vol}(\mathbb{B}_k(\mathbf{x}, \varrho))} \cdot \frac{\inf_{\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)} \mathcal{N}_k(\mathbf{z})}{\sup_{\mathbf{z} \in \mathbb{B}_k(\mathbf{x}, \varrho)} \mathcal{N}_k(\mathbf{z})} \\
&\geq \frac{\varrho \cos \theta (\varrho \sin \theta)^{k-1} (2\pi)^{(k-1)/2} k^{-((k-1)/2+1)}}{\varrho^k (2\pi/k)^{k/2}} \cdot \exp(-\varrho R/2) \\
&\geq \frac{(\sin \theta)^{k-1}}{2\sqrt{2\pi k}} \cdot e^{-\frac{1}{2}\varrho R} \geq \left(\frac{\beta}{Ck^2 R \ln(1/\beta)} \right)^k e^{-R/2}
\end{aligned}$$

Combining the two cases considered ($F(\mathbf{x}) = -1$ and $F(\mathbf{x}) = 1$), we obtain the desired result. \square

Finally, we show that there is a boundary proximity tester for the class of halfspace intersections.

Lemma E.13 (Boundary Proximity Tester for Halfspace Intersections). *Let \mathbb{D} be some class of distributions over \mathbb{R}^d such that for each distribution in \mathbb{D} , any one-dimensional marginal has density upper bounded by $C > 0$. Then, for any $\varrho \in (0, 1)$, there is a $(\varrho, 3Ck)$ -boundary proximity tester for the class of intersections of k halfspaces over \mathbb{R}^d with time and sample complexity $\text{poly}(d, k, 1/\varrho)$.*

Proof. The tester receives some intersection of halfspaces $f = 2 \prod_{i=1}^k \mathbb{1}\{\mathbf{w}^i \cdot \mathbf{x} - \tau_i\} - 1$ and $m_{\mathcal{T}}$ samples X from some unknown distribution over \mathbb{R}^d and does the following.

1. If for some $i \in [k]$ we have $\mathbb{P}_{\mathbf{x} \sim X} [|\mathbf{w}^i \cdot \mathbf{x} - \tau_i| \leq \varrho] > 3C\varrho$, then **reject**.
2. Otherwise, **accept**.

Soundness then follows from the fact that $\mathbb{P}_{\mathbf{x} \sim X} [\mathbf{x} \in \partial_{\varrho} f] \leq \sum_{i \in [k]} \mathbb{P}_{\mathbf{x} \sim X} [|\mathbf{w}^i \cdot \mathbf{x} - \tau_i| \leq \varrho]$ and a Hoeffding bound. Completeness follows from the fact that under any distribution \mathcal{D} in \mathbb{D} , we have $\mathbb{P}_{\mathbf{x} \sim \mathcal{D}} [|\mathbf{w}^i \cdot \mathbf{x} - \tau_i| \leq \varrho] \leq 2C\varrho$, due to the density upper bound in the direction \mathbf{w}^i and a Chernoff bound. \square

All of the ingredients of the proof of [Theorem E.11](#) are now in place.

Proof of [Theorem E.11](#). The theorem follows by combining either [Theorem E.6](#) or [Theorem E.7](#) with [Theorem E.11](#), [Lemma E.12](#) and [Lemma E.13](#). Note that since the parameter λ is unknown to the algorithm, we will run the corresponding discrepancy tester (either of [Theorem E.6](#) or of [Theorem E.7](#)) for all possible values of the parameter ϱ (of the discrepancy tester) within an $O(\epsilon/k^2)$ -net of the interval $[0, \frac{\beta}{Ck \log k}]$, where we know that the tester has to accept with high probability (we can amplify the success probability for each fixed value of ϱ through repetition). We accept if the (amplified) discrepancy tester accepts for all the values of ϱ in the net. In total, we will need $\text{poly}(k, 1/\epsilon)$ repetitions. \square

F NP-Hardness of Global Discrepancy Testing

In this section, we prove that there exist worst case pairs of distributions such that testing the globalized discrepancy between them with respect to the class of halfspaces is hard. These results also extend to the class of constant degree polynomial threshold functions. This motivates our study of localized notions of discrepancy. We now define the notion of discrepancy (globalized).

