A Game-Theoretic Analysis of the Empirical Revenue Maximization Algorithm with Endogenous Sampling

Anonymous Author(s)
Affiliation
Address
email

Abstract

The Empirical Revenue Maximization (ERM) is one of the most important price learning algorithms in auction design: as the literature shows it can learn approximately optimal reserve prices for revenue-maximizing auctioneers in both repeated auctions and uniform-price auctions. However, in these applications the agents who provide inputs to ERM have incentives to manipulate the inputs to lower the outputted price. We generalize the definition of an incentive-awareness measure proposed by Lavi et al (2019), to quantify the reduction of ERM's outputted price due to a change of $m \geq 1$ out of N input samples, and provide specific convergence rates of this measure to zero as N goes to infinity for different types of input distributions. By adopting this measure, we construct an efficient, approximately incentive-compatible, and revenue-optimal learning algorithm using ERM in repeated auctions against non-myopic bidders, and show approximate group incentive-compatibility in uniform-price auctions.

1 Introduction

In auction theory, it is well-known [30] that, when all buyers have values that are independently and identically drawn from a regular distribution F, the revenue-maximizing auction is simply the second price auction with anonymous reserve price $p^* = \arg\max\{p(1-F(p))\}$: if the highest bid is at least p^*, then the highest bidder wins the item and pays the maximum between the second highest bid and p^*. The computation of p^* requires the exact knowledge of the underlying value distribution, which is unrealistic because the value distribution is often unavailable in practice. Many works (e.g., [11, 15, 23]) on sample complexity in auctions have studied how to obtain a near-optimal reserve price based on samples from the distribution F instead of knowing the exact F. One of the most important (and most fundamental) price learning algorithms in those works is the *Empirical Revenue Maximization* (ERM) algorithm, which simply outputs the reserve price that is optimal on the uniform distribution over samples (plus some regularization to prevent overfitting).

Definition 1.1 (c-Guarded Empirical Revenue Maximization, ERM^c)**.** *Draw N samples from a distribution F and sort them non-increasingly, denoted by $v_1 \geq v_2 \geq \cdots \geq v_N$. Given some regularization parameter $0 \leq c < 1$, choose:*

$$i^* = \underset{cN < i \leq N}{\arg\max}\{i \cdot v_i\}, \qquad \textit{define } \mathrm{ERM}^c(v_1, \ldots, v_N) = v_{i^*}.$$

Assume that the smaller sample (with the larger index) is chosen in case of ties.

ERM^c was first proposed by Dhangwatnotai et al. [15] and then extensively studied by Huang et al. [23]. They show that the reserve price outputted by ERM^c is asymptotically optimal on the underlying distribution F as the number of samples N increases if F is bounded or has monotone hazard rate, with an appropriate choice of c. Other papers [5, 25] have continued to study ERM^c.

Submitted to 34th Conference on Neural Information Processing Systems (NeurIPS 2020). Do not distribute.

However, when ERM is put into practice, it is unclear how the samples can be obtained since many times there is no impartial sampling source. A natural solution is *endogenous sampling*. For example, in repeated second price auctions, the auctioneer can use the bids in previous auctions as samples and run ERM to set a reserve price at each round. But this solution has a challenge of strategic issue: since bidders can affect the determination of future reserve prices, they might have an incentive to underbid in order to increase utility in future auctions.

Another example of endogenous sampling is the uniform-price auction. In a uniform-price auction the auctioneer sells N copies of a good at some price p to N bidders with i.i.d. values $\boldsymbol{v}$ from F who submit bids $\boldsymbol{b}$. Bidders who bid at least p obtain one copy and pay p. The auctioneer can set the price to be $p = \mathrm{ERM}^0(\boldsymbol{v})$ to maximize revenue if bids are equal to values. But Goldberg et al. [17] show that this auction is not incentive-compatible as bidders can lower the price by strategic bidding. Therefore, the main question we consider in this paper is: *To what extent the presence of strategic agents undermines ERM with endogenous sampling?*

To formally answer the question, we adopt a notion called "incentive-awareness measure" originally proposed by Lavi et al. [26] under bitcoin's fee market context, which measures the reduction of a price learning function P due to a change of at most m samples out of the N input samples.

Definition 1.2 (Incentive-awareness measures)**.** *Let $P : \mathbb{R}_+^N \to \mathbb{R}_+$ be a function (e.g., ERM^c) that maps N samples to a reserve price. Draw N i.i.d. values $v_1, \ldots, v_N$ from a distribution F. Let $I \subseteq \{1, \ldots, N\}$ be an index set of size $|I| = m$, and $v_I = \{v_i \mid i \in I\}$, $v_{-I} = \{v_j \mid j \notin I\}$. A bidder can change v_I to any m non-negative bids b_I, hence change the price from $P(v_I, v_{-I})$ to $P(b_I, v_{-I})$. Define the* incentive-awareness measure*:*

$$\delta_I(v_I, v_{-I}) = 1 - \frac{\inf_{b_I \in \mathbb{R}_+^m} P(b_I, v_{-I})}{P(v_I, v_{-I})},$$

and worst-case incentive-awareness measures*:*

$$\delta_m^{\mathrm{worst}}(v_{-I}) = \sup_{v_I \in \mathbb{R}_+^m} [\delta_I(v_I, v_{-I})], \qquad \Delta_{N,m}^{\mathrm{worst}} = \mathbb{E}_{v_{-I} \sim F}[\delta_m^{\mathrm{worst}}(v_{-I})].$$

A smaller incentive-awareness measure means that the reserve price is decreased by a less amount when a bidder bids strategically. Since the reduction of reserve price usually increases bidders' utility, a smaller incentive-awareness measure implies that a bidder cannot benefit a lot from strategic bidding, hence the name "incentive-awareness measure".[1]

Lavi et al. [26] defined incentive-awareness measures only for $m = 1$ and showed that for any distribution F with a finite support size, $\Delta_{N,1}^{\mathrm{worst}} \to 0$ as $N \to \infty$. Later, Yao [31] showed that $\Delta_{N,1}^{\mathrm{worst}} \to 0$ for any continuous distribution with support included in $[1, D]$. They did not provide specific convergent rates of $\Delta_{N,1}^{\mathrm{worst}}$. We generalize their definition to allow $m \geq 1$, which is crucial in our two applications to be discussed. Our main theoretical contribution is to provide upper bounds on $\Delta_{N,m}^{\mathrm{worst}}$ for two types of value distributions F: the class of Monotone Hazard Rate (MHR) distributions where $\frac{f(v)}{1-F(v)}$ is non-decreasing over the support of the distribution (we use F to denote the CDF and f for PDF) and the class of bounded distributions which consists of all (continuous and discontinuous) distributions with support included in $[1, D]$. MHR distribution can be unbounded so we are the first to consider incentive-awareness measures for unbounded distributions.

Theorem 1.3 (Main)**.** *Let $P = \mathrm{ERM}^c$. The worst-case incentive-awareness measure is bounded by*

- *for MHR F, $\Delta_{N,m}^{\mathrm{worst}} = O\left(m \frac{\log^3 N}{\sqrt{N}}\right)$, if $m = o(\sqrt{N})$ and $\frac{m}{N} \leq c \leq \frac{1}{4e}$.*[2]
- *for bounded F, $\Delta_{N,m}^{\mathrm{worst}} = O\left(D^{8/3} m^{2/3} \frac{\log^2 N}{N^{1/3}}\right)$, if $m = o(\sqrt{N})$ and $\frac{m}{N} \leq c \leq \frac{1}{2D}$.*

The constants in the two big O's are independent of F and c.

This theorem implies that as long as the fraction of samples controlled by a bidder is relatively small, the strategic behavior of each bidder has little impact on ERM provided that other bidders are truthful. We will discuss intuitions and difficulties of the proof later and give an overview in Section 4.

[1] Lavi et al. [26] use the name "discount ratio" which we feel can be confused with the standard meaning of a discount ratio in repeated games.

[2] We use $a(n) = o(b(n))$ to denote $\lim_{n \to +\infty} \frac{a(n)}{b(n)} = 0$.

Repeated auctions against non-myopic bidders. Besides theoretical analysis, we apply the incentive-awareness measure to real-world scenarios to demonstrate the effect of strategic bidding on ERM. The main application we consider is repeated auctions where bidders participate in multiple auctions and have incentives to bid strategically to affect the auctions the seller will use in the future (Section 2). We consider a two-phase learning algorithm: the seller first runs second price auctions with no reserve for some time to collect samples, and then use these samples to set reserve prices by ERM in the second phase. The upper bound on the incentive-awareness measure of ERM implies that this algorithm is approximately incentive-compatible.

Kanoria and Nazerzadeh [25], Liu et al. [27], and Abernethy et al. [1] consider repeated auctions scenarios similar to ours. Kanoria and Nazerzadeh [25] set *personalized* reserve prices by ERM in repeated second-price auctions, so at least two bidders are needed in each auction and they will face different reserve prices. We use *anonymous* reserve price so we allow only one bidder to participate in the auctions and when there are more than one bidder they face the same price, thus preventing discrimination. Liu et al. [27] and Abernethy et al. [1] design approximately incentive-compatible algorithms using differential privacy techniques rather than pure ERM. Comparing with them, our two-phase ERM algorithm is more practical as it is much simpler, and their algorithms rely on the boundedness of value distributions while we allow unbounded distributions. Moreover, their results require a large number of auctions while ours need a large number of samples in the first phase which can be obtained by either few bids in many auctions, many bids in few auctions, or combined.

Uniform-price auctions and incentive-compatibility in the large. Another scenario to which we apply the incentive-awareness measure of ERM is uniform-price auctions (Section 3). Azevedo and Budish [4] show that, uniform-price auctions are *incentive-compatible in the large* in the sense that truthful bidding is an approximate equilibrium when there are many bidders in the auction. In fact, incentive-compatibility in the large is the intuition of Theorem 1.3: when N is large, no bidders can influence the learned price by much. The proof in [4] directly makes use of this intuition, showing that the bid of one bidder can affect the empirical distribution consisting of the N bids only by a little. However, their argument, which crucially relies on the assumption that bidders' value distribution has a *finite* support and bids must be chosen *from this finite support* as well, fails when the value distribution is *continuous* or *bids can be any real numbers*, as what we allow. We instead, appeal to some specific properties of ERM to show that it is incentive-compatible in the large.

Additional related works. Previous works on ERM mainly focus on its sample complexity, started by Cole and Roughgarden [11]. While ERM is suitable for the case of i.i.d. values (e.g., [23]), the literature on sample complexity has expanded to more general cases of non-i.i.d. values and multi-dimensional values, e.g. [29, 14, 19, 20], or considering non-truthful auctions, e.g. [21]. Babaioff et al. [5] study the performance of ERM with just two samples. While this literature assumes that samples are exogenous, our main contribution is to consider endogenous samples that are collected from bidders who are affected by the outcome of the learning algorithm.

Some works study repeated auctions but with myopic bidders [7, 28, 10, 9]. Existing works about non-myopic bidders focus on designing various learning algorithms to maximize revenue assuming bidders playing best responds [2, 3, 12, 18] or using no-regret learning algorithm [8]. We complement that line of works by showing that ERM, the most fundamental algorithm we believe, also has good performance in repeated auctions against strategic bidders.

Other works about incentive-aware learning (e.g., [13, 24, 6, 16]) consider settings different from ours. For example, [13] and [24] study repeated auctions where buyers' values are drawn from some distribution at first and then fixed throughout. The seller knows the distribution and tries to learn the exact values, which is different from our assumption that the distribution is unknown to the seller.

2 Main Application: A Two-Phase Model

Here we consider a *two-phase model* as a real-world scenario where strategic bidding affects ERM: the seller first runs second price auctions with no reserve for some time to collect samples, and then use these samples to set reserve prices by ERM in the second phase. This model can be regarded as an "exploration and exploitation" learning algorithm in repeated auctions, and we will show that this algorithm can be approximately incentive-compatible and revenue-optimal.

2.1 The Model

A two-phase model is denoted by $\mathrm{TP}(\mathcal{M}, P; F, \boldsymbol{T}, \boldsymbol{m}, \boldsymbol{K}, S)$, where $\mathcal{M}$ is a truthful, prior-independent mechanism, P is a price learning function, $\boldsymbol{T} = (T_1, T_2)$ are the numbers of auctions in the two phases, $\boldsymbol{m} = (m_1, m_2)$ are upper bounds on the number of auctions each bidder participates in, $\boldsymbol{K} = (K_1, K_2)$ are the number of bidders in auctions, $S = S_1 \times \cdots \times S_n$ is the strategy space, where $s_i \in S_i : \mathbb{R}_+^{m_{i,1}+m_{i,2}} \to \mathbb{R}_+^{m_{i,1}}$ is a strategy of bidder $i = 1, \ldots, n$. The procedure is:

- At the beginning, each bidder realizes $\boldsymbol{v}_i = (v_{i,1}, \ldots, v_{i,m_{i,1}+m_{i,2}})$ i.i.d. drawn from F. Let $\boldsymbol{v}_{-i}$ denote the values of bidders other than i. Bidder i knows $\boldsymbol{v}_i$ but does not know $\boldsymbol{v}_{-i}$.
- In the exploration phase, T_1 auctions are run using $\mathcal{M}$ and bidders bid according to some strategy $s \in S$. Each auction has K_1 bidders and each bidder i participants in $m_{i,1} \le m_1$ auctions. The auctioneer observes a random vector of bids $\boldsymbol{b} = (b_1, \ldots, b_{T_1 K_1})$ with the following distribution: let I be an index set corresponding to bidder i, with size $|I| = m_{i,1}$; then $\boldsymbol{b} = (b_I, b_{-I})$, where $b_I \sim s_i(\boldsymbol{v}_i)$, and $b_{-I} \sim s_{-i}(\boldsymbol{v}_{-i})$.
- In the exploitation phase, T_2 second price auctions ($K_2 \ge 2$) or posted price auctions ($K_2 = 1$) are run, with reserve price $p = P(\boldsymbol{b})$. Each auction has K_2 bidders and each bidder i participants in $m_{i,2} \le m_2$ auctions. The auctions in this phase are truthful because p has been fixed.

