1 Proof of Proposition 2.1

Lemma 1.1. $\forall i \in [0, d]$,

$$\binom{d}{i} \sum_{j=0}^{k} \frac{\binom{k}{j}}{\binom{d+k}{i+j}} = \frac{k+d+1}{d+1}.$$

Proof. Let

$$\frac{\binom{d}{i}\binom{k}{j}}{\binom{d+k}{i+j}} = \frac{d!k!(i+j)!(d+k-i-j)!}{i!(d-i)!j!(k-j)!(d+k)!} = \frac{\binom{d+k-i-j}{d-i}\binom{i+j}{i}}{\binom{d+k}{k}}$$

Following the Vandermonde's identity

$$\sum_{j=0}^{d-i} \binom{d+k-i-j}{d-i}\binom{i+j}{i} = \frac{k+d+1}{d+1}\binom{d+k}{k}.$$

Then we can conclude that

$$\sum_{j=0}^{k} \frac{\binom{k}{j}\binom{d}{i}}{\binom{d+k}{i+j}} = \sum_{j=0}^{k} \frac{\binom{d+k-i-j}{d-i}\binom{i+j}{i}}{\binom{d}{k}} = \frac{k+d+1}{d+1}$$

$\square$

Proof. The additivity comes directly from inner product is additive.

Scale invariant is also standard to prove, but more tedeious.

Without loss of generality, denote $p = a_0 y^0 + \cdots + a_i y^i + \cdots + a_d y^d$, and $q = p \odot (1+y)^k = b_0 y^0 + \cdots + b_j y^i + \cdots + b_{d+k} y^{d+k}$. Using definition of ψ, we have:

$$\psi(p) = \frac{1}{d+1} \sum_{i=0}^{d} a_i \frac{1}{\binom{d}{i}}$$

On the other hand, the coefficient of $(1+y)^k$ is also binomial coefficient, $\psi(q)$ can be simplified into

$$\psi(q) = \frac{1}{d+k+1} \sum_{i=0}^{d} a_i \sum_{j=0}^{k} \frac{\binom{k}{j}}{\binom{d+k}{i+j}}.$$

Observe the ratio between weights of a_i from both $\psi(p), \psi(q)$. Using Lemma 1.1, we can obtain

$$\frac{d+k+1}{d+1} \sum_{j=0}^{k} \frac{\binom{d}{i}\binom{k}{j}}{\binom{d+k}{i+j}} = 1.$$

Therefore, $\psi(p) = \psi(q)$.

$\square$