A Proofs

A.1 Preliminaries: Online optimization with time-dependent regularization

We give a brief background on Follow the Regularized Leader and Online Mirror Descent algorithmic templates, in the case where the regularization is varying and time-dependent.

The setup is the standard setup of online linear optimization. Let $\mathcal{W} \subseteq \mathbb{R}^d$ be a convex domain. On each prediction round $t = 1, \dots, T$, the learner has to produce a prediction $w_t \in \mathbb{R}^d$ based on $g_1, \dots, g_{t-1}$, and subsequently observes a new loss vector g_t and incurs the loss $w_t \cdot g_t$. The goal is to minimize the regret compared to any $w^\star \in \mathcal{W}$, given by $\sum_{t=1}^T g_t \cdot (w_t - w^\star)$.

Follow the Regularized Leader (FTRL). The FTRL template generates predictions $w_1, \dots, w_T \in \mathcal{W}$, for $t = 1, \dots, T$, as follows:

$$w_t = \operatorname*{argmin}_{w \in \mathcal{W}} \left\{ w \cdot \sum_{s=1}^{t-1} g_s + R_t(w) \right\}. \tag{6}$$

Here, $R_1, \dots, R_T : \mathcal{W} \to \mathbb{R}$ is a sequence of twice-differentiable, strictly convex functions.

The derivation and analysis of FTRL-type algorithms is standard; see, e.g., [29, 13, 25]. In our analysis, however, we require a particular regret bound that we could not find stated explicitly in the literature (similar bounds exist, however, and date back at least to [7]). For completeness, we provide the bound here with a proof in the full version of the paper [**?**].

Theorem 10. *Suppose that $R_t = \eta_t^{-1} R$ for all t for some strictly convex R, with $\eta_1 \geq \ldots \geq \eta_T > 0$. Then there exists a sequence of points $z_t \in [w_t, w_{t+1}]$ such that the following regret bound holds for all $w^\star \in \mathcal{W}$:*

$$\sum_{t=1}^T g_t \cdot (w_t - w^\star) \leq \frac{1}{\eta_1}\big(R(w^\star) - R(w_1)\big) + \sum_{t=1}^T \Big(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\Big)\big(R(w^\star) - R(w_{t+1})\big) + \frac{1}{2}\sum_{t=1}^T \eta_t \big(\|g_t\|_t^*\big)^2,$$

where $\|g\|_t^2 = g^\mathsf{T} \nabla^2 R(z_t) g$ is the local norm induced by R at an appropriate $z_t \in [w_t, w_{t+1}]$, and $\|\cdot\|_t^*$ is its dual norm.

Online Mirror Descent (OMD). The closely-related OMD framework produces predictions $w_1, \dots, w_T$ via the following procedure: initialize $w_1 = \operatorname{argmin}_{w \in \mathcal{W}} R_1(w)$, and for $t = 1, \dots, T$, compute

$$\begin{aligned} w'_{t+1} &= \operatorname*{argmin}_{w} \big\{ g_t \cdot w + D_{R_t}(w, w_t) \big\} = (\nabla R_t)^{-1}\big(\nabla R_t(w_t) - g_t\big); \\ w_{t+1} &= \operatorname*{argmin}_{w \in \mathcal{W}} D_{R_t}(w, w'_{t+1}). \end{aligned} \tag{7}$$

Here, $R_1, \dots, R_T : \mathcal{W} \to \mathbb{R}$ is a sequence of twice-differentiable, strictly convex functions and $D_R(w', w) = R(w') - R(w) - \nabla R(w) \cdot (w' - w)$ is the Bregman divergence of a convex function R at point $w \in \mathcal{W}$.

The proof of the following regret bound (which is again a somewhat specialized variant of standard bounds for OMD) appears in the full version of the paper [**?**].

Theorem 11. *Suppose that $R_t = \eta_t^{-1} R$ for all t for some strictly convex R, with $\eta_1 \geq \ldots \geq \eta_T > 0$. Then there exists a sequence of points $z_t \in [w_t, w'_{t+1}]$ such that the following regret bound holds for all $w^\star \in \mathcal{W}$:*

$$\sum_{t=1}^T g_t \cdot (w_t - w^\star) \leq \frac{1}{\eta_1}\big(R(w^\star) - R(w_1)\big) + \sum_{t=1}^{T-1} \Big(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\Big) D_R(w^\star, w_{t+1}) + \frac{1}{2}\sum_{t=1}^T \eta_t \big(\|g_t\|_t^*\big)^2,$$

where $\|\cdot\|_t$ is the local norm induced by R at an appropriate $z_t \in [w_t, w'_{t+1}]$, and $\|\cdot\|_t^*$ is its dual.