Utilities. Denote the utility of bidder i as:

$$U_i^{\mathrm{TP}}(\boldsymbol{v}_i, b_I, b_{-I}) = U_i^{\mathcal{M}}(\boldsymbol{v}_i, b_I, b_{-I}) + \sum_{t=m_{i,1}+1}^{m_{i,1}+m_{i,2}} u^{K_2}(v_{i,t}, P(b_I, b_{-I})), \tag{1}$$

where $U_i^{\mathcal{M}}(v_I, b_I, b_{-I})$ is the utility of bidder i in the first phase, and $u^{K_2}(v, p)$ is the interim utility of a bidder with value v in a second price auction with reserve price p among $K_2 \ge 1$ bidders:

$$u^{K_2}(v, p) = \mathbb{E}_{X_2, \ldots, X_{K_2} \sim F}\left[\big(v - \max\{p, X_2, \ldots, X_{K_2}\}\big) \cdot \mathbb{I}\big[v > \max\{p, X_2, \ldots, X_{K_2}\}\big]\right]. \tag{2}$$

The *interim utility* of bidder i in the two-phase model is $\mathbb{E}_{\boldsymbol{v}_{-i} \sim F}\left[U_i^{\mathrm{TP}}(\boldsymbol{v}_i, b_I, b_{-I})\right]$.

Approximate Bayesian incentive-compatibility. We use the additive version of the solution concept of an ϵ-Bayesian-Nash equilibrium (ϵ-BNE), i.e., in such a solution concept, no player can improve her utility by more than ϵ by deviating from the equilibrium strategy. We say a mechanism is ϵ-approximately Bayesian incentive-compatible (ϵ-BIC) if truthful bidding is an ϵ-BNE, i.e., if for any $\boldsymbol{v}_i \in \mathbb{R}_+^{m_{i,1}+m_{i,2}}$, any $b_I \in \mathbb{R}_+^{m_{i,1}}$,

$$\mathbb{E}_{\boldsymbol{v}_{-i} \sim F}\left[U_i^{\mathrm{TP}}(\boldsymbol{v}_i, b_I, v_{-I}) - U_i^{\mathrm{TP}}(\boldsymbol{v}_i, v_I, v_{-I})\right] \le \epsilon,$$

If a mechanism is ϵ-BIC and $\lim_{n\to\infty} \epsilon = 0$, then each bidder knows that if all other bidders are bidding truthfully then the gain from any deviation from truthful bidding is negligible for her. To realize that strategic bidding cannot benefit them much, bidders do not need to know the underlying distribution, but only the fact that the mechanism is ϵ-BIC. We are therefore going to assume in this paper that, in such a case, all bidders will bid truthfully.

Approximate revenue optimality. We say that a mechanism is $(1-\epsilon)$ revenue optimal, for some $0 < \epsilon < 1$, if its expected revenue is at least $(1-\epsilon)$ times the expected revenue of Myerson auction. Huang et al. [23] show that a one-bidder auction with posted price set by ERM^c (for an appropriate c) and with N samples from the value distribution is $(1-\epsilon)$ revenue optimal with $\epsilon = O((N^{-1}\log N)^{2/3})$ for MHR distributions and $\epsilon = O(\sqrt{DN^{-1}\log N})$ for bounded distributions.

The i.i.d. assumption. Our assumption of i.i.d. values is reasonable because in our scenario there is a large population of bidders, and we can regard this population as a distribution and each bidder as a sample from it. So from each bidder's perspective, the values of other bidders are i.i.d. from this distribution. Then the ϵ-BIC notion implies that when others bid truthfully, it is approximately optimal for bidder i to bid truthfully no matter what her value is.

2.2 Incentive-Compatibility and Revenue Optimality

Now we show that, as the incentive-awareness measure of P becomes lower, the price learning function becomes more incentive-aware in the sense that bidders gain less from non-truthful bidding:

Theorem 2.1. *In* $\mathrm{TP}(\mathcal{M}, P; F, \boldsymbol{T}, \boldsymbol{m}, \boldsymbol{K}, S)$*, truthful bidding is an* ϵ*-BNE, where,*

- *for any* P *and any bounded* F, $\epsilon = m_2 D \Delta^{\text{worst}}_{T_1K_1,m_1}$*, and*
- *for any MHR* F*, if we fix* $P = \mathrm{ERM}^c$ *with* $\frac{m_1}{T_1K_1} \leq c \leq \frac{1}{4e}$ *and* $m_1 = o(\sqrt{T_1K_1})$*, then* $\epsilon = O\left(m_2 v^* \Delta^{\text{worst}}_{T_1K_1,m_1}\right) + O\left(\frac{m_2 v^*}{\sqrt{T_1K_1}}\right)$*, where* $v^* = \arg\max_v \{v[1 - F(v)]\}$.

The constants in big O's are independent of F *and* c*.*

Combined with Theorem 1.3 which upper bounds the incentive-awareness measure, we can obtain explicit bounds on truthfulness of the two-phase model by plugging in $N = T_1K_1$ and $m = m_1$. Precisely, for any bounded F, $\epsilon = O\left(D^{11/3} m_2 m_1^{2/3} \frac{\log^2(T_1K_1)}{(T_1K_1)^{1/3}}\right)$ if $m_1 = o(\sqrt{T_1K_1})$ and $\frac{m_1}{T_1K_1} \leq c \leq \frac{1}{2D}$. For any MHR F, $\epsilon = O\left(v^* m_2 m_1 \frac{\log^3(T_1K_1)}{\sqrt{T_1K_1}}\right)$ if $m_1 = o(\sqrt{T_1K_1})$ and $\frac{m_1}{T_1K_1} \leq c \leq \frac{1}{4e}$. Thus, for both cases, keeping all the parameters except T_1 constant (in particular m_1 and m_2 are constants) implies that $\epsilon \to 0$ at a rate which is not slower than $O((T_1)^{-1/3} \log^3 T_1)$ as $T_1 \to +\infty$.

To simultaneously obtain both approximate BIC and approximate revenue optimality, a certain balance between the number of auctions in the two phases must be maintained. Few auctions in the first phase and many auctions in the second phase hurt truthfulness as the loss from non-truthful bidding (i.e., losing in the first phase) is small compared to the gain from manipulating the reserve price in the second phase. Many auctions in the first phase are problematic as we do not have any good revenue guarantees in the first phase (since we allow any truthful $\mathcal{M}$). Thus, a certain balance must be maintained, as expressed formally in the following theorem:

Theorem 2.2. *Assume that* $K_2 \geq K_1 \geq 1$ *and let* $\overline{m} = m_1 + m_2$*. In* $\mathrm{TP}(\mathcal{M}, \mathrm{ERM}^c; F, \boldsymbol{T}, \boldsymbol{m}, \boldsymbol{K}, S)$*, to simultaneously obtain* ϵ_1*-BIC and* $(1 - \epsilon_2)$ *revenue optimality (assuming truthful bidding), it suffices to set the parameters as follows:*

- *If* F *is an MHR distribution,* $\frac{\overline{m}}{T_1K_1} \leq c \leq \frac{1}{4e}$, $\overline{m} = o(\sqrt{T_1K_1})$*, then* $\epsilon_1 = O\left(v^* \overline{m}^2 \frac{\log^3(T_1K_1)}{\sqrt{T_1K_1}}\right)$*, and* $\epsilon_2 = O\left(\frac{T_1}{T} + \left[\frac{\log(T_1K_1)}{T_1K_1}\right]^{\frac{2}{3}}\right)$.
- *If* F *is bounded and regular,* $\frac{\overline{m}}{T_1K_1} \leq c \leq \frac{1}{2D}$, $\overline{m} = o(\sqrt{T_1K})$*, then* $\epsilon_1 = O\left(D^{11/3} \overline{m}^{5/3} \frac{\log^2(T_1K_1)}{(T_1K_1)^{1/3}}\right)$*, and* $\epsilon_2 = O\left(\frac{T_1}{T} + \sqrt{\frac{D \cdot \log(T_1K_1)}{T_1K_1}}\right)$.[3]

The proof is given in Appendix C.2. This theorem makes explicit the fact that in order to simultaneously obtain approximate BIC and approximate revenue optimality, T_1 cannot be too small nor too large: for approximate revenue optimality we need $T_1 \ll T$ and for approximate BIC we need, e.g., $T_1 \gg (v^*)^2 \overline{m}^4 \log^6(v^* \overline{m})/K_1$ for MHR distributions, and $T_1 \gg D^{11} \overline{m}^5 \log^6(D\overline{m})/K_1$ for bounded distributions. When setting the parameters in this way, both ϵ_1 and ϵ_2 go to 0 as $T \to \infty$.

2.3 Multi-Unit Extension

The auction in the exploitation phase can be generalized to a multi-unit Vickrey auction with anonymous reserve, where $k \geq 1$ identical units of an item are sold to K_2 unit-demand bidders and among those bidders whose bids are greater than the reserve price p, at most k bidders with largest bids win the units and pay the maximum between p and the $(k+1)$-th largest bid. The multi-unit Vickrey auction with an anonymous reserve price is revenue-optimal when the value distribution is regular, and the optimal reserve price does not depend on k or K_2 according to Myerson [30]. Thus the optimal reserve price can also be found by ERM^c. All our results concerning truthfulness, e.g.,

[3] The requirement that F is regular in addition to being bounded comes from the fact that ERM^c approximates the optimal revenue in an auction with many bidders only for regular distributions. In fact, the sample complexity literature on ERM^c only studies the case of one bidder (which is, in our notation, $K_2 = 1$). In this case, i.e., if the second phase uses posted price auctions, we do not need the regularity assumption. To capture the case of general K_2, we make a technical observation that for regular distributions $(1 - \epsilon)$ revenue optimality for a single buyer implies $(1 - \epsilon)$ revenue optimality for many buyers (Lemma C.1). We do not know if this is true without the regularity assumption or if this observation – which may be of independent interest – was previously known.

Theorem 2.1, still hold for the multi-unit extension with any $k \geq 1$. Moreover, Theorem 2.2 also holds because we have already considered the multi-unit extension in its proof in Appendix C.2.

2.4 Two-Phase ERM Algorithm in Repeated Auctions

The two-phase model with ERM as the price learning function can be seen as a learning algorithm in the following setting of repeated auctions against strategic bidders: there are T rounds of auctions, there are $K \geq 1$ bidders in each auction, and each bidder participates in at most $\overline{m}$ auctions. The algorithm, which we call "two-phase ERM", works as follows: in the first T_1 rounds, run any truthful, prior-independent auction $\mathcal{M}$ (e.g., the second price auction with no reserve); in the later $T_2 = T - T_1$ rounds, run second price auction with reserve $p = \mathrm{ERM}^c(b_1, \ldots, b_{T_1 K})$ where $b_1, \ldots, b_{T_1 K}$ are the bids from the first T_1 auctions. T_1 and c are adjustable parameters.

In repeated games, one may also consider ϵ-perfect Bayesian equilibrium (ϵ-PBE) as the solution concept besides ϵ-BNE. A formal definition is given in Appendix C.4 but roughly speaking, ϵ-PBE requires that the bidding of each bidder at each round of the auctions ϵ-approximately maximizes the total expected utility in all future rounds, conditioning on any observed history of allocations and payments. Note that the history may leak some information about the historical bids of other buyers and these bids will affect the seller's choice of mechanisms in future rounds. Similar to the ϵ-BNE notion, we can show that the two-phase ERM algorithm obtains: (1) truthful bidding is an $O\left(\log^2(T_1 K)\sqrt[3]{\frac{D^{11}K\overline{m}}{T_1}}\right)$-PBE; (2) $\left(1 - O\left(\frac{T_1}{T} + \sqrt{\frac{D\log(T_1 K)}{T_1 K}}\right)\right)$ revenue optimality, for bounded distributions; and similar results for MHR distributions. By choosing $T_1 = \tilde{O}(T^{\frac{2}{3}})$ to maximize revenue, we obtain $\tilde{O}(T^{-\frac{2}{9}})$-truthfulness and $(1 - \tilde{O}(T^{-\frac{1}{3}}))$ revenue optimality, where we assume D, $\overline{m}$, and K to be constant.[4]

Under the same setting, Liu et al. [27] and Abernethy et al. [1] design learning algorithms using differential privacy techniques. We can compare two-phase ERM and their algorithms. In terms of truthfulness notion, Liu et al. [27] assume that bidders play an exact PBE instead of ϵ-PBE, so their result is incomparable with ours. Their notion of exact PBE is too strong to be practical because bidders need to do a large amount of computation, while our notion guarantees bidders of approximately optimal utility as long as they bid truthfully. Although our truthfulness bound is worse than the $\tilde{O}(\frac{1}{\sqrt{T}})$-bound of [1], we emphasize that their ϵ-truthfulness notion is *weaker* than ours: in their definition, each bidder cannot gain more than ϵ in current and future rounds if she deviates from truthful bidding *only in the current round*, given any fixed future strategy. But in our definition, each bidder cannot gain more than ϵ if she deviates *in current and all future rounds*. Our algorithm is easier to implement and more time-efficient than theirs, and works for unbounded distribution while theirs only support bounded distributions because they need to discretize the value space.

3 A Second Application: Uniform-Price Auctions

The notion of an incentive-awareness measure (recall Definition 1.2) has implications regarding the classic uniform-price auction model, which we believe are of independent interest. In a static uniform-price auction we have N copies of a good and N unit-demand bidders with i.i.d. values $\boldsymbol{v}$ from F that submit bids $\boldsymbol{b}$. The auctioneer then sets a price $p = P(\boldsymbol{b})$. Each bidder i whose value v_i is above or equal to p receives a copy of the good and pays p, obtaining a utility of $v_i - p$; otherwise the utility is zero. Azevedo and Budish [4] show that this auction is "incentive-compatible in the large" which means that truthfulness is an ϵ-BNE and ϵ goes to zero as N goes to infinity. They assume bidders' value distribution has a finite support and their bids must be chosen from this finite support as well. They mention that allowing continuous supports and arbitrary bids is challenging.

In this context, taking $P = \mathrm{ERM}^c$ is very natural when the auctioneer aims to maximize revenue. Indeed, Goldberg et al. [17] suggest to use the uniform-price auction with $P = \mathrm{ERM}^c$, where $c = \frac{1}{N}$, as a revenue benchmark for evaluating other truthful auctions they design.

When the price function is $P = \mathrm{ERM}^{c=\frac{1}{N}}$, our analysis of the incentive-awareness measure generalizes the result of [4] to bounded and to MHR distributions. Moreover, we generalize their result to the case where coalitions of at most m bidders can coordinate bids and jointly deviate from truthfulness.

[4]The $\tilde{O}$ notation omits polylogarithmic terms.

