A Missing Proofs in Section 4

Lemma A.1. *For a given graph $G(\mathbb{V}, \mathbb{E}, w)$, let G' be the induced subgraph of G with vertex set $\mathbb{V} \setminus S$. For any pair $\{(u, v) | u \in \mathbb{V} \setminus S, v \in S\}$: 1) if the shortest path between u and v does not pass other nodes in S, then there exist a neighbor p of v such that $d_G(u, v) = d_{G'}(u, p) + w(p, v)$; 2) if the shortest path $P_{u,v}$ between u and v does pass other nodes in S except v, then there exist vertices $p \in S, p' \notin S$ such that $d_G(u, v) = d_G(u, p) + d_G(p, v)$ and $d_G(u, p) = d_{G'}(u, p') + w(p', p)$.*

Proof. 1) Note that the shortest path between u and v does not pass other nodes in S except v. Let the second last vertex from u to v be p, with $p \notin S$. Then we have $d_G(u, p) = d_{G'}(u, p)$, thus $d_G(u, v) = d_G(u, p) + w(p, v) = d_{G'}(u, p) + w(p, v)$. 2) Let p be the first vertex in S that $P_{u,v}$ passes through, and denote p' as the vertex just before p in path $P_{u,v}$. We have that $d_G(u, p') = d_{G'}(u, p')$. Then we know that $d_G(u, v) = d_G(u, p) + d_G(p, v)$ and $d_G(u, p) = d_{G'}(u, p') + w(p', p)$. □

A.1 Proof of Lemma 4.1

Proof. The privacy budget ϵ is divided into three parts:

(Part 1) For the distances between all pairs in $\mathbb{V} \setminus S$, our method achieves $(\epsilon', 0)$-DP by the result on trees from (Sealfon, 2016).

(Part 2) For the k^2 distances of all node pairs in S, by adding to each edge weights i.i.d. $Lap(0, \sigma_1)$ noises with $\sigma_1 = \sqrt{8n \log(1/\delta)}/\epsilon' = 2\sqrt{2}n^{1/2}\sqrt{\log(1/\delta)}/\epsilon'$, we can achieve (ϵ', δ)-DP according to Lemma 2.2.

(Part 3) For each edge (u, v) with $u \in \mathbb{V} \setminus S, v \in S$, obviously by Laplace mechanism $w'(u, v) := w(u, v) + Lap(1/\epsilon')$, we achieve $(\epsilon', 0)$-DP to release all the pairwise distances.

Composing three privacy budges up and using simple composition theorem of DP, we show that Algorithm 2 achieves (ϵ, δ)-DP. □

A.2 Proof of Theorem 4.2

Lemma A.2 (All pairwise distance on trees (Sealfon, 2016))**.** *For a tree T with non-negative edge weights w and $\epsilon > 0$, there is an ϵ-differentially private algorithm that releases APSP distances such that with probability $1 - \gamma$, all released distances have approximation error bounded $O(\log^{2.5} n \log(1/\gamma)/\epsilon)$.*

Proof. (of Theorem 4.2) For $u \in S, v \in S$, by adding $d(u, v)$ with $Lap(0, \sigma_1)$ noise where $\sigma_1 = 2\sqrt{2}n^{1/2}\sqrt{\log(1/\delta)}/\epsilon'$, with probability $1 - \gamma$, $|\hat{d}(u, v) - d(u, v)| = O(k\sqrt{\log(1/\delta)} \log(k/\gamma)$ $\forall u, v \in S$. Based on Lemma A.2, we have that with probability $1 - \gamma$, all released distances in G' have approximation error $O(\log^{2.5} n \log(1/\gamma)/\epsilon)$. For those edges (u, v) with $u \in \mathbb{V} \setminus S, v \in S$, we add to each $w(u, v)$ a Laplace noise according to $Lap(0, \sigma_0)$ with $\sigma_0 = 1/\epsilon'$. Thus, with another probability $1 - \gamma$, $|w'(u, v) - w(u, v)| \leq O(\log(n/\gamma)/\epsilon)$ $\forall \{u \in \mathbb{V} \setminus S, v \in S\}$. Union bound implies that, with probability $1 - 3\gamma$, the total error is bounded by $O(\epsilon^{-1} k \log(k/\gamma)\sqrt{\log(1/\delta)}) + O(\epsilon^{-1}(\log^{2.5} n) \log(1/\gamma))$ as claimed. □