A.2 Upper bounds for FTRL

Proof of Lemma 6. We observe that Eq. (4) is an instantiation of FTRL with $R_t(p) = \eta_t^{-1} R(p)$ as regularizations, where $R(p) = -H(p) = \sum_{i=1}^N p_i \log p_i$ is the negative entropy. Hence, we can invoke

Theorem 10 to bound the regret compared to any probability distribution $p^\star$. It suffices to bound the regret for $p^\star$ that minimizes $\sum_{t=1}^T p \cdot \ell_t$, which is always a point-mass on a single expert $i^\star$, for which $R(p^\star) = 0$. Therefore, Theorem 10 in our case reads

$$\sum_{t=1}^T \sum_{i=1}^N p_{t,i}\big(g_{t,i} - g_{t,i^\star}\big) \le -\frac{1}{\eta_1} R(p_1) - \sum_{t=1}^T \Big(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\Big) R(p_{t+1}) + \frac{1}{2} \sum_{t=1}^T \eta_t \big(\|g_t\|_t^*\big)^2.$$

Now set $\eta_t = \sqrt{\log(N)/t}$. For the first two terms in the bound, observe that $R(p_1) = -\log N$, and further, that

$$\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} = \frac{1}{\sqrt{\log N}} \frac{1}{\sqrt{t} + \sqrt{t+1}} \le \frac{1}{2\sqrt{t \log N}} = \frac{\eta_t}{2 \log N}. \tag{8}$$

For the final sum, we have to evaluate the Hessian $\nabla^2 R(p'_t)$ at a point $p'_t \in [p_t, p_{t+1}]$. A straightforward differentiation shows that this matrix is diagonal, with diagonal elements $\nabla^2 R(p'_t)_{ii} = 1/p'_{t,i}$. Thus,

$$\big(\|g_t\|_t^*\big)^2 = g_t^\top \big(\nabla^2 R(p'_t)\big)^{-1} g_t = p'_t \cdot g_t^2. \tag{9}$$

The final sum can be divided and bounded as follows

$$\begin{aligned}\sum_{t=1}^T \eta_t \big(p'_t \cdot g_t^2\big) &= \sum_{t=1}^{4\log N} \eta_t \big(p'_t \cdot g_t^2\big) + \sum_{t=1+4\log N}^T \eta_t \big(p'_t \cdot g_t^2\big) \\ &\le 4 \log N + \sum_{t=1+\log N}^T \eta_t \big(p'_t \cdot g_t^2\big).\end{aligned}$$

Where we used the fact that $\sum_{s=1}^t \eta_s = \sum_{s=1}^t \sqrt{\log(N)/s} \le 2\sqrt{t \log N}$. To conclude the proof it suffices to show that $p'_{t,i} \le 9 p_{t,i}$ for $t \ge 4 \log N$. To see this, denote $G_t = \sum_{s=1}^{t-1} g_s$ and write

$$\frac{e^{-\eta_{t+1} G_{t+1,i}}}{e^{-\eta_t G_{t,i}}} = e^{-\eta_{t+1} g_{t,i}}\, e^{(\eta_t - \eta_{t+1}) G_{t,i}}.$$

For $t \ge 4 \log N$, the following relations hold:

$$\begin{aligned}&0 < \eta_{t+1} |g_{t,i}| \le \eta_{t+1} \le \frac{1}{2}; \\ &0 < (\eta_t - \eta_{t+1}) |G_{t,i}| \le \sqrt{\log N} \frac{\sqrt{t+1} - \sqrt{t}}{\sqrt{t(t+1)}} t \le \frac{\sqrt{\log N}}{\sqrt{t} + \sqrt{t+1}} \le \eta_t \le \frac{1}{2}.\end{aligned}$$

Hence, for $t \ge 4 \log N$ we have

$$\frac{1}{3} \le \frac{e^{-\eta_{t+1} G_{t+1,i}}}{e^{-\eta_t G_{t,i}}} \le 3,$$

and consequently

$$p_{t+1,i} = \frac{e^{-\eta_{t+1} G_{t+1,i}}}{\sum_{j=1}^N e^{-\eta_{t+1} G_{t+1,j}}} \le 9 \frac{e^{-\eta_t G_{t,i}}}{\sum_{j=1}^N e^{-\eta_t G_{t,j}}} = 9 p_{t,i}.$$