Theorem 3.1. *In the uniform-price auction, suppose that any m bidders can jointly deviate from truthful bidding, then no bidder can obtain ϵ more utility (we call this (m, ϵ)-group BIC), where,*

- *for any P and any bounded F, $\epsilon = D\Delta_{N,m}^{\mathrm{worst}}$, and*
- *for any MHR distribution F, if we fix $P = \mathrm{ERM}^c$ with $\frac{m}{N} \le c \le \frac{1}{4e}$ and $m = o(\sqrt{N})$, then $\epsilon = O\left(v^* \Delta_{N,m}^{\mathrm{worst}}\right) + O\left(\frac{v^*}{\sqrt{N}}\right)$, where $v^* = \arg\max_v \{v[1 - F(v)]\}$.*

The constants in big O's are independent of F and c.

Proof of Theorem 3.1 for bounded distributions. Denote a coalition of m bidders by an index set $I \subseteq \{1, \ldots, N\}$, and the true values of all bidders by (v_I, v_{-I}). When other bidders bid v_{-I} truthfully, and the coalition bids b_I instead of v_I, the reduction of price is at most

$$P(v_I, v_{-I}) - P(b_I, v_{-I}) \le P(v_I, v_{-I})\delta_I(v_I, v_{-I}) \le P(v_I, v_{-I})\delta_m^{\mathrm{worst}}(v_{-I}) \le D\delta_m^{\mathrm{worst}}(v_{-I}),$$

by Definition 1.2 and by the fact that all values are upper-bounded by D. Then for each bidder $i \in I$, the increase of her utility by such a joint deviation is no larger than the reduction of price, i.e.

$$\begin{aligned}\mathbb{E}_{v_{-I}}\left[u_i(v_I, P(b_I, v_{-I})) - u_i(v_I, P(v_I, v_{-I}))\right] &\le \mathbb{E}_{v_{-I}}\left[P(v_I, v_{-I}) - P(b_I, v_{-I})\right] \\ &\le D\mathbb{E}_{v_{-I}}\left[\delta_m^{\mathrm{worst}}(v_{-I})\right] = D\Delta_{N,m}^{\mathrm{worst}}.\end{aligned}$$

□

The proof of this theorem for MHR distributions is similar to the proof of Theorem 2.1, thus omitted.

Combining with Theorem 1.3, we conclude that the uniform-price auction with $P = \mathrm{ERM}^c$ (for the c's mentioned there) is (m, ϵ)-group BIC with ϵ converging to zero at a rate not slower than $O(m^{2/3}\frac{\log^2 N}{N^{1/3}})$ for bounded distributions and $O(m\frac{\log^3 N}{\sqrt{N}})$ for MHR distributions (constants in these big O's depend on distributions).

Theorem 3.1 also generalizes the result in [26] which is only for bounded distributions and $m = 1$.

4 More Discussions on Incentive-awareness Measures

4.1 Overview of the Proof for Upper Bounds on $\Delta_{N,m}^{\mathrm{worst}}$

Here we provide an overview of the proof of Theorem 1.3. Details are in Appendix B.

Firstly, we show an important property of ERM^c: suppose $c \ge \frac{m}{N}$, for any m values v_I, any $N - m$ values v_{-I}, and any m values $\overline{v}_I$ that are greater than or equal to the maximum value in v_{-I}, we have $\mathrm{ERM}^c(\overline{v}_I, v_{-I}) \ge \mathrm{ERM}^c(v_I, v_{-I})$. As a consequence, $\delta_m^{\mathrm{worst}}(v_{-I}) = \delta_I(\overline{v}_I, v_{-I})$.

Based on this property, we transfer the expectation in the incentive-awareness measure in the following way:

$$\begin{aligned}&\Delta_{N,m}^{\mathrm{worst}} = \mathbb{E}[\delta_m^{\mathrm{worst}}(v_{-I})] = \mathbb{E}[\delta_I(\overline{v}_I, v_{-I})] = \int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta]\mathrm{d}\eta \\ &\le \int_0^1 \left(\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \mid \overline{\mathrm{E}}]\Pr[\overline{\mathrm{E}}] + \Pr[\mathrm{E}]\right)\mathrm{d}\eta = \int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}]\mathrm{d}\eta + \Pr[\mathrm{E}],\end{aligned}$$

where E denotes the event that the index $k^* = \arg\max_{i > cN}\{iv_i\}$ (which is the index selected by ERM^c) satisfies $k^* \le dN$, $\overline{\mathrm{E}}$ denotes the complement of E, and the probability in $\Pr[\mathrm{E}]$ is taken over the random draw of $N - m$ i.i.d. samples from F, with other m samples fixed to be the upper bound (can be $+\infty$) of the distribution. For any value distribution, we prove that the first part

$$\int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}]\mathrm{d}\eta \le O\left(\sqrt[3]{\frac{m^2}{d^8}}\frac{\log^2 N}{\sqrt[3]{N}}\right),$$

with some constructions of auxiliary events and involved probabilistic argument. And we further tighten this bound to $O\left(\frac{m}{d^{7/2}}\frac{\log^3 N}{\sqrt{N}}\right)$ for MHR distribution by leveraging its properties.

The final part of the proof is to bound $\Pr[\mathrm{E}]$. For bounded distribution, we choose $d = 1/D$. Since the support of the distribution is bounded by $[1, D]$, $N \cdot v_N \geq N$, while for any $k \leq dN$, $kv_k \leq (\frac{1}{D}N)D = N \leq Nv_N$. ERM^c therefore never chooses an index $k \leq dN$ (recall that in case of a tie, ERM^c picks the larger index). This implies $\Pr[\mathrm{E}] = \Pr[k^* \leq dN] = 0$ for bounded distribution. For MHR distribution, we choose $d = \frac{1}{2e}$ and show $\Pr[\mathrm{E}] = O\left(\frac{1}{N}\right)$. As the corresponding proof is quite complicated, we omit it here.

4.2 Lower Bounds on $\Delta_{N,m}^{\mathrm{worst}}$ and on the Approximate BIC Parameter, ϵ_1

Theorem 1.3 gives an upper bound on $\Delta_{N,m}^{\mathrm{worst}}$ for bounded and MHR distributions and for a specific range of c's. Here we briefly discuss the lower bound, with details given in Appendix F.

Lavi et al. [26] show that for the two-point distribution $v = 1$ and $v = 2$, each w.p. 0.5, $\Delta_{N,1}^{\mathrm{worst}} = \Omega(N^{-1/2})$, when $c = 1/N$. We adopt their analysis to provide a similar lower bound for $[1, D]$-bounded distributions and the corresponding range of c's. Let F be a two-point distribution where for $X \sim F$, $\Pr[X = 1] = 1 - 1/D$ and $\Pr[X = D] = 1/D$.

Theorem 4.1. *For the above F, for any $c \in [\frac{m}{N}, \frac{1}{2D}]$, ERM^c gives $\Delta_{N,m}^{\mathrm{worst}} = \Omega(\frac{1}{\sqrt{N}})$ where the constant in Ω depends on D.*

Note that $\Delta_{N,m}^{\mathrm{worst}}$ only upper bounds the ϵ_1-BIC parameter ϵ_1 in the two-phase model: a lower bound on $\Delta_{N,m}^{\mathrm{worst}}$ does not immediately implies a lower bound on ϵ_1. Still, a direct argument will show that the above distribution F gives the same lower bound on ϵ_1. For simplicity let $K_1 = K_2 = 2$ and suppose bidder i participates in m_1 and m_2 auctions in the two phases, respectively. Let $N = T_1 K_1$ and assume $m_1 = o(\sqrt{N})$. Suppose the first-phase mechanism $\mathcal{M}$ is the second price auction with no reserve price. Then in the two-phase model with ERM^c, ϵ_1 must be $\Omega(\frac{m_2}{\sqrt{N}})$ to guarantee ϵ_1-BIC.

It remains open to prove a lower bound for MHR distributions, and to close the gap between our $O(N^{-1/3}\log^2 N)$ upper bound and the $\Omega(N^{-1/2})$ lower bound for bounded distributions.

4.3 Unbounded Regular Distributions

Theorem 2.2 shows that, in the two-phase model, approximate incentive-compatibility and revenue optimality can be obtained simultaneously for bounded (regular) distributions and for MHR distributions. A natural question would then be: what is the largest class of value distribution we can consider? Note that for non-regular distributions, Myerson [30] shows that revenue optimality cannot be guaranteed by anonymous reserve price, so ERM is not a correct choice. Thus we generalize our results to the class of regular distributions that are unbounded and not MHR. Here we provide a sketch, with details given in Appendix G.

Our results can be generalized to α-strongly regular distributions with $\alpha > 0$. As defined in [11], a distribution F with positive density function f on its support $[A, B]$ where $0 \leq A \leq B \leq +\infty$ is *α-strongly regular* if the virtual value function $\phi(x) = x - \frac{1-F(x)}{f(x)}$ satisfies $\phi(y) - \phi(x) \geq \alpha(y - x)$ whenever $y > x$ (or $\phi'(x) \geq \alpha$ if $\phi(x)$ is differentiable). As special cases, regular and MHR distributions are 0-strongly and 1-strongly regular distributions, respectively. For any $\alpha > 0$, we obtain bounds similar to MHR distributions on $\Delta_{N,m}^{\mathrm{worst}}$ and on approximate incentive-compatibility in the two-phase model and the uniform-price auction. Specifically, if F is α-strongly regular then $\Delta_{N,m}^{\mathrm{worst}} = O\left(m\frac{\log^3 N}{\sqrt{N}}\right)$, if $m = o(\sqrt{N})$ and $\frac{m}{N} \leq \left(\frac{\log N}{N}\right)^{1/3} \leq c \leq \frac{\alpha^{1/(1-\alpha)}}{4}$.

It remains an open problem for future research whether ERM^c is incentive-compatible in the large for regular but not α-strongly regular distributions for any $\alpha > 0$. For these distributions the choice of c must be more sophisticated since it creates a clash between approximate incentive-compatibility and approximate revenue optimality. Intuitively, a large c (for example, a constant) will hurt revenue optimality and a too small c will hurt incentive-compatibility. In Appendix G, we provide examples and proofs to formally illustrate such a fact, and further discuss our conjecture that some intermediate c can maintain the balance between incentive-compatibility and revenue optimality.

Broader Impact

This work is mainly theoretical. It provides some intuitions and guidelines for potential practice, but does not have immediate societal consequences. A possible positive consequence is: the auction we consider uses an anonymous reserve price, while most of the related works on repeated auctions use unfair personalized prices. We do not see negative consequences.

References

[1] Jacob D Abernethy, Rachel Cummings, Bhuvesh Kumar, Sam Taggart, and Jamie H Morgenstern. Learning Auctions with Robust Incentive Guarantees. In *Advances in Neural Information Processing Systems 32*, pages 11587–11597. Curran Associates, Inc., 2019. URL `http://papers.nips.cc/paper/9334-learning-auctions-with-robust-incentive-guarantees.pdf`.

[2] Kareem Amin, Afshin Rostamizadeh, and Umar Syed. Learning prices for repeated auctions with strategic buyers. In *Advances in Neural Information Processing Systems*, pages 1169–1177, 2013.

[3] Kareem Amin, Afshin Rostamizadeh, and Umar Syed. Repeated contextual auctions with strategic buyers. In *Advances in Neural Information Processing Systems*, pages 622–630, 2014.

[4] Eduardo M Azevedo and Eric Budish. Strategy-proofness in the large. *The Review of Economic Studies*, 86(1):81–116, 2018.

[5] Moshe Babaioff, Yannai A. Gonczarowski, Yishay Mansour, and Shay Moran. Are two (samples) really better than one? In *Proceedings of the 2018 ACM Conference on Economics and Computation*, EC '18, pages 175–175, New York, NY, USA, 2018. ACM. ISBN 978-1-4503-5829-3. doi: 10.1145/3219166.3219187. URL `http://doi.acm.org/10.1145/3219166.3219187`.

[6] Maria-Florina Balcan, Avrim Blum, Jason D Hartline, and Yishay Mansour. Reducing mechanism design to algorithm design via machine learning. *Journal of Computer and System Sciences*, 74(8):1245–1270, 2008.

[7] Avrim Blum and Jason D Hartline. Near-optimal online auctions. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 1156–1163. Society for Industrial and Applied Mathematics, 2005.

[8] Mark Braverman, Jieming Mao, Jon Schneider, and Matt Weinberg. Selling to a no-regret buyer. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 523–538. ACM. ISBN 978-1-4503-5829-3. doi: 10.1145/3219166.3219233. URL `https://dl.acm.org/doi/10.1145/3219166.3219233`.

[9] Sebastien Bubeck, Nikhil R Devanur, Zhiyi Huang, and Rad Niazadeh. Online auctions and multi-scale online learning. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, pages 497–514. ACM, 2017.

[10] Nicolo Cesa-Bianchi, Claudio Gentile, and Yishay Mansour. Regret minimization for reserve prices in second-price auctions. *IEEE Transactions on Information Theory*, 61(1):549–564, 2015.

[11] Richard Cole and Tim Roughgarden. The sample complexity of revenue maximization. *symposium on the theory of computing*, pages 243–252, 2014.

[12] Yuan Deng, Sébastien Lahaie, and Vahab Mirrokni. A Robust Non-Clairvoyant Dynamic Mechanism for Contextual Auctions. In *Advances in Neural Information Processing Systems 32*, pages 8654–8664. Curran Associates, Inc., 2019. URL `http://papers.nips.cc/paper/9071-a-robust-non-clairvoyant-dynamic-mechanism-for-contextual-auctions.pdf`.

[13] Nikhil R. Devanur, Yuval Peres, and Balasubramanian Sivan. Perfect bayesian equilibria in repeated sales. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '15, pages 983–1002. Society for Industrial and Applied Mathematics. event-place: San Diego, California.

[14] Nikhil R Devanur, Zhiyi Huang, and Christos-Alexandros Psomas. The sample complexity of auctions with side information. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pages 426–439. ACM, 2016.

[15] Peerapong Dhangwatnotai, Tim Roughgarden, and Qiqi Yan. Revenue maximization with a single sample. *Games and Economic Behavior*, 91:318–333, 2015.

[16] Alessandro Epasto, Mohammad Mahdian, Vahab Mirrokni, and Song Zuo. Incentive-aware learning for large markets. In *Proceedings of the 2018 World Wide Web Conference on World Wide Web*, pages 1369–1378. International World Wide Web Conferences Steering Committee, 2018.