Definition F.1 (Discrepancy). Let D_1, D_2 be two distributions on \mathbb{R}^d and let \mathcal{F} be a set of boolean functions on \mathbb{R}^d . We say that the discrepancy between D_1 and D_2 with respect to \mathcal{F} , denoted by $\text{disc}_{\mathcal{F}}(D_1, D_2)$ is,

$$\text{disc}_{\mathcal{F}}(D_1, D_2) = \sup_{f_1, f_2 \in \mathcal{F}} \left(\left| \mathbb{P}_{\mathbf{x} \sim D_1} [f_1(\mathbf{x}) \neq f_2(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim D_2} [f_1(\mathbf{x}) \neq f_2(\mathbf{x})] \right| \right)$$

We prove our hardness result by reducing the following problem of learning constant degree PTFs with noise to the problem of identifying if the discrepancy between two distributions is large/small.

Definition F.2. For constants $\epsilon > 0, k \in \mathbb{N}$, let $\text{PTF-MA}(k, \epsilon)$ refers to the following promise problem: Given a set of tuples $\{\mathbf{x}_i, y_i\}_{i \in [n]}$ where $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{\pm 1\}$ for all $i \in [n]$, distinguish between the following two cases:

- There exists a halfspace h such that $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{h(\mathbf{x}_i) = y_i\} \geq 1 - \epsilon$,
- For every degree k PTF g , we have that $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{g(\mathbf{x}_i) = y_i\} \leq \frac{1}{2} + \epsilon$

This problem is known to be NP hard through a reduction from label cover.

Lemma F.3 ([BGS18]). *For any constant $k \in \mathbb{N}, \epsilon > 0$, $\text{PTF-MA}(k, \epsilon)$ is NP-hard.*

Given a set $S \subseteq \mathbb{R}^d$, let U_S denote the uniform distribution on that set. We define decision version of the problem of discrepancy testing for which we prove our NP-hardness result.

Definition F.4. For constants $\epsilon > 0$ and a class \mathcal{F} of boolean functions on \mathbb{R}^d , let $\text{DISC}(\mathcal{F}, \epsilon)$ be the following promise problem: Given sets $S, S' \subseteq \mathbb{R}^d$, distinguish between the two cases:

- $\text{disc}_{\mathcal{F}}(U_S, U_{S'}) \geq 1 - \epsilon$
- $\text{disc}_{\mathcal{F}}(U_S, U_{S'}) \leq \epsilon$

We are now ready to state and prove our result on the NP-hardness of $\text{DISC}(\mathcal{F}, \epsilon)$ when \mathcal{F} is the class of constant degree polynomial threshold functions.

Theorem F.5. *Let $k \in \mathbb{N}$ and $\epsilon > 0$. Let \mathcal{F} be the class of PTFs of degree k . The problem $\text{DISC}(\mathcal{F}, \epsilon)$ is NP-hard.*

Proof. We give a reduction from $\text{PTF-MA}(2k, \epsilon)$ to $\text{DISC}(\mathcal{F}, 8\epsilon)$. The input to $\text{PTF-MA}(2k, \epsilon)$ is a set of tuples $\{\mathbf{x}_i, y_i\}_{i \in [n]}$ where $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{\pm 1\}$ for all $i \in [n]$. Let $S^+ = \{\mathbf{x}_i \mid y_i = +1, i \in [n]\}$ and $S^- = \{\mathbf{x}_i \mid y_i = -1, i \in [n]\}$. We assume that $\left| \frac{|S^+|}{n} - \frac{1}{2} \right| \leq \epsilon$ and $\left| \frac{|S^-|}{n} - \frac{1}{2} \right| \leq \epsilon$. Otherwise, there exists a trivial halfspace (taking constant value) that achieves success probability greater than $\frac{1}{2} + \epsilon$ and this can easily be checked in polynomial time. We say that S^+, S^- are ϵ -unbiased if the above property holds. We now complete the proof by proving the following two claims and using [Lemma F.3](#).