Since $p'_t \in [p_t, p_{t+1}]$, the same inequality holds for p'_t; that is, $p'_{t,i} \le 9 p_{t,i}$ for all i, and the proof is complete. ∎

Lemma 12. *For the adaptive MW algorithm in Eq. (4) with loss vectors $g_t = \tilde{\ell}_{t,i}$, we have*

$$\sum_{t=1}^T \eta_t \sum_{i=1}^N p_{t,i} \big(\tilde{\ell}_{t,i} - \tilde{\ell}_{t,i^\star}\big)^2 \le \frac{16 \log N}{\Delta} + \frac{1}{8} \overline{\mathcal{R}}_T.$$

Proof. By setting $t_0 = 64\Delta^{-2}\log N$ and $\eta_t = \sqrt{\log(N)/t}$ we obtain

$$\begin{aligned}\sum_{t=1}^{T}\eta_t\sum_{i=1}^{N}p_{t,i}\big(\tilde{\ell}_{t,i}-\tilde{\ell}_{t,i^\star}\big)^2 &\le \sum_{t=1}^{t_0}\eta_t + \sum_{t=t_0+1}^{T}\eta_{t_0}\sum_{i=1}^{N}p_{t,i}\big(\tilde{\ell}_{t,i}-\tilde{\ell}_{t,i^\star}\big)^2\\ &\le 2\sqrt{\log(N)}\sqrt{t_0}+\frac{\Delta}{8}\sum_{t=t_0+1}^{T}\sum_{i=1}^{N}p_{t,i}\big(\tilde{\ell}_{t,i}-\tilde{\ell}_{t,i^\star}\big)^2\\ &\le \frac{16\log N}{\Delta}+\frac{1}{8}\sum_{t=t_0+1}^{T}\sum_{i=1}^{N}p_{t,i}(\mu_i-\mu_{i^\star}),\end{aligned}$$

where in the final inequality we used observation 3. To conclude we note that $p_{t,i}(\mu_i-\mu_{i^\star})\ge 0$, thus we can modify the last summation to range over $t=1,\dots,T$. ■

Lemma 13. *For the adaptive MW algorithm in Eq. (4), we have*

$$\frac{1}{\log N}\sum_{t=1}^{T}\eta_t H(p_{t+1}) \le \frac{50\log N}{\Delta}+\frac{5}{8}\overline{\mathcal{R}}_T .$$

Proof. First we split the sum as follows,

$$\frac{1}{\log N}\sum_{t=1}^{T}\eta_t H(p_{t+1}) = \frac{1}{\log N}\sum_{t=1}^{t_0}\eta_t H(p_{t+1}) + \frac{1}{\log N}\sum_{t=t_0+1}^{T}\eta_t H(p_{t+1}),$$

where $t_0 = 64\Delta^{-2}\log N$. For the summation of $t=\{t_0+1,\dots,T\}$ we use Lemma 7 with $\tau = t\log N \ge t_0\log N = 64\Delta^{-2}\log^2 N$ to obtain

$$\begin{aligned}\frac{1}{\log N}\sum_{t=t_0+1}^{T}\eta_t H(p_{t+1}) &= \sum_{t=t_0+1}^{T}\frac{1}{\sqrt{t\log N}}\sum_{i=1}^{N}p_{t+1,i}\log\frac{1}{p_{t+1,i}}\\ &\le \frac{5}{8}\sum_{t=t_0+1}^{T}\sum_{i\ne i^\star}p_{t+1,i}\Delta + 2\sum_{t=t_0+1}^{T}\frac{1}{\sqrt{t\log N}}e^{-\frac{1}{8}\Delta\sqrt{t\log N}}\\ &\le \frac{5}{8}\sum_{t=t_0+1}^{T}\sum_{i=1}^{N}p_{t,i}(\mu_i-\mu_{i^\star})+\Delta+2\sum_{t=t_0+1}^{T}\frac{1}{\sqrt{t\log N}}e^{-\frac{1}{8}\Delta\sqrt{t\log N}},\end{aligned}$$

where the last inequality follows for reordering terms in the summation and that $\Delta \le \mu_i - \mu_{i^\star}$ for $i \ne i^\star$. Using the fact that $p_{t,i}(\mu_i-\mu_{i^\star})\ge 0$ we get

$$\frac{1}{\log N}\sum_{t=t_0+1}^{T}\eta_t H(p_{t+1}) \le \frac{5}{8}\sum_{t=1}^{T}\sum_{i=1}^{N}p_{t,i}(\mu_i-\mu_{i^\star})+\Delta+2\sum_{t=t_0+1}^{T}\frac{1}{\sqrt{t\log N}}e^{-\frac{1}{8}\Delta\sqrt{t\log N}}. \tag{10}$$