[17] Andrew V Goldberg, Jason D Hartline, Anna R Karlin, Michael Saks, and Andrew Wright. Competitive auctions. *Games and Economic Behavior*, 55(2):242–269, 2006.

[18] Negin Golrezaei, Adel Javanmard, and Vahab Mirrokni. Dynamic Incentive-Aware Learning: Robust Pricing in Contextual Auctions. In *Advances in Neural Information Processing Systems 32*, pages 9756–9766. Curran Associates, Inc., 2019. URL `http://papers.nips.cc/paper/9169-dynamic-incentive-aware-learning-robust-pricing-in-contextual-auctions.pdf`.

[19] Yannai A Gonczarowski and S Matthew Weinberg. The sample complexity of up-to-ε multi-dimensional revenue maximization. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 416–426. IEEE, 2018.

[20] Chenghao Guo, Zhiyi Huang, and Xinzhi Zhang. Settling the sample complexity of single-parameter revenue maximization. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 662–673. ACM, 2019.

[21] Jason Hartline and Samuel Taggart. Sample complexity for non-truthful mechanisms. In *Proceedings of the 20th ACM Conference on Economics and Computation*, pages 399–416. ACM, 2019.

[22] Jason Hartline, Vahab Mirrokni, and Mukund Sundararajan. Optimal marketing strategies over social networks. In *Proceedings of the 17th international conference on World Wide Web*, pages 189–198. ACM, 2008.

[23] Zhiyi Huang, Yishay Mansour, and Tim Roughgarden. Making the most of your samples. *SIAM Journal on Computing*, 47(3):651–674, 2018.

[24] Nicole Immorlica, Brendan Lucier, Emmanouil Pountourakis, and Samuel Taggart. Repeated sales with multiple strategic buyers. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, pages 167–168. ACM, 2017.

[25] Yash Kanoria and Hamid Nazerzadeh. Incentive-compatible learning of reserve prices for repeated auctions. In *Companion Proceedings of The 2019 World Wide Web Conference*, pages 932–933. ACM, 2019.

[26] Ron Lavi, Or Sattath, and Aviv Zohar. Redesigning bitcoin's fee market. In *The World Wide Web Conference*, pages 2950–2956. ACM, 2019.

[27] Jinyan Liu, Zhiyi Huang, and Xiangning Wang. Learning optimal reserve price against non-myopic bidders. In *Advances in Neural Information Processing Systems*, pages 2042–2052, 2018.

[28] Andres M Medina and Mehryar Mohri. Learning theory and algorithms for revenue optimization in second price auctions with reserve. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 262–270, 2014.

[29] Jamie H Morgenstern and Tim Roughgarden. On the pseudo-dimension of nearly optimal auctions. In *Advances in Neural Information Processing Systems*, pages 136–144, 2015.

[30] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.

[31] Andrew Chi-Chih Yao. An incentive analysis of some bitcoin fee design. *arXiv preprint arXiv:1811.02351*, 2018.

A Useful Facts

In this section we present some facts about ERM^c and incentive-awareness measures, some definitions about value distributions, and some useful lemmas that will be used throughout.

A.1 Facts about ERM^c and Incentive-Awareness Measures

Claim A.1. *Let $P = \mathrm{ERM}^c$, where $c \geq \frac{m}{N}$. For any $v_I \in \mathbb{R}_+^m$, $v_{-I} \in \mathbb{R}_+^{N-m}$, let $\overline{v}_I$ denote m values such that $\min \overline{v}_I \geq \max v_{-I}$. Then we have $P(\overline{v}_I, v_{-I}) \geq P(v_I, v_{-I})$.*

Proof. Let $\overline{v} := \max v_{-I}$ be the largest value in v_{-I} and $\overline{v}_I$ be m copies of $\overline{v}$. It suffices to show that $P(\overline{v}_I, v_{-I}) \geq P(v_I, v_{-I})$ since $P = \mathrm{ERM}^c$ ignores the largest m samples, given $cN \geq m$. If $v_i \geq \overline{v}$ for each $i \in I$, then we have $P(\overline{v}_I, v_{-I}) = P(v_I, v_{-I})$ directly. If there exists some $i \in I$ such that $v_i < \overline{v}$, then we increase v_i to $\overline{v}$ and show that for any such i and (v_i, v_{-i}), $P(\overline{v}, v_{-i}) \geq P(v_i, v_{-i})$. Let $v' = P(v_i, v_{-i})$, then one can verify (assuming ERM^c picks the smaller value when there are ties) that (1) $P(v', v_{-i}) = P(v_i, v_{-i})$, and (2) $P(v', v_{-i}) \leq P(\overline{v}, v_{-i})$, implying $P(\overline{v}, v_{-i}) \geq P(v_i, v_{-i})$. □

Claim A.2. *Let $P = \mathrm{ERM}^c$, where $c \geq \frac{m}{N}$. For any $v_I \in \mathbb{R}_+^m$, $v_{-I} \in \mathbb{R}_+^{N-m}$, let $\overline{v}_I$ be any m values that are greater than or equal to the maximal value in v_{-I}. Then $\delta_m^{\mathrm{worst}}(v_{-I}) = \delta_I(\overline{v}_I, v_{-I})$.*

Proof. Recall the definition

$$\delta_m^{\mathrm{worst}}(v_{-I}) = \sup_{v_I \in \mathbb{R}_+^m} \delta_I(v_I, v_{-I}) = \sup_{v_I \in \mathbb{R}_+^m} \left\{ 1 - \frac{\inf_{b_I \in \mathbb{R}_+^m} \mathrm{ERM}^c(b_I, v_{-I})}{\mathrm{ERM}^c(v_I, v_{-I})} \right\}.$$

Claim A.1 immediately implies $\delta_m^{\mathrm{worst}}(v_{-I}) = \lim_{v_I \to +\infty} \delta_I(v_I, v_{-I})$. Moreover, since ERM^c ignores the highest $cN \geq m$ values, we have $\mathrm{ERM}^c(\overline{v}_I, v_{-I}) = \mathrm{ERM}^c(\overline{v}'_I, v_{-I})$ as long as both $\overline{v}_I$ and $\overline{v}'_I$ are greater than or equal to $\max v_{-I}$, no matter what they are exactly. Thus $\delta_m^{\mathrm{worst}}(v_{-I}) = \delta_I(\overline{v}_I, v_{-I}) = \delta_I(\overline{v}'_I, v_{-I})$. □

Therefore, we will use $\overline{v}_I$ to denote any m values that are greater than or equal to $\max v_{-I}$, for example, m copies of $\max v_{-I}$ or m copies of "$+\infty$". We always have $\delta_m^{\mathrm{worst}}(v_{-I}) = \delta_I(\overline{v}_I, v_{-I})$.

A.2 Quantiles and Revenue Curves of Value Distributions

For a distribution $F(v)$, define the *quantile* $q(v) = 1 - F(v)$ as a mapping from value space to quantile space. Inversely, $v(q) = q^{-1}(v) = F^{-1}(1-q)$ is the mapping from quantile space to value space (i.e., w.p. q a buyer's value will be at least $v(q)$). Define the *revenue curve* $R(q) = qv(q)$ as the expected revenue for the seller by posting price $v(q)$. Let $R^* = \max_q\{R(q)\}$ denote the optimal revenue the seller can obtain with one bidder, and $q^* = \arg\max_q R(q), v^* = v(q^*)$. When there are several i.i.d. bidders with a regular value distribution, v^* is the optimal reserve price in a second price auction, and such an auction is revenue optimal [30]. Any bounded distribution satisfies $q^* \geq \frac{1}{D}$ because for any $q < \frac{1}{D}$, $qv(q) \leq qD < 1 \leq 1v(1) \leq R^*$. Any MHR distribution has a unique q^* and $q^* \geq \frac{1}{e}$ [22].

A.3 Concentration Inequality

For a distribution F, draw N samples and sort them non-increasingly, $v_1 \geq v_2 \geq \cdots \geq v_N$. Let $q_j = q(v_j)$ denote their quantiles. The ratio j/N is the *empirical quantile* of value v_j since j/N is the quantile of v_j in the uniform distribution over $\{v_1, \ldots, v_N\}$. The following concentration inequality shows that for each value v_j, its empirical quantile j/N is close to its true quantile q_j with high probability, when m samples are fixed to be $+\infty$ while other $N-m$ samples are i.i.d. drawn from F.

Lemma A.3. *Draw $N-m$ i.i.d. samples from a distribution F, and fix m samples to be $+\infty$. Sort these samples non-increasingly: $+\infty = v_1 = \cdots = v_m > v_{m+1} \geq \cdots \geq v_N$. With probability at least $1-\delta$ over the random draw of samples, we have for any $j > m$,*

$$\left| q_j - \frac{j}{N} \right| \leq \sqrt{\frac{2\ln(2(N-m)\delta^{-1})}{N-m}} + \frac{\ln(2(N-m)\delta^{-1})}{N-m} + \frac{m}{N}.$$

Proof. The value v_j $(j > m)$ is the $(j-m)$th largest value in $N-m$ i.i.d. samples from F, by using Bernstein inequality (see e.g., Lemma 5 in Guo et al. [20]), we know that with probability at least $1-\delta$, $\left| q_j - \frac{j-m}{N-m} \right| \leq \sqrt{\frac{2\ln(2(N-m)\delta^{-1})}{N-m}} + \frac{\ln(2(N-m)\delta^{-1})}{N-m}$. Also note that $|\frac{j}{N} - \frac{j-m}{N-m}| = \frac{(N-j)}{N-m}\frac{m}{N} < \frac{m}{N}$. By triangular inequality, $\left| q_j - \frac{j}{N} \right| \leq \sqrt{\frac{2\ln(2(N-m)\delta^{-1})}{N-m}} + \frac{\ln(2(N-m)\delta^{-1})}{N-m} + \frac{m}{N}$. □

B Main Proof: Upper Bounds on Incentive-Awareness Measures

B.1 Proof of Theorem 1.3

Recall the setting of Definition 1.2: we draw N i.i.d. values $v_1, \dots, v_N$ from F, and we have an additional parameter m which is the number of bids that can be changed in the input of the price learning function. Theorem 1.3 then states an upper bound on $\Delta_{N,m}^{\text{worst}}$ for $P = \text{ERM}^c$. For bounded distributions, the theorem follows immediately from the next lemma which is our main technical lemma. Note that this lemma is useful in establishing the bound on $\Delta_{N,m}^{\text{worst}}$ not only for bounded distributions but also for all other distributions. Throughout, we assume that $v_1, \dots, v_N$ are sorted, so that $v_1 \geq \cdots \geq v_N$.

Lemma B.1 (Main Lemma)**.** *Suppose $m = o(\sqrt{N})$. Let d be a constant, $0 < d < 1$. Suppose $\frac{m}{N} \leq c \leq \frac{d}{2}$. Let E be the event that the index $k^* = \arg\max_{i > cN}\{iv_i\}$ (which is the index selected by ERM^c) satisfies $k^* \leq dN$. For any non-negative distribution F,*

$$\Delta_{N,m}^{\text{worst}} \leq O\left(\sqrt[3]{\frac{m^2}{d^8}}\frac{\log^2 N}{\sqrt[3]{N}}\right) + \Pr[\mathrm{E}],$$

where the probability in $\Pr[\mathrm{E}]$ is taken over the random draw of $N-m$ i.i.d. samples from F, with other m samples fixed to be $+\infty$.

To see that this lemma immediately implies the theorem for bounded distribution, choose $d = 1/D$. Since the support of the distribution is bounded by $[1, D]$, $N \cdot v_N \geq N$ while for any $k \leq dN$,

$$kv_k \leq (\frac{1}{D}N)D = N \leq Nv_N.$$

ERM^c therefore never chooses an index $k \leq dN$ (recall that in case of a tie, ERM^c picks the larger index). This implies $\Pr[\mathrm{E}] = \Pr[k^* \leq dN] = 0$ and we have the bound in the theorem.

Remark. For MHR distributions, Lemma D.5 shows that $\Pr[\mathrm{E}] = O\left(\frac{1}{N}\right)$ if we choose $d = \frac{1}{2e}$, so Lemma B.1 already gives an upper bound on $\Delta_{N,m}^{\text{worst}}$. However, we can use some additional properties of MHR distributions to strengthen the bound on the first term in the main lemma to $O\left(\frac{m}{d^{7/2}}\frac{\log^3 N}{\sqrt{N}}\right)$, as explained in Appendix D.3.

B.2 Proof of the Main Lemma (Lemma B.1)

Let $\overline{v}_I$ be any m values that are greater than the maximal value in v_{-I}. By Claim A.2, $\delta_m^{\text{worst}}(v_{-I}) = \delta_I(\overline{v}_I, v_{-I})$. Thus,

$$\begin{aligned}\Delta_{N,m}^{\text{worst}} &= \mathbb{E}[\delta_m^{\text{worst}}(v_{-I})] = \mathbb{E}[\delta_I(\overline{v}_I, v_{-I})] = \int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta]\mathrm{d}\eta \\ &\leq \int_0^1 \left(\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \mid \overline{\mathrm{E}}]\Pr[\overline{\mathrm{E}}] \ + \ \Pr[\mathrm{E}]\right)\mathrm{d}\eta \\ &= \int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}]\mathrm{d}\eta \ + \ \Pr[\mathrm{E}], \end{aligned} \tag{3}$$

where $\overline{\mathrm{E}}$ denotes the complement of E. Then the main effort is to upper-bound $\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}]$. After the random draw of v_{-I}, we sort all values non-increasingly, denoted by $\overline{v}_1 = \cdots = \overline{v}_m \geq v_{m+1} \geq \cdots \geq v_N$, and let $\overline{q}_1 = \cdots = \overline{q}_m \leq q_{m+1} \leq \cdots \leq q_N$ be their quantiles, where $q_j = q(v_j)$. We use a concentration inequality (Lemma A.3) to argue that for each value v_j, its empirical quantile j/N should be close to its true quantile q_j with high probability, as follows:

Claim B.2. *Define event* Conc*:*

$$\text{Conc} = \left[\forall j > m, \left|q_j - \frac{j}{N}\right| \leq 2\sqrt{\frac{4\ln(2(N-m))}{N-m}} + \frac{m}{N}\right],$$

then $\Pr[\overline{\text{Conc}}] \leq \frac{1}{N-m}$, where the probability is over the random draw of the $N-m$ samples v_{-I}.