Claim (Completeness). *Let S^+, S^- be ϵ -unbiased. If there exists a halfspace h such that $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{h(\mathbf{x}_i) = y_i\} \geq 1 - \epsilon$, then $\text{disc}_{\mathcal{F}}(U_{S^+}, U_{S^-}) \geq 1 - 8\epsilon$.*

Proof. We have that $\frac{|S^+|}{n} \mathbb{P}_{\mathbf{x} \sim U_{S^+}} [h(\mathbf{x}) = 1] + \frac{|S^-|}{n} \mathbb{P}_{\mathbf{x} \sim U_{S^+}} [h(\mathbf{x}) = 0] \geq 1 - \epsilon$. Thus, simplifying some terms, we obtain that

$$\begin{aligned} 1 - \epsilon &\leq \frac{|S^-|}{n} + \frac{|S^+|}{n} \cdot \mathbb{P}_{\mathbf{x} \sim U_{S^+}} [h(\mathbf{x}) = 1] - \frac{|S^-|}{n} \cdot \mathbb{P}_{\mathbf{x} \sim U_{S^-}} [h(\mathbf{x}) = 1] \\ &\leq \frac{1}{2} + \frac{1}{2} \cdot \left(\mathbb{P}_{\mathbf{x} \sim U_{S^+}} [h(\mathbf{x}) = 1] - \mathbb{P}_{\mathbf{x} \sim U_{S^-}} [h(\mathbf{x}) = 1] \right) + 3\epsilon \end{aligned}$$

where the last inequality follows from the fact that S^+, S^- are ϵ -unbiased. Thus, we obtain that $(\mathbb{P}_{\mathbf{x} \sim U_{S^+}}[h(\mathbf{x}) = 1] - \mathbb{P}_{\mathbf{x} \sim U_{S^-}}[h(\mathbf{x}) = 1]) \geq 1 - 8\epsilon$. Let g be the the halfspace that always outputs -1 . Clearly, we have that $\text{disc}_{\mathcal{F}}(U_{S^+}, U_{S^-}) \geq (\mathbb{P}_{\mathbf{x} \sim U_{S^+}}[h(\mathbf{x}) \neq g(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim U_{S^-}}[h(\mathbf{x}) \neq g(\mathbf{x})]) \geq 1 - 8\epsilon$. \square

Claim (Soundness). *Let S^+, S^- be ϵ -unbiased. If there exists no degree $2k$ PTF h such that $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{h(\mathbf{x}_i) = y_i\} \geq \frac{1}{2} + \epsilon$, then $\text{disc}_{\mathcal{F}}(U_{S^+}, U_{S^-}) \leq 8\epsilon$.*

Proof. Say $\text{disc}_{\mathcal{F}}(U_{S^+}, U_{S^-}) \geq 8\epsilon$. Since \mathcal{F} is closed under complements, we obtain without loss of generality that there exist two PTFs h_1, h_2 of degree d such that $\mathbb{P}_{\mathbf{x} \sim U_{S^-}}[h_1(\mathbf{x}) \neq h_2(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim U_{S^+}}[h_1(\mathbf{x}) \neq h_2(\mathbf{x})] \geq \frac{1}{2} + \epsilon$. Consider the function $g(\mathbf{x}) = h_1(\mathbf{x}) \cdot h_2(\mathbf{x})$. We have that g is a degree $2k$ PTF. Thus, we obtain that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{g(\mathbf{x}) = y\} &= \frac{|S^-|}{n} \cdot \mathbb{P}_{\mathbf{x} \sim U_{S^-}}[g(\mathbf{x}) = -1] + \frac{|S^+|}{n} \cdot \mathbb{P}_{\mathbf{x} \sim U_{S^+}}[g(\mathbf{x}) = 1] \\ &= \frac{|S^-|}{n} \cdot \mathbb{P}_{\mathbf{x} \sim U_{S^-}}[h_1(\mathbf{x}) \neq h_2(\mathbf{x})] + \frac{|S^+|}{n} \cdot (1 - \mathbb{P}_{\mathbf{x} \sim U_{S^+}}[h_1(\mathbf{x}) \neq h_2(\mathbf{x})]) \\ &\geq \frac{1}{2} + \frac{1}{2} \left(\mathbb{P}_{\mathbf{x} \sim U_{S^-}}[h_1(\mathbf{x}) \neq h_2(\mathbf{x})] - \mathbb{P}_{\mathbf{x} \sim U_{S^+}}[h_1(\mathbf{x}) \neq h_2(\mathbf{x})] \right) - 3\epsilon \\ &\geq \frac{1}{2} + \epsilon \end{aligned}$$

where the penultimate inequality follows from the fact that S^+, S^- are ϵ -unbiased and the last inequality follows from our lower bound on the discrepancy. Since there exists no PTF of degree $2k$ that succeeds with probability $\frac{1}{2} + \epsilon$, we have a contradiction. \square