Moreover, we have

$$\begin{aligned}\sum_{t=t_0+1}^{T}\frac{1}{\sqrt{t\log N}}e^{-\frac{1}{8}\Delta\sqrt{\log N}\sqrt{t}} &\le \frac{1}{\sqrt{\log N}}\int_{t_0}^{T}\frac{1}{\sqrt{t}}e^{-\frac{1}{8}\Delta\sqrt{\log N}\sqrt{t}}dt\\ &= \frac{1}{\sqrt{\log N}}\cdot\frac{16}{\Delta\sqrt{\log N}}e^{-\frac{1}{8}\Delta\sqrt{\log N}\sqrt{t}}\Big|_{T}^{t_0}\\ &\le \frac{16}{\Delta\log N}\\ &\le \frac{16}{\Delta}.\end{aligned} \tag{11}$$

Lastly, for the summation of $t=\{1,\dots,t_0\}$ we get

$$\frac{1}{\log N}\sum_{t=1}^{t_0}\eta_t H(p_{t+1}) \le 2\sqrt{t_0\log N} = \frac{16\log N}{\Delta} \tag{12}$$

which follows from $H(p)\le\log N$ and $\sum_{t=1}^{t_0}1/\sqrt{t}\le 2\sqrt{t_0}$. Combining Eqs. (10) to (12), the proof is concluded. ■

A.3 Lower bound for OMD

Proof of Theorem 9. Let q_t denote the probability that MW-OMD chooses the best expert (i.e., expert #1) on round t. For $t \le C$, the best expert suffers higher losses than the other expert, thus $\mathbf{E}[q_t] \le 1/2$. For $t > C$, it holds that

$$q_t = \frac{e^{-\sum_{s=1}^{t-1} \eta_s (\tilde{\ell}_{s,1} - \tilde{\ell}_{s,2})}}{1 + e^{-\sum_{s=1}^{t-1} \eta_s (\tilde{\ell}_{s,1} - \tilde{\ell}_{s,2})}} \le e^{-\sum_{s=1}^{t-1} \eta_s (\tilde{\ell}_{s,1} - \tilde{\ell}_{s,2})} = e^{-\sum_{s=1}^{C} \eta_s} \exp\left(\sum_{s=C+1}^{t-1} \eta_s (\ell_{s,2} - \ell_{s,1}) \right).$$

Now, observe that

$$\sum_{s=1}^{C} \eta_s \ge C\eta_C = \alpha\sqrt{C}.$$

Also, by a standard application of Hoeffding's lemma (e.g., Appendix A of [3]),

$$\begin{aligned}
\mathbf{E} \exp\left(\sum_{s=C+1}^{t-1} \eta_s (\ell_{s,2} - \ell_{s,1}) \right) &= \prod_{s=C+1}^{t-1} \mathbf{E} e^{\eta_s (\ell_{s,2} - \ell_{s,1})} \\
&\le \prod_{s=C+1}^{t-1} e^{\eta_s \Delta + \eta_s^2/8} \\
&\le \exp\left(\Delta \sum_{s=1}^{t-1} \eta_s \right) \exp\left(\frac{1}{8} \sum_{s=1}^{t-1} \eta_s^2 \right) \\
&\le \exp\left(2\alpha\Delta\sqrt{t} + \alpha^2 \log t \right).
\end{aligned}$$

Overall, we have shown that for $t > C$,

$$\mathbf{E}[q_t] \le \exp\left(-\alpha\sqrt{C} + 2\alpha\Delta\sqrt{t} + \alpha^2 \log t \right).$$

Whenever $t \le t_1 := \min\left\{ 2^{-6} C/\Delta^2, \exp\left(\tfrac{1}{4}\sqrt{C}/\alpha \right) \right\}$, the right hand side is $\le \exp(-\tfrac{1}{2}\alpha\sqrt{C}) \le \tfrac{1}{2}$ for $\alpha \ge 1/\sqrt{C}$. Hence, in that case,

$$\mathcal{R}_T \ge \sum_{s=1}^{t_1} \Delta \mathbf{E}[1 - q_s] \ge \sum_{s=1}^{t_1} \tfrac{1}{2}\Delta \ge \tfrac{1}{2}\Delta t_1. \qquad \blacksquare$$