Proof. Set $\delta = \frac{1}{N-m}$ in Lemma A.3. □

Now, define $G(\eta) = \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}} \ \wedge \ \text{Conc}]$ for $0 \leq \eta \leq 1$. We have

$$\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}] \leq G(\eta) + \frac{1}{N-m}. \tag{4}$$

Lemma B.3. *There exists a constant $C = \Theta\left(\frac{m \log^3 N}{d^4 \sqrt{N}}\right)$ such that $\eta > C^{2/3} \Rightarrow G(\eta) \leq \frac{C}{\eta^{3/2}}$.*

Finally we upper-bound the integral in (3):

$$\int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}]\mathrm{d}\eta \leq \int_0^1 \left(G(\eta) + \frac{1}{N-m}\right)\mathrm{d}\eta \qquad \text{By (4)}$$
$$\leq \int_0^{C^{\frac{2}{3}}} 1\mathrm{d}\eta + \int_{C^{\frac{2}{3}}}^1 \frac{C}{\eta^{\frac{3}{2}}}\mathrm{d}\eta + \frac{1}{N-m} \qquad \text{By Lemma B.3}$$
$$\leq 3C^{\frac{2}{3}} + \frac{1}{N-m} = O\left(\sqrt[3]{\frac{m^2 \log^6 N}{d^8 N}}\right) + \frac{1}{N-m} = O\left(\sqrt[3]{\frac{m^2}{d^8}}\frac{\log^2 N}{\sqrt[3]{N}}\right),$$

which, together with (3), concludes the proof of Lemma B.1.

B.3 Proof of Lemma B.3

Recall that we need to upper-bound $G(\eta) = \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}]$ by $\Theta\left(\frac{m \log^3 N}{d^4\sqrt{N}}\frac{1}{\eta^{3/2}}\right)$. We do this via a union bound of $M+1$ events, where M is a number to be chosen later. Each event is parameterized by η_t, θ_t for $t = 0, \ldots, M$ which are chosen to satisfy the following conditions:

- $\eta_0 = \frac{1}{2}\eta, \eta_1 = \eta, \eta_2 = 2\eta$.
- For $t \geq 3$, η_t can be chosen arbitrarily, as long as $\eta_2 < \eta_3 < \cdots < \eta_M < 1$.
- $\eta_{M+1} = 1$.
- $\theta_0 = 1,$ and $\theta_t = \frac{\eta}{2\eta_{t+1}}$ for $t = 1, \ldots, M$.

Define the following $M+1$ events $\mathrm{Bad}(\eta_t, \theta_t)$, where $t = 0, \ldots, M$:

$$\mathrm{Bad}(\eta_t, \theta_t) = \left[\text{there exists } j \geq k^* \text{ such that } \begin{cases} v_j \leq (1-\eta_t)v_{k^*} \\ jv_j \geq k^* v_{k^*} - \frac{m}{\theta_t}v_{k^*} \end{cases}\right] \qquad (5)$$

The next lemma shows that the union of these events contains the event $[\delta_I(\overline{v}_I, v_{-I}) > \eta] \wedge \overline{\mathrm{E}}$.

Lemma B.4. *Suppose $\frac{2m}{dN} < \eta < 1$ and that the parameters η_t, θ_t satisfy the above conditions. If $\delta_I(\overline{v}_I, v_{-I}) > \eta$ and $k^* > dN$, then there exists $t \in \{0, \ldots, M\}$ such that the event $\mathrm{Bad}(\eta_t, \theta_t)$ holds.*

The proof of this lemma is given in Appendix B.4. Moreover, the next lemma upper-bounds the probability of each of these bad events, when assuming that Conc holds as well.

Lemma B.5. *If η_t and θ_t are at least $\Omega\left(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}}\right)$ (for some constant in Ω to be detailed in the proof), then $\Pr[\mathrm{Bad}(\eta_t, \theta_t) \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}] = O\left(\frac{m \log^2 N}{d^4 \theta_t \sqrt{\eta_t^3 N}}\right)$.*

The proof of Lemma B.5 is in Appendix B.5. Now,

$$\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}] \leq \sum_{t=0}^{M} \Pr[\mathrm{Bad}(\eta_t, \theta_t) \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}] \qquad \text{Lemma B.4}$$
$$= \sum_{t=0}^{M} O\left(\frac{m \log^2 N}{d^4 \theta_t \sqrt{\eta_t^3 N}}\right) \qquad \text{Lemma B.5}$$
$$= O\left(\frac{m \log^2 N}{d^4 \sqrt{N}} \cdot \sum_{t=0}^{M} \frac{\eta_{t+1}}{\eta}\frac{1}{\eta_t^{3/2}}\right) \qquad \text{Definition of } \theta_t$$

Note that because $\eta_t, \theta_t \geq \frac{\eta}{2}$, the condition of Lemma B.5 is satisfied under the assumption that $\eta \geq \Theta\left(\left(\frac{m \log^3 N}{d^4\sqrt{N}}\right)^{2/3}\right)$ in Lemma B.3. Finally, we choose a sequence of $\{\eta_t\}$ to make the above summation small enough:

Claim B.6. *There exist an integer M and parameters $\eta_3, \ldots, \eta_M$ that satisfy the conditions described above, such that*

$$\sum_{t=0}^{M} \frac{\eta_{t+1}}{\eta} \frac{1}{\eta_t^{3/2}} = O\left(\frac{\log\log(N-m)}{\eta^{3/2}}\right),$$

assuming $\eta = \Omega\left(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}}\right)$.

The proof of this claim is given in Appendix B.7. To conclude the proof,

$$\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}] \le O\left(\frac{m\log^2 N}{d^4\sqrt{N}} \cdot \frac{\log\log(N-m)}{\eta^{3/2}}\right) = O\left(\frac{m\log^3 N}{d^4\sqrt{N}} \frac{1}{\eta^{3/2}}\right).$$

Remark. This proof is inspired by a proof in Yao [31]. We improve upon that proof in two aspects: (1) Our definition of a sequence of bad events (Lemma B.4) improves upon similar single bad events defined in Yao [31] and Lavi et al. [26]; (2) Yao [31] only considers bounded and continuous distributions, while our proof works for arbitrary distributions. This is because Yao [31] works in the value space when upper-bounding the probability of bad events over the random draw of values v_{-I} (Lemma B.5), but we work in the quantile space, which circumvents the boundedness assumption and deals with discontinuity. To argue in the quantile space, we need Conc to show that $q_j v_j$ approximates $j v_j$ in the proof of Lemma B.5.

B.4 Proof of Lemma B.4

Suppose $\delta_I(\overline{v}_I, v_{-I}) > \eta$ and $k^* > dN$. By definition, there exist m bids $b_I \in \mathbb{R}_+^m$ such that $\mathrm{ERM}^c(b_I, v_{-I}) < (1-\eta)v_{k^*}$. Without loss of generality, we can assume that all m bids are identical, as shown in the following claim:

Claim B.7. *For any sorted $\boldsymbol{v} = (v_1 \ge \cdots \ge v_N)$. Let $I = \{1, \ldots, m\}$, let*

$$b^* = \arg\min_{b \in \mathbb{R}_+} \mathrm{ERM}^c(\overbrace{b, \ldots, b}^{m \text{ copies}}, v_{-I}).$$

then $b^* = \mathrm{ERM}^c(b^*, \ldots, b^*, v_{-I}) = \min_{b_I \in \mathbb{R}_+^m} \mathrm{ERM}^c(b_I, v_{-I})$.

Proof. We will show that for any vector of m bids $b_I = (b_1, \ldots, b_m)$ that minimizes $\mathrm{ERM}^c(b_I, v_{-I})$, we can construct another vector $b'_I = (b, \ldots, b)$ such that $\mathrm{ERM}^c(b'_I, v_{-I}) = \mathrm{ERM}^c(b_I, v_{-I}) = b$. Because b_I minimizes $\mathrm{ERM}^c(b_I, v_{-I})$, we can assume that there is a bid b_{i^*} such that $\mathrm{ERM}^c(b_I, v_{-I}) = b_{i^*}$ (otherwise, we can decrease the bids in b_I without increasing the price). Let $b = b_{i^*}$. For any $b_j > b$, decrease b_j to b, then the price does not change. For any $b_j < b$, increase b_j to b, then the price does not increase; and if the price decreases, then it contradicts the fact that $\mathrm{ERM}^c(b_I, v_{-I})$ is minimized. In this way, we change all bids in b_I to b, without affecting the price. ☐

By Claim B.7, there exists $b \in \mathbb{R}_+$ which equals $\mathrm{ERM}^c(b, \ldots, b, v_{-I})$ and satisfies

$$b < (1-\eta)v_{k^*}. \tag{6}$$

Choose index i for which $v_i \ge b > v_{i+1}$. Assume for now $i \le N-1$, we will postpone the analysis for $i = N$ to the end. Now we show that setting $j = i$ or $i+1$ will satisfy the lemma. Clearly $i \ge k^*$. The change of the bids vector caused by b is:

$$(\overline{v}_1, \ldots, \overline{v}_m, v_{m+1}, \ldots, v_{k^*}, \ldots, v_i, v_{i+1}, \ldots, v_N) \to (v_{m+1}, \ldots, v_{k^*}, \ldots, v_i, \overbrace{b, \ldots, b}^{m \text{ times}}, v_{i+1}, \ldots, v_N).$$

Note that $k^* - m > dN - m \ge cN$, so v_{k^*} will not be ignored by ERM^c after the change of bids. Then in order for b to be chosen by ERM^c, we need:

$$i \cdot b \ge (k^* - m) \cdot v_{k^*} = k^* v_{k^*} - m v_{k^*}. \tag{7}$$

We will choose j depending on how large v_i is:

(a) If $v_i < (1 - \frac{1}{2}\eta)v_{k^*} = (1-\eta_0)v_{k^*}$, we set $j = i$ and $t = 0$. Clearly, $iv_i \ge ib \ge k^* v_{k^*} - m v_{k^*}$.

(b) If $v_i \ge (1 - \frac{1}{2}\eta)v_{k^*}$, then we set $j = i+1$ and choose the t ($1 \le t \le M$) such that

$$(1-\eta_t)v_{k^*} \ge v_{i+1} > (1-\eta_{t+1})v_{k^*}. \tag{8}$$

To see why $(i+1)v_{i+1} \geq k^* v_{k^*} - \frac{2\eta_{t+1}}{\eta} m v_{k^*}$ holds, first we write b as a convex combination of v_i and v_{i+1}: $b = (1-\lambda)v_i + \lambda v_{i+1}$. From (6) and (7), we immediately get

$$(1-\eta)v_{k^*} > (1-\lambda)v_i + \lambda v_{i+1}, \tag{9}$$

$$(1-\lambda)iv_i + \lambda(i+1)v_{i+1} \geq k^* v_{k^*} - m v_{k^*}. \tag{10}$$

Equation (10) further implies $\lambda(i+1)v_{i+1} \geq k^* v_{k^*} - (1-\lambda)k^* v_{k^*} - m v_{k^*}$. Divide by λ,

$$(i+1)v_{i+1} \geq k^* v_{k^*} - \frac{m}{\lambda} v_{k^*}.$$

Then it remains to lower-bound λ by θ_t. Intuitively, since v_i is larger than $(1-\frac{\eta}{2})v_{k^*}$ but $b < (1-\eta)v_{k^*}$, the coefficient of v_{i+1} cannot be too small. Formally, from (9) and (8), we have:

$$(1-\eta)v_{k^*} > (1-\lambda)(1-\frac{1}{2}\eta)v_{k^*} + \lambda(1-\eta_{t+1})v_{k^*},$$

$$\implies 1-\eta > 1 - \frac{1}{2}\eta - \lambda(\eta_{t+1} - \frac{1}{2}\eta)$$

$$\implies \lambda > \frac{\frac{1}{2}\eta}{\eta_{t+1} - \frac{1}{2}\eta} \geq \frac{\eta}{2\eta_{t+1}} = \theta_t,$$

concluding the proof of this case.

Finally we return to the analysis for $i = N$. If $\frac{k^*}{N} < 1 - \frac{1}{2}\eta$, then $v_N \leq \frac{k^* v_{k^*}}{N} < (1-\frac{1}{2}\eta)v_{k^*}$, so the above argument (a) can be reused. Otherwise, from (6) and (7), we have:

$$(1-\eta)v_{k^*} > b \geq \frac{(k^*-m)v_{k^*}}{N} \geq (1 - \frac{1}{2}\eta - \frac{m}{N})v_{k^*},$$

which contradicts the assumption that $\eta > \frac{2m}{dN}$.

B.5 Proof of Lemma B.5

For convenience we drop the subscript t and just write $\eta = \eta_t, \theta = \theta_t$. Recall that we need to upper-bound $\Pr[\mathrm{Bad}(\eta,\theta) \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}]$ where:

- $\mathrm{Bad}(\eta,\theta)$ implies that there exists $j \geq k^*$ such that $v_j \leq (1-\eta)v_{k^*}$ and $jv_j \geq k^* v_{k^*} - \frac{m}{\theta} v_{k^*}$.
- $\overline{\mathrm{E}}$ is $k^* \geq dN$.
- Conc requires that $|q_j - \frac{j}{N}| \leq 2\sqrt{\frac{4\ln(2(N-m))}{N-m}} + \frac{m}{N}$ for any $j > m$.

Define

$$H = \frac{m}{d\theta - \frac{m}{N}} \qquad \text{and} \qquad h = \frac{1}{2}\left(d\eta - 4\sqrt{\frac{4\ln(2(N-m))}{N-m}} - \frac{4m}{N\theta}\right).$$

Assume $H, h > 0$, which can be satisfied when η and θ are at least $\Omega\left(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}}\right)$.

Claim B.8. *The event* $[\mathrm{Bad}(\eta,\theta) \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}]$ *implies that there exists* $j \geq k^*$ *which satisfies:*

1. $jv_j \leq k^* v_{k^*} \leq (j+H)v_j$;
2. $q_j - q_{k^*} \geq 2h$.

Proof of Claim B.8. Choose the j in $\mathrm{Bad}(\eta,\theta)$ which satisfies $jv_j \geq k^* v_{k^*} - \frac{m}{\theta} v_{k^*}$. To see why the first inequality holds, note that $dNv_{k^*} \leq k^* v_{k^*} \leq jv_j + \frac{m}{\theta} v_{k^*} \leq Nv_j + \frac{m}{\theta} v_{k^*}$, subtracting the first and forth term, we get $(dN - \frac{m}{\theta})v_{k^*} \leq Nv_j$, further implying $k^* v_{k^*} \leq jv_j + \frac{m}{\theta}\frac{Nv_j}{(dN - m/\theta)}$, which is the first inequality.