This concludes the proof of [Theorem F.5](#). \square

NeurIPS Paper Checklist

1. Claims

Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

Answer: [Yes]

Justification: We provide detailed proofs and/or references for all the claims made in the abstract and introduction.

Guidelines:

- The answer NA means that the abstract and introduction do not include the claims made in the paper.
- The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
- The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
- It is fine to include aspirational goals as motivation as long as it is clear that these goals are not attained by the paper.

2. Limitations

Question: Does the paper discuss the limitations of the work performed by the authors?

Answer: [Yes]

Justification: We include a dedicated section in the end of the main paper.

Guidelines:

- The answer NA means that the paper has no limitation while the answer No means that the paper has limitations, but those are not discussed in the paper.
- The authors are encouraged to create a separate "Limitations" section in their paper.
- The paper should point out any strong assumptions and how robust the results are to violations of these assumptions (e.g., independence assumptions, noiseless settings, model well-specification, asymptotic approximations only holding locally). The authors should reflect on how these assumptions might be violated in practice and what the implications would be.
- The authors should reflect on the scope of the claims made, e.g., if the approach was only tested on a few datasets or with a few runs. In general, empirical results often depend on implicit assumptions, which should be articulated.
- The authors should reflect on the factors that influence the performance of the approach. For example, a facial recognition algorithm may perform poorly when image resolution is low or images are taken in low lighting. Or a speech-to-text system might not be used reliably to provide closed captions for online lectures because it fails to handle technical jargon.
- The authors should discuss the computational efficiency of the proposed algorithms and how they scale with dataset size.
- If applicable, the authors should discuss possible limitations of their approach to address problems of privacy and fairness.
- While the authors might fear that complete honesty about limitations might be used by reviewers as grounds for rejection, a worse outcome might be that reviewers discover limitations that aren't acknowledged in the paper. The authors should use their best judgment and recognize that individual actions in favor of transparency play an important role in developing norms that preserve the integrity of the community. Reviewers will be specifically instructed to not penalize honesty concerning limitations.

3. Theory Assumptions and Proofs

Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

Answer: [Yes]

Justification: We provide full proofs for all of our results in the appendix.

Guidelines:

- The answer NA means that the paper does not include theoretical results.
- All the theorems, formulas, and proofs in the paper should be numbered and cross-referenced.
- All assumptions should be clearly stated or referenced in the statement of any theorems.
- The proofs can either appear in the main paper or the supplemental material, but if they appear in the supplemental material, the authors are encouraged to provide a short proof sketch to provide intuition.
- Inversely, any informal proof provided in the core of the paper should be complemented by formal proofs provided in appendix or supplemental material.
- Theorems and Lemmas that the proof relies upon should be properly referenced.

4. Experimental Result Reproducibility

Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

Answer: [NA]

Justification: Our paper does not have any experiments.

Guidelines:

- The answer NA means that the paper does not include experiments.
- If the paper includes experiments, a No answer to this question will not be perceived well by the reviewers: Making the paper reproducible is important, regardless of whether the code and data are provided or not.
- If the contribution is a dataset and/or model, the authors should describe the steps taken to make their results reproducible or verifiable.
- Depending on the contribution, reproducibility can be accomplished in various ways. For example, if the contribution is a novel architecture, describing the architecture fully might suffice, or if the contribution is a specific model and empirical evaluation, it may be necessary to either make it possible for others to replicate the model with the same dataset, or provide access to the model. In general, releasing code and data is often one good way to accomplish this, but reproducibility can also be provided via detailed instructions for how to replicate the results, access to a hosted model (e.g., in the case of a large language model), releasing of a model checkpoint, or other means that are appropriate to the research performed.
- While NeurIPS does not require releasing code, the conference does require all submissions to provide some reasonable avenue for reproducibility, which may depend on the nature of the contribution. For example
 - (a) If the contribution is primarily a new algorithm, the paper should make it clear how to reproduce that algorithm.
 - (b) If the contribution is primarily a new model architecture, the paper should describe the architecture clearly and fully.
 - (c) If the contribution is a new model (e.g., a large language model), then there should either be a way to access this model for reproducing the results or a way to reproduce the model (e.g., with an open-source dataset or instructions for how to construct the dataset).
 - (d) We recognize that reproducibility may be tricky in some cases, in which case authors are welcome to describe the particular way they provide for reproducibility. In the case of closed-source models, it may be that access to the model is limited in some way (e.g., to registered users), but it should be possible for other researchers to have some path to reproducing or verifying the results.