B Analysis of OMD in the Purely Stochastic Case

Proof of Theorem 8. Applying Theorem 11 for the experts setting we get

$$\mathcal{R}_T \le \frac{1}{\eta_1}\left(H(p_1) - H(p^\star) \right) + \sum_{t=1}^{T-1} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \sum_{i=1}^{N} p_i^\star \log \frac{p_i^\star}{p_{t+1,i}} + \frac{1}{2} \sum_{t=1}^{T} \eta_t \left(\|\ell_t\|_t^* \right)^2,$$

where we used the fact that the Bregman divergence of the negative entropy is the KL divergence. In addition, using similar observations as in the proof of Lemma 6 (e.g., Eqs. (8) and (9)) and setting $\eta_t = c/\sqrt{t}$ we obtain

$$\mathcal{R}_T \le \frac{\log N}{c} + \frac{1}{2c^2} \sum_{t=1}^{T-1} \eta_t \log \frac{1}{p_{t+1,i^\star}} + \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N} \eta_t p_{t,i} \ell_{t,i}^2.$$

Applying additive translation we get,

$$\mathcal{R}_T \le \frac{\log N}{c} + \frac{1}{2c^2} \sum_{t=1}^{T-1} \eta_t \log \frac{1}{p_{t+1,i^\star}} + \frac{1}{2} \sum_{t=1}^{T} \eta_t \sum_{i=1}^{N} p_{t,i} (\ell_{t,i} - \ell_{t,i^\star})^2. \tag{13}$$

Similarly to Lemma 12 we can bound the third term by

$$\frac{1}{2} \sum_{t=1}^{T} \eta_t \sum_{i=1}^{N} p_{t,i} (\ell_{t,i} - \ell_{t,i^\star})^2 \le \frac{c^2}{\Delta} + \frac{1}{2}\overline{\mathcal{R}}_T = \frac{\log N}{\Delta} + \frac{1}{2}\overline{\mathcal{R}}_T. \tag{14}$$

We now examine the second term. Using the MW algorithm defined in Eq. (5) we have,

$$\log \frac{1}{p_{t+1,i^\star}} = \log \frac{\sum_{i=1}^N e^{-\sum_{s=1}^{t-1} \eta_s \ell_{t,i}}}{e^{-\sum_{s=1}^{t-1} \eta_s \ell_{s,i^\star}}} = \log\Big(1 + \sum_{i \neq i^\star} e^{-\sum_{s=1}^{t-1} \eta_s (\ell_{s,i} - \ell_{s,i^\star})}\Big).$$

Plugging it back to the original term we get

$$\frac{1}{2c^2} \sum_{t=1}^{T-1} \eta_t \log \frac{1}{p_{t+1,i^\star}} = \frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + \sum_{i \neq i^\star} e^{-\sum_{s=1}^{t-1} \eta_s (\ell_{s,i} - \ell_{s,i^\star})}\Big).$$

By taking the expectation and using its linearity property we obtain

$$\begin{aligned}
\frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \mathbf{E}\Big[\log\Big(1 + \sum_{i \neq i^\star} e^{-\sum_{s=1}^{t-1} \eta_s (\ell_{t,i} - \ell_{t,i^\star})}\Big)\Big] &\le \frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + \sum_{i \neq i^\star} \mathbf{E}\big[e^{-\sum_{s=1}^{t-1} \eta_s (\ell_{s,i} - \ell_{s,i^\star})}\big]\Big) \\
&\le \frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + \sum_{i \neq i^\star} \prod_{s=1}^{t-1} \mathbf{E}\big[e^{-\eta_s (\ell_{s,i} - \ell_{s,i^\star})}\big]\Big),
\end{aligned}$$

where we used Jensen inequality for concave functions for the first inequality and the fact that $x_t := \ell_{t,i} - \ell_{t,i^\star}$ are i.i.d. for the second inequality. Applying Hoeffding's Lemma yields,

$$\frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + \sum_{i \neq i^\star} \prod_{s=1}^{t-1} \mathbf{E}\big[e^{-\eta_s (\ell_{s,i} - \ell_{s,i^\star})}\big]\Big) \le \frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + N \exp\Big(\sum_{s=1}^{t-1} \big(\tfrac{1}{2}\eta_s^2 - \eta_s \Delta\big)\Big)\Big).$$