Now consider the second inequality. Since $\mathrm{Bad}(\eta,\theta)$ requires $jv_j \geq k^* v_{k^*} - \frac{m}{\theta} v_{k^*}$ and $v_j \leq (1-\eta)v_{k^*}$, we have

$$j \geq \frac{k^* v_{k^*} - \frac{m}{\theta} v_{k^*}}{(1-\eta)v_{k^*}} = \frac{k^* - \frac{m}{\theta}}{1-\eta} \geq (k^* - \frac{m}{\theta})(1+\eta) = k^* + k^*\eta - \frac{m}{\theta}(1+\eta) \geq k^* + dN\eta - 2\frac{m}{\theta},$$

and dividing by N,

$$\frac{j}{N} - \frac{k^*}{N} \geq d\eta - 2\frac{m}{N\theta}.$$

Using the condition Conc on j and k^*, we can derive the relationship between q_j and q_{k^*} by simple calculation:

$$q_j - q_{k^*} \geq \left(d\eta - 2\frac{m}{N\theta}\right) - 2\left(2\sqrt{\frac{4\ln(2(N-m))}{N-m}} + \frac{m}{N}\right) \geq 2h.$$

$\square$

Divide the quantile space $[0,1]$ into $[0, d/2]$ and $(1-d/2)/h$ equal-length intervals with length h,

$$[0,1] = [0, \frac{d}{2}] \cup I_1 \cup I_2 \cup \cdots \cup I_{\frac{1-d/2}{h}}, \tag{11}$$

where $I_l = (d/2 + (l-1)h, d/2 + lh]$. Thus a uniformly random draw of quantile falls into I_l with probability h. Define i_l^* and $i_{<(l+1)}^*$:

$$i_l^* = \arg\max_{i>cN} \left\{ iv_i \mid q_i \in I_l \right\} \text{ or } i_l^* = \emptyset \text{ if there is no such } i.$$

$$i_{<(l+1)}^* = \arg\max_{i>cN} \left\{ iv_i \mid q_i \in I_1 \cup \cdots \cup I_l \right\} \text{ or } i_{<(l+1)}^* = \emptyset \text{ if there is no such } i.$$

And $A_l \stackrel{\text{def}}{=} i_l^* v_{i_l^*}$, $A_{<(l+1)} \stackrel{\text{def}}{=} i_{<(l+1)}^* v_{i_{<(l+1)}^*}$. Moreover, define event W_l for each l,

$$W_l = \left[\{i_{l+2}^* \neq \emptyset\} \wedge \{i_{<(l+1)}^* \neq \emptyset\} \wedge \left\{A_{l+2} \leq A_{<(l+1)} \leq A_{l+2} + H\widetilde{v}_{l+2}\right\} \right],$$

where $\widetilde{v}_{l+2} \stackrel{\text{def}}{=} v(d/2 + (l+1)h)$ is the upper bound on the values with quantiles in I_{l+2}. We argue that if the event $[\text{Bad}(\eta,\theta) \wedge \overline{\text{E}} \wedge \text{Conc}]$ holds then there must exist an index l such that W_l holds. To see this, consider the index j that is promised to exist in $\text{Bad}(\eta,\theta)$ and choose the index l such that $q_j \in I_{l+2}$. Note that $[\overline{\text{E}} \wedge \text{Conc}]$ implies $q_j \geq q_{k^*} > d/2$, so both q_j and q_{k^*} must fall in $I_1 \cup I_2 \cup \cdots$. To see why W_l must hold, note that:

- $A_{l+2} \leq A_{<(l+1)}$ since $q_j - q_{k^*} > 2h$, implying $q_{k^*} \in I_{<(l+1)}$ and $A_{<(l+1)} = k^* v_{k^*}$.
- $A_{<(l+1)} \leq A_{l+2} + H\widetilde{v}_{l+2}$ since $k^* v_{k^*} \leq (j+H)v_j \leq i_{l+2}^* v_{i_{l+2}^*} + H\widetilde{v}_{l+2}$.

Therefore, a union bound over $\Pr[W_l]$ suffices to prove that $\Pr[\text{Bad}(\eta,\theta) \wedge \overline{\text{E}} \wedge \text{Conc}]$ is small. The idea to bound $\Pr[W_l]$ is a refinement of Yao [31]: Note that there is an interval I_{l+1} with length h between I_{l+2} and $I_{<(l+1)}$ and consider the number X of quantiles falling into I_{l+1}. There is enough randomness in X as its variance is $\Omega(hN)$, implying that the difference between the rankings of any pair of quantiles in I_{l+2} and $I_{<(l+1)}$ varies broadly. As a result, it's unlikely that $A_{<(l+1)}$ will fall in the short interval $[A_{l+2}, A_{l+2} + H\widetilde{v}_{l+2}]$. Formally, we will prove that

Lemma B.9. *For any l, $\Pr[W_l] \leq O(\frac{H\log^2 N}{\sqrt{hd^3N}})$.*

The proof of Lemma B.9 is in Appendix B.6. To conclude,

$$\Pr[\text{Bad}(\eta,\theta) \wedge \overline{\text{E}} \wedge \text{Conc}] \leq \sum_{l=1}^{\frac{1-d/2}{h}} \Pr[W_l] \leq \frac{1}{h} O\left(\frac{H\log^2 N}{\sqrt{hd^3N}}\right) = O\left(\frac{m\log^2 N}{d\theta\sqrt{(d\eta)^3 d^3 N}}\right),$$

where the last equality is because $H = O(\frac{m}{d\theta})$ and $h = \Omega(d\eta)$ under the assumption that η and θ are at least $\Omega(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}})$.

B.6 Proof of Lemma B.9

We need to upper-bound $\Pr[W_l]$ over the random draw of v_{-I}, or in quantile space, q_{-I}, which are $N-m$ i.i.d. random draws from $\text{Uniform}[0,1]$. Let N_L be the number of quantile draws that are in $L \stackrel{\text{def}}{=} [0, d/2] \cup I_1 \cup \cdots \cup I_{l+1}$. Suppose we draw the quantiles in the following procedure: first determine N_L, then draw $N - m - N_L$ quantiles that are not in L; finally draw N_L quantiles that are in L.

Note that N_L follows a binomial distribution, and a Chernoff bound implies that

$$\Pr[N_L \geq \frac{d}{4}(N-m)] \geq 1 - \exp\left(-\frac{d(N-m)}{16}\right). \tag{12}$$

We thus assume $N_L \geq d(N-m)/4$.

Then draw $N - m - N_L$ quantiles from $[0,1]\backslash L$, so $i^*_{l+2}, v_{i^*_{l+2}}$ and A_{l+2} are determined. Suppose $i^*_{l+2} \neq \emptyset$; otherwise, W_l does not hold.

Now we draw N_L quantiles, $q^{(1)}, \ldots, q^{(N_L)}$ from L. Consider the increment of $A_{<(l+1)}$, as a sequence $A^{(t)}, t = 1, \ldots, N_L$. After the time $t-1$ when $A^{(t-1)} \geq A_{l+2}$, the index $i^*_{<(l+1)}$ is no longer $\emptyset$. When one more sample $q^{(t)}$ is generated, there are three cases:

1. If $q^{(t)} \in [0, d/2]$, $A^{(t-1)}$ increases by at least $\widetilde{v}_{l+2}$. This is because each term $iv_i^{(t-1)}$ increases to $(i+1)v_i^{(t)} \geq iv_i^{(t-1)} + \widetilde{v}_{l+2}$, for any i such that $q_i \in I_1, \ldots, I_l$.
2. If $q^{(t)} \in I_1 \cup \cdots \cup I_l$, then $A^{(t-1)}$ does not decrease.
3. If $q^{(t)} \in I_{l+1}$, $A^{(t-1)}$ does not change.

Let s be the number of quantiles that are not in I_{l+1}, and $A^{(t_1)}, \ldots, A^{(t_s)}$ be those steps, and write $B^{(i)} \stackrel{\text{def}}{=} (A^{(t_i)} - A_{l+2})/\widetilde{v}_{l+2}$ for $i = 1, \ldots, s$. We have $A_{<(l+1)} = \widetilde{v}_{l+2} B^{(s)} + A_{l+2}$. Then our task is to analyze the probability that $B^{(s)} \in [0, H]$.

We can think of the generation of $B^{(s)}$ as follows: regardless of s, first generate an infinite sequence $B^{(1)}, B^{(2)}, \ldots$, where at each step i the value $B^{(i)}$ is increased by 1 with probability at least $\Pr[q \in [0, d/2] \mid q \in L] \geq d/2$. Then pick an index s by a binomial distribution $Bin(N_L, 1 - \Pr[q \in I_{l+1} \mid q \in L])$. Then the s-th value in the infinite sequence $\{B^{(i)}\}$ is chosen as $B^{(s)}$. Note that $Bin(N_L, 1 - \Pr[q \in I_{l+1} \mid q \in L])$ is dominated by $Bin(N_L, 1-h)$, so the probability that $B^{(s)}$ takes on any one of the values in the sequence $\{B^{(i)}\}$ is at most $\Pr[s = i] = O(1/\sqrt{hN_L})$.

Then we consider the length of the sub-sequence where $B^{(i)} \in [0, H]$. Intuitively, the expected number of steps for $B^{(i)}$ to increase by H, is at most $H/(d/2)$. The probability that it takes more than $2H(\log N_L)^2/d$ steps implies that the sum of $2H(\log N_L)^2/d$ i.i.d. Bernoulli variables whose success probability is at least $d/2$ does not reach H, which can be bounded by a Chernoff bound:

$$
\begin{aligned}
\Pr[\text{Length} > \frac{2H(\log N_L)^2}{d}] &\leq \Pr[Bin\left(\frac{2H(\log N_L)^2}{d}, \frac{d}{2}\right) < H] \\
&\leq \exp\left(-\frac{1}{2}H(\log N_L)^2\left(1 - \frac{1}{(\log N_L)^2}\right)^2\right) \\
&= O\left(\exp\left(-\frac{1}{8}(\log N_L)^2\right)\right) \\
&= O\left(N_L^{-\frac{1}{8}\log N_L}\right).
\end{aligned}
$$

Assuming Length $\leq (2H(\log N_L)^2)/d$, the probability that $B^{(s)} \in [0, H]$ can be bounded by a union bound:

$$
\sum_{i: B^{(i)} \in [0,H]} \Pr[s = i] \leq \text{Length} \cdot O\left(\frac{1}{\sqrt{hN_L}}\right) \leq O\left(\frac{H(\log N_L)^2}{d\sqrt{hN_L}}\right).
$$

Therefore,

$$
\begin{aligned}
\Pr[W_l] &\leq O\left(\frac{H(\log N_L)^2}{d\sqrt{hN_L}}\right) + O\left(N_L^{-\frac{1}{8}\log N_L}\right) \\
&\leq O\left(\frac{H(\log N)^2}{d\sqrt{hdN}}\right) + \left(\frac{d}{4}(N-m)\right)^{-\frac{1}{8}\log\frac{d}{4}(N-m)} + \exp\left(-\frac{d(N-m)}{16}\right) \qquad \text{By (12)} \\
&= O\left(\frac{H(\log N)^2}{\sqrt{hd^3N}}\right).
\end{aligned}
$$

B.7 Proof of Claim B.6

We need to show that there exist an integer M and parameters $\eta_0 = \frac{1}{2}\eta < \eta_1 = \eta < \eta_2 = 2\eta < \eta_3 < \cdots < \eta_M < \eta_{M+1} = 1$, such that

$$
\sum_{t=0}^{M} \frac{\eta_{t+1}}{\eta} \frac{1}{\eta_t^{3/2}} = O\left(\frac{\log\log(N-m)}{\eta^{3/2}}\right).
$$

We start with:

$$\sum_{t=0}^{M} \frac{\eta_{t+1}}{\eta} \frac{1}{\eta_t^{3/2}} = \frac{1}{\eta^{3/2}} \sum_{t=0}^{M} \frac{\eta_{t+1}/\eta}{(\eta_t/\eta)^{3/2}} = \frac{1}{\eta^{3/2}} \left(O(1) + \sum_{t=2}^{M} \frac{\eta_{t+1}/\eta}{(\eta_t/\eta)^{3/2}} \right) \tag{13}$$

Let $\eta_{t+1}/\eta = (\eta_t/\eta)^{3/2}$ for any $t \geq 2$. We can recursively compute η_t until the maximum step $t = M$ which satisfies $\eta_M < 1$. Then (13) is upper-bounded by $\frac{1}{\eta^{3/2}}(O(1) + M)$. By our construction of $\{\eta_t\}$, we have

$$\frac{\eta_M}{\eta} = (\frac{\eta_2}{\eta})^{\frac{3}{2}^{M-2}} = 2^{\frac{3}{2}^{M-2}} < \frac{1}{\eta}.$$

Thus,

$$M < \log_{3/2} \log_2 \frac{1}{\eta} + 2 = O(\log \log \frac{1}{\eta}) = O(\log \log(N - m)),$$

where the last equality follows from the assumption that $\eta = \Omega\left(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}}\right)$. Thus, the summation (13) becomes

$$\sum_{t=0}^{M} \frac{\eta_{t+1}}{\eta} \frac{1}{\eta_t^{3/2}} = O\left(\frac{\log \log(N - m)}{\eta^{3/2}}\right).$$

as required.