5. Open access to data and code

Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

Answer: [NA]

Justification: Our paper does not have any experiments.

Guidelines:

- The answer NA means that paper does not include experiments requiring code.
- Please see the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- While we encourage the release of code and data, we understand that this might not be possible, so “No” is an acceptable answer. Papers cannot be rejected simply for not including code, unless this is central to the contribution (e.g., for a new open-source benchmark).
- The instructions should contain the exact command and environment needed to run to reproduce the results. See the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- The authors should provide instructions on data access and preparation, including how to access the raw data, preprocessed data, intermediate data, and generated data, etc.
- The authors should provide scripts to reproduce all experimental results for the new proposed method and baselines. If only a subset of experiments are reproducible, they should state which ones are omitted from the script and why.
- At submission time, to preserve anonymity, the authors should release anonymized versions (if applicable).
- Providing as much information as possible in supplemental material (appended to the paper) is recommended, but including URLs to data and code is permitted.

6. Experimental Setting/Details

Question: Does the paper specify all the training and test details (e.g., data splits, hyper-parameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

Answer: [NA]

Justification: Our paper does not have any experiments.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The experimental setting should be presented in the core of the paper to a level of detail that is necessary to appreciate the results and make sense of them.
- The full details can be provided either with the code, in appendix, or as supplemental material.

7. Experiment Statistical Significance

Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

Answer: [NA]

Justification: Our paper does not have any experiments.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The authors should answer "Yes" if the results are accompanied by error bars, confidence intervals, or statistical significance tests, at least for the experiments that support the main claims of the paper.
- The factors of variability that the error bars are capturing should be clearly stated (for example, train/test split, initialization, random drawing of some parameter, or overall run with given experimental conditions).
- The method for calculating the error bars should be explained (closed form formula, call to a library function, bootstrap, etc.)
- The assumptions made should be given (e.g., Normally distributed errors).
- It should be clear whether the error bar is the standard deviation or the standard error of the mean.

- It is OK to report 1-sigma error bars, but one should state it. The authors should preferably report a 2-sigma error bar than state that they have a 96% CI, if the hypothesis of Normality of errors is not verified.
- For asymmetric distributions, the authors should be careful not to show in tables or figures symmetric error bars that would yield results that are out of range (e.g. negative error rates).
- If error bars are reported in tables or plots, The authors should explain in the text how they were calculated and reference the corresponding figures or tables in the text.

8. Experiments Compute Resources

Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

Answer: [NA]

Justification: Our paper does not have any experiments.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The paper should indicate the type of compute workers CPU or GPU, internal cluster, or cloud provider, including relevant memory and storage.
- The paper should provide the amount of compute required for each of the individual experimental runs as well as estimate the total compute.
- The paper should disclose whether the full research project required more compute than the experiments reported in the paper (e.g., preliminary or failed experiments that didn't make it into the paper).

9. Code Of Ethics

Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics <https://neurips.cc/public/EthicsGuidelines?>

Answer: [Yes]

Justification: We read the code of ethics and strongly believe that our work conforms with stated code.

Guidelines:

- The answer NA means that the authors have not reviewed the NeurIPS Code of Ethics.
- If the authors answer No, they should explain the special circumstances that require a deviation from the Code of Ethics.
- The authors should make sure to preserve anonymity (e.g., if there is a special consideration due to laws or regulations in their jurisdiction).

10. Broader Impacts

Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

Answer: [Yes]

Justification: We include a discussion on the broader impacts of our work at the end of the main paper.