Next, we bound the argument of the exponent

$$\begin{aligned}
\sum_{s=1}^{t-1} \big(\tfrac{1}{2}\eta_s^2 - \eta_s \Delta\big) &\le \frac{c^2}{2} \sum_{s=1}^{t-1} \frac{1}{s} - c\Delta \sum_{s=1}^{t-1} \frac{1}{\sqrt{s}} \\
&\le \frac{c^2}{2}(1 + \log t) - c\Delta\sqrt{t} \\
&\le c^2 \log t - c\Delta\sqrt{t},
\end{aligned}$$

where we bounded the summations by their integrals. Therefore we have

$$\frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + N e^{\sum_{s=1}^{t-1} (\frac{\eta_s^2}{2} - \eta_s \Delta)}\Big) \le \frac{1}{2c} \sum_{t=1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + N e^{c^2 \log t - c\Delta\sqrt{t}}\Big).$$

First we examine the sum from t_1 onward, while we require that for $t \ge t_1$ it holds

$$c^2 \log t \le \tfrac{1}{2} c\Delta\sqrt{t}. \tag{15}$$

To satisfy Eq. (15) it suffices to take

$$t_1 = \Big(\frac{8c}{\Delta}\Big)^2 \log^2 \frac{8c}{\Delta}.$$

Therefore,

$$\begin{aligned}
\frac{1}{2c} \sum_{t=t_1+1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + N e^{c^2 \log t - 2c\Delta\sqrt{t}}\Big) &\le \frac{1}{2c} \sum_{t=t_1+1}^{T-1} \frac{1}{\sqrt{t}} \log\Big(1 + N e^{-\frac{1}{2}c\Delta\sqrt{t}}\Big) \\
&\le \frac{N}{2c} \sum_{t=t_1+1}^{T-1} \frac{1}{\sqrt{t}} e^{-\frac{1}{2}c\Delta\sqrt{t}} && (\log(1+x) \le x) \\
&\le \frac{N}{2c} \int_{t_1}^{T-1} \frac{1}{\sqrt{t}} e^{-\frac{1}{2}c\Delta\sqrt{t}} dt \\
&\le \frac{N}{c^2\Delta} e^{-\frac{1}{2}c\Delta\sqrt{t_1}} dt \\
&\le \frac{2N}{c^2\Delta} e^{-c^2 \log t_1} && (c^2 \log t_1 \le \tfrac{1}{2}c\Delta\sqrt{t_1}) \\
&\le \frac{2}{\Delta \log N}. && (t_1 > 1 \text{ and } c = \sqrt{\log N})
\end{aligned}$$

To conclude we examine the bound up to t_1,

$$\begin{aligned}
\frac{1}{2c}\sum_{t=1}^{t_1}\frac{1}{\sqrt{t}}\log\Big(1+Ne^{c^2\log t-2c\Delta\sqrt{t}}\Big) &\le \frac{1}{2c}\sum_{t=1}^{t_1}\frac{1}{\sqrt{t}}\log\Big(2Ne^{c^2\log t-2c\Delta\sqrt{t}}\Big)\\
&\le \frac{1}{2c}\sum_{t=1}^{t_1}\frac{1}{\sqrt{t}}\Big(\log 2N+c^2\log t\Big)\\
&\le \frac{\log 2N+c^2\log t_1}{2c}\sum_{t=1}^{t_1}\frac{1}{\sqrt{t}} \qquad\qquad (\log t\le\log t_1)\\
&\le \frac{\log 2N+c^2\log t_1}{c}\sqrt{t_1}.
\end{aligned}$$

Since $c=\sqrt{\log N}$, for $t_1\ge\lceil e^2\rceil$ we have $c^2\log t_1=\log N\log t_1\ge\log 2N$, and also Eq. (15) still holds. This implies

$$\begin{aligned}
\frac{\log 2N+c^2\log t_1}{c}\sqrt{t_1} &\le 2c\log t_1\sqrt{t_1}\\
&\le 2\Delta t_1\\
&\le 128\frac{\log N}{\Delta}\log^2\Big(\frac{8\sqrt{\log N}}{\Delta}\Big).
\end{aligned}$$

when we used the fact that $c\log t_1\le\Delta\sqrt{t_1}$ for the last inequality. Adding both results(up to t_1 and from t_1 onward) we obtain,

$$\frac{1}{2c^2}\sum_{t=1}^{T-1}\eta_t\log\frac{1}{p_{t+1,i^\star}}\le 128\frac{\log N}{\Delta}\log^2\Big(\frac{8\sqrt{\log N}}{\Delta}\Big)+\frac{2}{\Delta\log N} \tag{16}$$

Finally, plugging Eqs. (14) and (16) into Eq. (13), taking the expectation and rearranging terms we get

$$\mathbf{E}\big[\overline{\mathcal{R}}_T\big]\le\frac{256\log N}{\Delta}\log^2\Big(\frac{8\log N}{\Delta}\Big)+\frac{8\log N}{\Delta}. \qquad\blacksquare$$