C Missing Proofs From Section 2

C.1 Proof of Theorem 2.1 (for Bounded Distributions)

Proof of Theorem 2.1 for bounded Distributions. First consider the reduction of reserve price caused by the deviation of bidder i. The true values of all bidders in the first phase are (v_I, v_{-I}), where bidder i's true values are $v_I \in \mathbb{R}_+^{m_{i,1}}$. When other bidders bid v_{-I} truthfully, and bidder i bids b_I instead, the reserve price p changes from $P(v_I, v_{-I})$ to $P(b_I, v_{-I})$ and the change is at most

$$P(v_I, v_{-I}) - P(b_I, v_{-I}) \leq P(v_I, v_{-I})\delta_I(v_I, v_{-I}) \leq P(v_I, v_{-I})\delta^{\text{worst}}_{m_{i,1}}(v_{-I}) \leq D\delta^{\text{worst}}_{m_1}(v_{-I}),$$

by Definition 1.2 and by the fact that all values are upper-bounded by D. Consider the increase of utility in the second phase. We claim that for any two possible reserve prices $p_2 \leq p_1$, for any $v \in \mathbb{R}_+$, for any $K_2 \geq 1$, we have

$$u^{K_2}(v, p_2) - u^{K_2}(v, p_1) \leq p_1 - p_2. \tag{14}$$

To see this, first re-write $u^{K_2}(v, p)$ in (2) as

$$u^{K_2}(v, p) = \mathbb{E}_{X_2,\ldots,X_{K_2}\sim F}\left[(v - \max\{p, X^*\})^+\right],$$

where $X^* \stackrel{\text{def}}{=} \max\{X_2, \ldots, X_{K_2}\}$ and $(x)^+ \stackrel{\text{def}}{=} \max\{x, 0\}$. Note that $(x)^+ - (y)^+ \leq |x - y|$, thus

$$\begin{aligned} u^{K_2}(v, p_2) - u^{K_2}(v, p_1) &= \mathbb{E}\left[(v - \max\{p_2, X^*\})^+ - (v - \max\{p_1, X^*\})^+\right] \\ &\leq \mathbb{E}\left[|\max\{p_1, X^*\} - \max\{p_2, X^*\}|\right] \leq \mathbb{E}\left[|p_1 - p_2|\right] = p_1 - p_2. \end{aligned}$$

For the first phase we have $U_i^{\mathcal{M}}(\boldsymbol{v}_i, b_I, v_{-I}) \leq U_i^{\mathcal{M}}(\boldsymbol{v}_i, v_I, v_{-I})$ since $\mathcal{M}$ is truthful. Thus, by (1) and (14) the difference in interim utilities is at most

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{v}_{-i}}\left[U_i^{\text{TP}}(\boldsymbol{v}_i, b_I, v_{-I}) - U_i^{\text{TP}}(\boldsymbol{v}_i, v_I, v_{-I})\right] \\ &\leq \mathbb{E}_{\boldsymbol{v}_{-i}}\left[U_i^{\mathcal{M}}(\boldsymbol{v}_i, b_I, v_{-I}) - U_i^{\mathcal{M}}(\boldsymbol{v}_i, v_I, v_{-I})\right] \\ &\quad + \mathbb{E}_{\boldsymbol{v}_{-i}}\left[\sum_{t=m_{i,1}+1}^{m_{i,1}+m_{i,2}}\left[u^{K_2}\left(v_{i,t}, P(b_I, v_{-I})\right) - u^{K_2}\left(v_{i,t}, P(v_I, v_{-I})\right)\right]\right] \\ &\leq 0 + \mathbb{E}_{\boldsymbol{v}_{-i}}\left[m_2(P(v_I, v_{-I}) - P(b_I, v_{-I}))\right] \leq \mathbb{E}_{\boldsymbol{v}_{-i}}\left[m_2 D\delta^{\text{worst}}_{m_1}(v_{-I})\right] = m_2 D\Delta^{\text{worst}}_{T_1K_1,m_1}, \end{aligned}$$

which indicates that truthful bidding is an ϵ-BNE, where $\epsilon = m_2 D\Delta^{\text{worst}}_{T_1K_1,m_1}$. This concludes the proof for bounded distributions. □

Remark. The proof for MHR distributions is trickier since the difference $P(v_I, v_{-I}) - P(b_I, v_{-I})$ can be unbounded. Intuitively, the probability that $P(v_I, v_{-I})$ will be higher than $(1 + o(1))v^*$ (v^* is defined in the statement of the lemma) is exponentially small, and the main effort is to show that the expected difference $P(v_I, v_{-I}) - P(b_I, v_{-I})$ multiplied by this exponentially small probability is negligible. Full details are given Appendix D.2.

C.2 Proof of Theorem 2.2

The bound on approximate truthfulness, i.e., ϵ_1, follows from Theorem 2.1 and Theorem 1.3, where we first obtain the bound on $\Delta^{\text{worst}}_{T_1K_1,m_1}$ from Theorem 1.3 by setting $N = T_1K_1$ and $m = m_1$ and then replace m_2m_1 with $O(\overline{m}^2)$ for MHR distribution and replacing $m_2m_1^{2/3}$ with $O(\overline{m}^{5/3})$ for bounded distribution.

It remains to consider revenue, where we will use sample complexity results to obtain the convergence rate of the revenue loss, i.e., ϵ_2. Let rev_1, rev_2 be the expected revenues of the two phases in $\text{TP}(\mathcal{M}, P; \boldsymbol{T}, \boldsymbol{m}, \boldsymbol{K}, S)$, and rev^* be the revenue obtained by using Myerson's auction in all rounds, i.e., $rev^* = T_1\text{Mye}^{K_1} + T_2\text{Mye}^{K_2}$ where Mye^K is the revenue of Myerson's auction with K i.i.d. bidders from F. For rev_1, we only have $rev_1 \geq 0$ since we do not any revenue guarantee for the arbitrary first-phase mechanism $\mathcal{M}$. Now consider rev_2, let $r^{K_2}(p)$ denote the expected revenue of a second price auction with reserve price p. Since the values in the two phases are independent, we have

$$rev_2 = T_2 \cdot \mathbb{E}_{v_1,\ldots,v_{T_1K_1}\sim F}\left[r^{K_2}(\text{ERM}^c(v_1,\ldots,v_{T_1K_1}))\right].$$

We need to compare $r^{K_2}(\text{ERM}^c(v_1,\ldots,v_{T_1K_1}))$ with Mye^{K_2}. Since bidders have i.i.d. regular value distributions, Myerson's auction is exactly the second price auction with reserve price $p = v^*$. When $K_2 = 1$, Myerson's auction becomes a post-price auction. Let $\epsilon^{sample}(\cdot)$ be the inverse function of the required number of samples for ERM^c to guarantee $(1-\epsilon^{sample})$-optimal revenue (as obtained in Huang et al. [23]) in the posted-price auction, i.e., the expected revenue of a one-bidder auction with a posted price p determined by ERM^c with N samples is at least $(1-\epsilon^{sample}(N))$ times the optimal expected revenue. Then for the one-bidder case, we have

$$rev_2 = T_2(1-\epsilon^{sample}(T_1K_1))\text{Mye}^1.$$

For general K_2, while the sample complexity literature does not analyze the revenue of the same reserve price $p = \text{ERM}^c(v_1,\ldots,v_{T_1K_1})$ in a second price auction with $K_2 \geq 2$ bidders, we are able to generalize the existing revenue guarantee to the case of multiple bidders (and multiple units) under the assumption that the distribution is regular. The generalization is made by the following lemma, which we believe is of independent interest:

Lemma C.1. *For any regular distribution F, if the expected revenue of a posted price auction with price p and with one bidder whose value is drawn from F is $(1-\epsilon)$-optimal, then the revenue of a Vickrey auction with reserve price p selling at most $k \geq 1$ units of a item to $K \geq 2$ i.i.d. unit-demand bidders with values from F is also $(1-\epsilon)$-optimal.*

The proof is in Appendix C.3. Thus for $K_2 \geq 2$, we also have: $rev_2 = T_2(1-\epsilon^{sample}(T_1K_1))\text{Mye}^{K_2}$.

Finally,

$$\begin{aligned}
1-\epsilon_2 = \frac{rev_1 + rev_2}{rev^*} &\geq \frac{0 + T_2\text{Mye}^{K_2}\cdot(1-\epsilon^{sample}(T_1K_1))}{T_1\text{Mye}^{K_1} + T_2\text{Mye}^{K_2}} \\
&\geq 1 - \frac{T_1\text{Mye}^{K_1}}{T_1\text{Mye}^{K_1} + T_2\text{Mye}^{K_2}} - \epsilon^{sample}(T_1K_1) \\
&\geq 1 - \frac{T_1}{T} - \epsilon^{sample}(T_1K_1) \qquad (\text{Mye}^{K_2} \geq \text{Mye}^{K_1} \text{ since } K_2 \geq K_1)
\end{aligned}$$

From Huang et al. [23], we know that for bounded distributions, $\epsilon^{sample}(N) = O(\sqrt{\frac{D\cdot\log N}{N}})$ when $c \leq \frac{1}{2D}$, and for MHR distributions (MHR implies regularity), $\epsilon^{sample}(N) = O([\frac{\log N}{N}]^{\frac{2}{3}})$ when $c \leq \frac{1}{4e}$. This implies the bounds on ϵ_2 as stated in the theorem, and concludes the proof.

C.3 Proof of Lemma C.1

It's more convenient to work in the quantile space. Let $v(q)$, $R(q) = qv(q)$ be the value curve and revenue curve of F. It's well-known that the derivative $R'(q)$ equals to the virtual value $\phi(v(q)) = v - \frac{1-F(v)}{f(v)}$ and by Myerson's Lemma, the expected revenue with allocation rule $x(\cdot)$ equals to the virtual surplus:

$$rev = \sum_{i=1}^{K}\mathbb{E}[R'(q)x_i(q)].$$

Let $x^{K,k}(q)$ be the allocation to (the probability of winning of) a bidder whose value has quantile q in a Vickrey auction selling k units to K bidders without reserve price. Specifically, $x^{1,1}(q) = 1$; for general K,

$x^{K,1}(q) = (1-q)^{K-1}$; for general K, k, $x^{K,k}(q) = \sum_{i=0}^{k-1} \binom{K-1}{i} q^i (1-q)^{K-1-i}$. With reserve price p_0, let $q_0 = q(p_0)$, then the allocation becomes $x_{q_0}^{K,k}(q) = x^{K,k}(q)$ for $q < q_0$ and $x_{q_0}^{K,k}(q) = 0$ otherwise. So the revenue of p_0 is

$$rev(K,k) = K \int_0^{q_0} R'(q) x^{K,k}(q) \mathrm{d}q,$$

and the optimal revenue is:

$$rev^*(K,k) = K \int_0^{q^*} R'(q^*) x^{K,k}(q) \mathrm{d}q,$$

where q^* satisfies: $R'(q) \geq 0, \forall q \leq q^*$ and $R'(q) \geq 0, \forall q \geq q^*$. And define:

$$loss(K,k) = rev^*(K,k) - rev(K,k) = K \int_{q_0}^{q^*} R'(q) x^{K,k}(q) \mathrm{d}q.$$

Since p_0 is $(1-\epsilon)$-optimal with $K=1, k=1$ (the posted-price auction), we have:

$$loss(1,1) \leq \epsilon \cdot rev^*(1,1).$$

Now for general K, k:

- If $q_0 > q^*$. The loss:

$$\begin{aligned} loss(K,k) &= K \int_{q^*}^{q_0} -R'(q) x^{K,k}(q) \mathrm{d}q \\ &\leq K \int_{q^*}^{q_0} -R'(q) x^{K,k}(q^*) \mathrm{d}q \\ &= K x^{K,k}(q^*) \int_{q^*}^{q_0} -R'(q) \mathrm{d}q = K x^{K,k}(q^*) loss(1,1), \end{aligned}$$

  since $x^{K,k}(q)$ is non-increasing in q (actually, the monotonicity of $x^{K,k}(q)$ is the only property that is used throughout the proof), and the optimal revenue:

$$\begin{aligned} rev^*(K,k) &\geq K \int_0^{q^*} R'(q) x^{K,k}(q^*) \mathrm{d}q \\ &= K x^{K,k}(q^*) \int_0^{q^*} R'(q) \mathrm{d}q = K x^{K,k}(q^*) rev^*(1,1), \end{aligned}$$

  which gives: $\frac{loss(K,k)}{rev^*(K,k)} \leq \frac{loss(1,1)}{rev^*(1,1)} \leq \epsilon$.

- If $q_0 < q^*$. The loss:

$$loss(K,k) = K \int_{q_0}^{q^*} R'(q) x^{K,k}(q) \mathrm{d}q \leq K x^{K,k}(q_0) loss(1,1),$$

  and the optimal revenue:

$$\begin{aligned} rev^*(K,k) &= K \int_0^{q_0} R'(q) x^{K,k}(q) \mathrm{d}q + K \int_{q_0}^{q^*} R'(q) x^{K,k}(q) \mathrm{d}q \\ &\geq K \int_0^{q_0} R'(q) x^{K,k}(q_0) \mathrm{d}q + K \int_{q_0}^{q^*} R'(q) x^{K,k}(q) \mathrm{d}q \\ &= K x^{K,k}(q_0) rev(1,1) + loss(K,k), \end{aligned}$$

  which gives:

$$\begin{aligned} \frac{loss(K,k)}{rev^*(K,k)} &\leq \frac{loss(K,k)}{K x^{K,k}(q_0) rev(1,1) + loss(K,k)} = \frac{1}{\frac{K x^{K,k}(q_0) rev(1,1)}{loss(K,k)} + 1} \\ &\leq \frac{1}{\frac{rev(1,1)}{loss(1,1)} + 1} \leq \frac{1}{\frac{1-\epsilon}{\epsilon} + 1} = \epsilon. \end{aligned}$$

C.4 Perfect Bayesian Equilibrium

Here we consider the setting of T-round repeated auctions where each auction contains $K \geq 1$ bidders and each bidder participates in at most $\overline{m}$ rounds of auctions. We use $\boldsymbol{v}_i = (v_i^t)$ to denote bidder i's profile of values, where v_i^t is her value at round t if she participates in that round. Similarly denote by $\boldsymbol{b}_i = (b_i^t)$ the bids of bidder i. Values are i.i.d. samples from some distribution F.

In repeated auctions, the seller can adjust the mechanism dynamically based on the bidding history of buyers, and buyers can use historical information to adjust their bidding strategies. The solution concept of an ϵ-perfect Bayesian equilibrium (ϵ-PBE) captures this dynamic nature. For each bidder i, we use h_i^t to denote the history she can observe at the start of round t. For example, h_i^t includes her bid $b_i^{t'}$, whether she receives the item, how much she pays, etc, at round $t' < t$ if she participates in round t'. We assume that bidder i cannot observe the bids in the auctions she does not participate in. We allow bidder i to anticipate her values in future rounds, so she can make decision on her entire value profile $\boldsymbol{v}_i = (v_i^t)$. Bidder i's strategy is thus denoted by $\sigma_i = (\sigma_i^t)$ where σ_i^t maps $\boldsymbol{v}_i$ and h_i^t to a bid $b_i^t = \sigma_i^t(\boldsymbol{v}_i, h_i^t)$. Let $U_i^{[t:T]}(\sigma; \boldsymbol{v}_i, h_i^t)$ be the total expected utility of bidder i in rounds $t, t+1, \ldots, T$, given her value profile $\boldsymbol{v}_i$, the history h_i^t at round t, and bidders playing $\sigma = (\sigma_i, \sigma_{-i})$.