Guidelines:

- The answer NA means that there is no societal impact of the work performed.
- If the authors answer NA or No, they should explain why their work has no societal impact or why the paper does not address societal impact.
- Examples of negative societal impacts include potential malicious or unintended uses (e.g., disinformation, generating fake profiles, surveillance), fairness considerations (e.g., deployment of technologies that could make decisions that unfairly impact specific groups), privacy considerations, and security considerations.

- The conference expects that many papers will be foundational research and not tied to particular applications, let alone deployments. However, if there is a direct path to any negative applications, the authors should point it out. For example, it is legitimate to point out that an improvement in the quality of generative models could be used to generate deepfakes for disinformation. On the other hand, it is not needed to point out that a generic algorithm for optimizing neural networks could enable people to train models that generate Deepfakes faster.
- The authors should consider possible harms that could arise when the technology is being used as intended and functioning correctly, harms that could arise when the technology is being used as intended but gives incorrect results, and harms following from (intentional or unintentional) misuse of the technology.
- If there are negative societal impacts, the authors could also discuss possible mitigation strategies (e.g., gated release of models, providing defenses in addition to attacks, mechanisms for monitoring misuse, mechanisms to monitor how a system learns from feedback over time, improving the efficiency and accessibility of ML).

11. Safeguards

Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pretrained language models, image generators, or scraped datasets)?

Answer: [NA]

Justification: We do not release any data and models.

Guidelines:

- The answer NA means that the paper poses no such risks.
- Released models that have a high risk for misuse or dual-use should be released with necessary safeguards to allow for controlled use of the model, for example by requiring that users adhere to usage guidelines or restrictions to access the model or implementing safety filters.
- Datasets that have been scraped from the Internet could pose safety risks. The authors should describe how they avoided releasing unsafe images.
- We recognize that providing effective safeguards is challenging, and many papers do not require this, but we encourage authors to take this into account and make a best faith effort.

12. Licenses for existing assets

Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

Answer: [NA]

Justification: Our paper does not use any existing code, data or models.

Guidelines:

- The answer NA means that the paper does not use existing assets.
- The authors should cite the original paper that produced the code package or dataset.
- The authors should state which version of the asset is used and, if possible, include a URL.
- The name of the license (e.g., CC-BY 4.0) should be included for each asset.
- For scraped data from a particular source (e.g., website), the copyright and terms of service of that source should be provided.
- If assets are released, the license, copyright information, and terms of use in the package should be provided. For popular datasets, paperswithcode.com/datasets has curated licenses for some datasets. Their licensing guide can help determine the license of a dataset.
- For existing datasets that are re-packaged, both the original license and the license of the derived asset (if it has changed) should be provided.

- If this information is not available online, the authors are encouraged to reach out to the asset’s creators.

13. **New Assets**

Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

Answer: [NA]

Justification: Our paper does not release any new assets.

Guidelines:

- The answer NA means that the paper does not release new assets.
- Researchers should communicate the details of the dataset/code/model as part of their submissions via structured templates. This includes details about training, license, limitations, etc.
- The paper should discuss whether and how consent was obtained from people whose asset is used.
- At submission time, remember to anonymize your assets (if applicable). You can either create an anonymized URL or include an anonymized zip file.

14. **Crowdsourcing and Research with Human Subjects**

Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

Answer: [NA]

Justification: Our paper does not involve crowdsourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Including this information in the supplemental material is fine, but if the main contribution of the paper involves human subjects, then as much detail as possible should be included in the main paper.
- According to the NeurIPS Code of Ethics, workers involved in data collection, curation, or other labor should be paid at least the minimum wage in the country of the data collector.

15. **Institutional Review Board (IRB) Approvals or Equivalent for Research with Human Subjects**

Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

Answer: [NA]

Justification: Our paper does not involve crowd-sourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Depending on the country in which research is conducted, IRB approval (or equivalent) may be required for any human subjects research. If you obtained IRB approval, you should clearly state this in the paper.
- We recognize that the procedures for this may vary significantly between institutions and locations, and we expect authors to adhere to the NeurIPS Code of Ethics and the guidelines for their institution.
- For initial submissions, do not include any information that would break anonymity (if applicable), such as the institution conducting the review.