C Analysis of Time-varying Regularization Algorithms

In this section, we assume the setup of online (linear) optimization, with the notation established in Section 4.1. For the proofs below, we recall the notion of a Bregman divergence. For a continuously differentiable and strictly convex function $F:\mathcal{W}\to\mathbb{R}$ defined on a closed convex set $\mathcal{W}$, the Bregman divergence associated with F at a point $w\in\mathcal{W}$ is defined by

$$\forall w'\in\mathcal{W},\qquad D_F(w',w)=F(w')-F(w)-\nabla F(w)\cdot(w'-w).$$

C.1 Follow the Regularized Leader

First, we present a general analysis for Follow the Regularized Leader, described in Eq. (6), and later establish Theorem 10.

Theorem 14. *There exists a sequence of points $z_t\in[w_t,w_{t+1}]$ such that, for all $w^\star\in\mathcal{W}$,*

$$\sum_{t=1}^{T}g_t\cdot(w_t-w^\star)\le R_{T+1}(w^\star)-R_1(w_1)+\sum_{t=1}^{T}\big(R_t(w_{t+1})-R_{t+1}(w_{t+1})\big)+\frac{1}{2}\sum_{t=1}^{T}\big(\|g_t\|_t^*\big)^2.$$

Here $\|w\|_t=\sqrt{w^\mathsf{T}\nabla^2R_t(z_t)w}$ is the local norm induced by R_t at z_t, and $\|\cdot\|_t^*$ is its dual.

Proof. Denote $\Phi_t(w)=w\cdot\sum_{s=1}^{t-1}g_s+R_t(w)$, so that $w_t=\operatorname{argmin}_{w\in\mathcal{W}}\Phi_t(w)$. We first write

$$\begin{aligned}
\sum_{t=1}^{T}g_t\cdot w_{t+1} &= \sum_{t=1}^{T}\big(\Phi_{t+1}(w_{t+1})-\Phi_t(w_{t+1})\big)+\sum_{t=1}^{T}\big(R_t(w_{t+1})-R_{t+1}(w_{t+1})\big)\\
&=\Phi_{T+1}(w_{T+1})-\Phi_1(w_1)+\sum_{t=1}^{T}\big(\Phi_t(w_t)-\Phi_t(w_{t+1})\big)+\sum_{t=1}^{T}\big(R_t(w_{t+1})-R_{t+1}(w_{t+1})\big).
\end{aligned}$$

Since w_t is the minimizer of Φ_t over $\mathcal{W}$, first-order optimality conditions imply

$$\Phi_t(w_t) - \Phi_t(w_{t+1}) = -\nabla\Phi_t(w_t) \cdot (w_{t+1} - w_t) - D_{\Phi_t}(w_{t+1}, w_t) \le -D_{\Phi_t}(w_{t+1}, w_t) = -D_{R_t}(w_{t+1}, w_t),$$

where we have used the fact that the Bregman divergence is invariant to linear terms. On the other hand, since w_{T+1} is the minimizer of Φ_{T+1}, we have that

$$\sum_{t=1}^{T} g_t \cdot w^\star = \Phi_{T+1}(w^\star) - R_{T+1}(w^\star) \ge \Phi_{T+1}(w_{T+1}) - R_{T+1}(w^\star).$$

Combining inequalities and observing that $\Phi_1(w_1) = R_1(w_1)$, we obtain

$$\sum_{t=1}^{T} g_t \cdot (w_{t+1} - w^\star) \le R_{T+1}(w^\star) - R_1(w_1) + \sum_{t=1}^{T} \big(R_t(w_{t+1}) - R_{t+1}(w_{t+1})\big) - \sum_{t=1}^{T} D_{R_t}(w_{t+1}, w_t).$$

On the other hand, a Taylor expansion of $R_t(\cdot)$ around w_t with an explicit second-order remainder term implies that, for some intermediate point $z_t \in [w_t, w_{t+1}]$, it holds that

$$D_{R_t}(w_{t+1}, w_t) = \tfrac{1}{2}(w_{t+1} - w_t)^\mathsf{T} \, \nabla^2 R_t(z_t)\, (w_{t+1} - w_t) = \tfrac{1}{2}\|w_{t+1} - w_t\|_t^2.$$

An application of Holder's inequality then gives

$$g_t \cdot (w_t - w_{t+1}) \le \|g_t\|_t^* \, \|w_t - w_{t+1}\|_t \le \tfrac{1}{2}\big(\|g_t\|_t^*\big)^2 + \tfrac{1}{2}\|w_t - w_{t+1}\|_t^2 = \tfrac{1}{2}\big(\|g_t\|_t^*\big)^2 + D_{R_t}(w_{t+1}, w_t).$$

The proof is finalized by summing over $t = 1, \ldots, T$ and adding to the inequality above. ■

Proof of Theorem 10. Fix any $w^\star \in \mathcal{W}$. Observe that FTRL with regularizations $R_t(w) = \eta_t^{-1} R(w)$ is equivalent to FTRL with $R_t(w) = \eta_t^{-1}(R(w) - R(w^\star))$. Applying Theorem 14 for the latter and rearranging, we obtain the claimed bound. ■

C.2 Online Mirror Descent

We next consider Online Mirror Descent (see Eq. (7)), and prove the following general bound from which Theorem 11 directly follows.

Lemma 15. *There exist points $z_t \in [w_t, w'_{t+1}]$ such that for all $w^\star \in \mathcal{W}$,*

$$\sum_{t=1}^{T} g_t \cdot (w_t - w^\star) \le R_1(w^\star) - R_1(w_1) + \sum_{t=1}^{T-1} \big(D_{R_{t+1}}(w^\star, w_{t+1}) - D_{R_t}(w^\star, w_{t+1})\big) + \frac{1}{2}\sum_{t=1}^{T} \big(\|g_t\|_t^*\big)^2.$$

Here $\|w\|_t = \sqrt{w^\mathsf{T} \nabla^2 R_t(z_t) w}$ is the local norm induced by R_t at z_t, and $\|\cdot\|_t^*$ is its dual.

Proof. Fix any $w^\star \in \mathcal{W}$. We will bound each of the terms $g_t \cdot (w_t - w^\star)$. First, from the update rule of Mirror Descent and the three-point property of the Bregman divergence, we have

$$\begin{aligned} g_t \cdot (w'_{t+1} - w^\star) &= (\nabla R(w_t) - \nabla R(w'_{t+1})) \cdot (w'_{t+1} - w^\star) \\ &= D_{R_t}(w^\star, w_t) - D_{R_t}(w^\star, w'_{t+1}) - D_{R_t}(w'_{t+1}, w_t). \end{aligned}$$

Now, a Taylor expansion of R_t at x_t (with an explicit Lagrange remainder term) shows that there exists $z_t \in [w_t, w_{t+1}]$ for which

$$D_{R_t}(w'_{t+1}, w_t) = \tfrac{1}{2}(w'_{t+1} - w_t)^\mathsf{T} \, \nabla^2 R_t(z_t)\, (w'_{t+1} - w_t) = \tfrac{1}{2}\|w'_{t+1} - w_t\|_t^2.$$

Also, since w_{t+1} is the projection (with respect to the Bregman divergence R_t) of the point w'_{t+1} onto the set $\mathcal{W}$ that contains $w^\star$, it holds that $D_{R_t}(w^\star, w_{t+1}) \le D_{R_t}(x^\star, w'_{t+1})$. Putting things together, we obtain

$$g_t \cdot (w'_{t+1} - w^\star) \le D_{R_t}(w^\star, w_t) - D_{R_t}(w^\star, w_{t+1}) - \tfrac{1}{2}\|w'_{t+1} - w_t\|_{z_t}^2. \tag{17}$$

On the other hand, Hölder's inequality and the fact that $ab \le \frac{1}{2}(a^2 + b^2)$ yield

$$g_t \cdot (w_t - w'_{t+1}) \le \|g_t\|_t^* \cdot \|w_t - w'_{t+1}\|_t \le \tfrac{1}{2}(\|g_t\|_t^*)^2 + \tfrac{1}{2}\|w_t - w'_{t+1}\|_t^2. \tag{18}$$

Summing Eqs. (17) and (18) together over $t = 1, \ldots, T$ gives the regret bound

$$\sum_{t=1}^{T} g_t \cdot (w_t - w^\star) \le \sum_{t=1}^{T} \big(D_{R_t}(w^\star, w_t) - D_{R_t}(w^\star, w_{t+1})\big) + \frac{1}{2}\sum_{t=1}^{T} \big(\|g_t\|_t^*\big)^2.$$

Rearranging the first summation and using the facts that $D_{R_T}(w^\star, w_{T+1}) \ge 0$ and $D_{R_1}(w^\star, w_1) \le R_1(w^\star) - R_1(w_1)$ (the latter follows since w_1 is the minimizer of R_1, and so $\nabla R_1(w_1) \cdot (w^\star - w_1) \ge 0$) gives the stated regret bound. ■