Definition C.2. *A profile of strategy $\sigma = (\sigma_i, \sigma_{-i})$ is an ϵ-*perfect Bayesian equilibrium (ϵ-PBE) *if for each bidder i, each round t, any history h_i^t, any values $\boldsymbol{v}_i$, the strategy σ_i approximately maximizes bidder i's expected utility from round t to round T up to ϵ error, i.e., $U_i^{[t:T]}(\sigma; \boldsymbol{v}_i, h_i^t) \geq U_i^{[t:T]}(\sigma_i', \sigma_{-i}; \boldsymbol{v}_i, h_i^t) - \epsilon$ for any alternative strategy σ_i'.*

Definition C.3. *The seller's mechanism (or auction learning algorithm) is ϵ-*perfect Bayesian incentive-compatible (ϵ-PBIC) *if truthful bidding (i.e., $\sigma_i^t(\boldsymbol{v}_i, h_i^t) = v_i^t$) is an ϵ-PBE.*

We emphasize that the expected utility $U_i^{[t:T]}(\sigma; \boldsymbol{v}_i, h_i^t)$ is conditioned on h_i^t. This is because, the history h_i^t which includes the allocation of item and the payment of bidder i can leak information about other bidders' bids (or values). Other bidders' bids will influence the mechanism the seller will use in future rounds. Thus, based on this information, bidder i can update her belief about other bidders' bids and the seller's choice of mechanisms by Bayesian rule, then she can compute her expected utility on her updated belief.

PBIC of the two-phase ERM algorithm. As discussed, the two-phase ERM algorithm is a learning algorithm that learns approximately revenue-optimal auctions in an approximately incentive-compatible way against strategic bidders in repeated auctions. The algorithm, obtained by adopting the two-phase model with ERM as the price learning function and setting $K_1 = K_2 = K$, works as follows:

- in the first T_1 rounds, run any truthful, prior-independent auction $\mathcal{M}$, e.g., the second price auction with no reserve;
- in the later $T_2 = T - T_1$ rounds, run second price auction with reserve $p = \mathrm{ERM}^c(b_1, \ldots, b_{T_1 K})$ where $b_1, \ldots, b_{T_1 K}$ are the bids from the first T_1 auctions.

T_1 and c are adjustable parameters of the two-phase ERM algorithm.

Theorem C.4. *The two-phase ERM algorithm is ϵ-PBIC, where,*

- *for any bounded F, $\epsilon = \overline{m} D \Delta^{\mathrm{worst}}_{T_1 K, \overline{m} K}$, and*
- *for any MHR F, if $\frac{\overline{m}}{T_1} \leq c \leq \frac{1}{4e}$ and $\overline{m} K = o(\sqrt{T_1 K})$, then $\epsilon = O\left(\overline{m} v^* \Delta^{\mathrm{worst}}_{T_1 K, \overline{m} K}\right) + O\left(\frac{\overline{m} v^*}{\sqrt{T_1 K}}\right)$, where $v^* = \arg\max_v \{v[1 - F(v)]\}$.*

The constants in big O's are independent of F and c.

Combining with the bounds on $\Delta^{\mathrm{worst}}_{N,m}$ in Theorem 1.3, we have

Corollary C.5. *The two-phase ERM algorithm is ϵ_1-PBIC, where,*

- *for any bounded F,*
$$\epsilon_1 = O\left(\log^2(T_1 K) \sqrt[3]{\frac{D^{11} K \overline{m}^5}{T_1}}\right)$$
if $\frac{\overline{m}}{T_1} \leq c \leq \frac{1}{2D}$ and $\overline{m} K = o(\sqrt{T_1 K})$;
- *for any MHR F,*
$$\epsilon_1 = O\left(\log^3(T_1 K) v^* \overline{m}^2 \sqrt{\frac{K}{T_1}},\right)$$
if $\frac{\overline{m}}{T_1} \leq c \leq \frac{1}{4e}$ and $\overline{m} K = o(\sqrt{T_1 K})$.

The constants in big O's are independent of F and c.

The guarantee of $(1 - \epsilon_2)$ revenue optimality is the same as Theorem 2.2:

- *For bounded and regular distribution,* $\epsilon_2 = O\left(\frac{T_1}{T} + \sqrt{\frac{D\cdot\log(T_1K)}{T_1K}}\right)$.
- *For MHR distribution,* $\epsilon_2 = O\left(\frac{T_1}{T} + \left[\frac{\log(T_1K)}{T_1K}\right]^{\frac{2}{3}}\right)$.

In the rest of this section we prove Theorem C.4 for bounded distributions. The proof is similar to that of Theorem 2.1 except that we need to consider the effect of history h_i^t on the conditional distribution of the values of other bidders when considering perfect Bayesian equilibrium. The extension to MHR distributions is similar to the extension of Theorem 2.1 to MHR distributions (discussed in Appendix D.2) and hence omitted.

Proof of Theorem C.4 for bounded distributions. Assume that other bidders bid truthfully and bidder i deviates from truthful bidding to other strategy, consider the increase of bidder i's total expected utility from round t to T, for each t, given any history h_i^t and any values $\boldsymbol{v}_i$. If $t > T_1$, then the auctions in t and later rounds are in the second phase and never change due to the deviation of bidder i, thus deviation does not increase her utility.

Then we consider $t \le T_1$. Strategic bidding does not increase bidder i's utility in rounds $t, t+1, \ldots, T_1$ because these rounds are in the first phase and the mechanism in the first phase is truthful. Thus, strategic bidding can increase her utility only in the second phase. Let $t' > T_1$ be a second-phase round in which she participates. The auction at round t' is a second-price auction with reserve price determined by ERM^c from bids in rounds 1 to T_1. Denote by $\boldsymbol{v}^{[1:T_1]}$ the values of all bidders in rounds 1 to T_1. If bidder i bid truthfully, then the reserve price at round t' is $p_1 = \mathrm{ERM}^c(\boldsymbol{v}^{[1:T_1]})$. Let $A \subseteq [1:T_1]$ be the set of rounds in which bidder i participates from round 1 to round T_1. Then we can partition $\boldsymbol{v}^{[1:T_1]}$ into two parts: $\boldsymbol{v}^A$ and $\boldsymbol{v}^{[1:T_1]\setminus A}$, where $\boldsymbol{v}^A$ denotes bidders' values in the rounds in A, and $\boldsymbol{v}^{[1:T_1]\setminus A}$ denotes the values in the rounds not in A. There are $|A|K$ values in $\boldsymbol{v}^A$, $|A|$ of which are bidder i's values. By deviating, bidder i can change her values in $\boldsymbol{v}^A$ to some arbitrary bids. We denote by $\boldsymbol{b}^A$ the bids of bidder i and the values of other bidders in $\boldsymbol{v}^A$. After deviation, the reserve price is changed to $p_2 = \mathrm{ERM}^c(\boldsymbol{b}^A, \boldsymbol{v}^{[1:T_1]\setminus A})$. By (14), the increase of bidder i's utility due to the change of reserve price is at most $p_1 - p_2$, which is further upper-bounded by

$$\begin{aligned} p_1 - p_2 &= \mathrm{ERM}^c(\boldsymbol{v}^A, \boldsymbol{v}^{[1:T_1]\setminus A}) - \mathrm{ERM}^c(\boldsymbol{b}^A, \boldsymbol{v}^{[1:T_1]\setminus A}) \\ &\le \mathrm{ERM}^c(\boldsymbol{v}^A, \boldsymbol{v}^{[1:T_1]\setminus A}) \cdot \delta_I(\boldsymbol{v}^A, \boldsymbol{v}^{[1:T_1]\setminus A}) \\ &\le \mathrm{ERM}^c(\boldsymbol{v}^A, \boldsymbol{v}^{[1:T_1]\setminus A}) \cdot \delta_{|A|K}^{\text{worst}}(\boldsymbol{v}^{[1:T_1]\setminus A}) \\ &\le D \cdot \delta_{|A|K}^{\text{worst}}(\boldsymbol{v}^{[1:T_1]\setminus A}). \end{aligned}$$

We then argue that given any history h_i^t, $\boldsymbol{v}^{[1:T_1]\setminus A}$ are still i.i.d. samples from F, from bidder i's perspective. Note that bidder i does not participate in the auctions in rounds $[1:T_1]\setminus A$, and the auctions she does participate in before round t is prior-independent, which implies that the allocation of item and the payments of bidders in any round depend only on the bids of bidders in that round but not on any information like bids from other rounds. Moreover, other bidders' values across different rounds are independent. Therefore, the auctions bidder i participates in leaks no information about other bidders' values in rounds $[1:T_1]\setminus A$.

Therefore, the increase of bidder i's expected utility at round t' is at most

$$\begin{aligned} \mathbb{E}\left[p_1 - p_2 \mid h_i^t, \boldsymbol{v}_i\right] &\le \mathbb{E}\left[D \cdot \delta_{|A|K}^{\text{worst}}(\boldsymbol{v}^{[1:T_1]\setminus A}) \mid h_i^t, \boldsymbol{v}_i\right] \\ &= \mathbb{E}_{\boldsymbol{v}^{[1:T_1]\setminus A}\sim F}\left[D \cdot \delta_{|A|K}^{\text{worst}}(\boldsymbol{v}^{[1:T_1]\setminus A})\right] \\ &= D \cdot \Delta_{T_1K,|A|K}^{\text{worst}} \\ &\le D \cdot \Delta_{T_1K,\overline{m}K}^{\text{worst}}, \end{aligned}$$

where the last inequality is because $|A| \le \overline{m}$ and $\Delta_{N,m_1}^{\text{worst}} \ge \Delta_{N,m_2}^{\text{worst}}$ for $m_1 \ge m_2$.

Since bidder i participates in at most $\overline{m}$ auctions, the sum of increases of expected utility from round t to T is at most $\overline{m}D\Delta_{T_1K,\overline{m}K}^{\text{worst}}$. □

D Analysis for MHR Distributions

Recall that a distribution F is MHR if its hazard rate $\frac{f(x)}{1-F(x)}$ is monotone non-decreasing.

D.1 Properties of MHR Distributions

Recall that $R(q) = qv(q)$ is the revenue curve of distribution F, where $q(v) = 1 - F(v)$. And $q^* = \arg\max_q R(q)$ is the quantile of the optimal reserve price $v^* = \arg\max_v [1 - F(v)]v = v(q^*)$.

For MHR distributions, we first introduce a lemma which says that q^* is bounded away from 0 by a constant.

Lemma D.1 (Hartline et al. [22])**.** *Any MHR distribution has a unique q^*, and $q^* \geq \frac{1}{e}$.*

Moreover, the revenue curve decreases quadratically from q^*.

Lemma D.2 (Huang et al. [23], Lemma 3.3)**.** *For any MHR F, for any $0 \leq q \leq 1$, $R(q^*) - R(q) \geq \frac{1}{4}(q^* - q)^2 R(q^*)$.*

The following lemma shows that samples from an MHR distribution are rarely too large.

Lemma D.3. *Let F be an MHR distribution. Let $X = \max\{v_1, \ldots, v_N\}$ where $v_1, \ldots, v_N$ are N i.i.d. samples from F. For any $x \geq v^*$, we have $\Pr[X > x] \leq N e^{-x/v^*+1}$.*

Proof. Note that $1 - F(x) = \exp\left\{-\int_0^x \frac{f(v)}{1-F(v)} \mathrm{d}v\right\} \leq \exp\left\{-\int_{v^*}^x \frac{f(v)}{1-F(v)} \mathrm{d}v\right\}$. By the definition of v^* we know $(v^*[1 - F(v^*)])' = 0$, or $\frac{f(v^*)}{1-F(v^*)} = \frac{1}{v^*}$. By the definition of MHR, we have $\frac{f(x)}{1-F(x)} \geq \frac{1}{v^*}$ for any $x \geq v^*$, thus

$$1 - F(x) \leq \exp\left\{-\int_{v^*}^x \frac{1}{v^*} \mathrm{d}v\right\} = \exp\left\{-\frac{x - v^*}{v^*}\right\}.$$

Then the lemma follows from a simple union bound:

$$\Pr[X > x] = \Pr[\exists i, v_i > x] \leq N[1 - F(x)] \leq N \exp\left\{-\frac{x}{v^*} + 1\right\}.$$

□

We will use above lemmas to prove some further lemmas which characterize the behavior of ERM^c on samples from a MHR distribution, where c can be any value between m/N and $1/(2e)$. The samples we consider consist of m copies of $+\infty$, denoted by v_I, and $N - m$ random draws from F. We sort the samples non-increasingly and use

$$v_{-I} = (v_{m+1} \geq v_{m+2} \geq \cdots \geq v_N)$$

to denote the random draws. Let $q_{m+1} \leq q_{m+2} \leq \cdots \leq q_N$ denote their quantiles where $q_j = q(v_j)$.

Lemma D.4. *Let F be an MHR distribution. Suppose $m = o(\sqrt{N})$. Fix m values v_I to be $+\infty$, and randomly draw $N - m$ values v_{-I} from F. Let $k^* = \arg\max_{i > cN}\{i v_i\}$, i.e., the index selected by ERM^c, where $\frac{m}{N} \leq c \leq \frac{1}{2e}$. Then we have*

$$R(q_{k^*}) \geq \left(1 - O\left(\sqrt{\frac{\log N}{N}}\right)\right) R(q^*),$$

with probability at least $1 - O\left(\frac{1}{N}\right)$.

Proof. Let $\gamma \stackrel{\text{def}}{=} 2\sqrt{\frac{4\ln(2(N-m))}{N-m}} + \frac{m}{N} = O\left(\sqrt{\frac{\log N}{N}}\right)$ as in Claim B.2. We have $|q_j - \frac{j}{N}| \leq \gamma$ for any $j > m$ with probability at least $1 - \frac{1}{N-m}$. We thus assume $|q_j - \frac{j}{N}| \leq \gamma$.

The intuition is follows: The product $j v_j$ divided by N approximates $R(q_j) = q_j v_j$ up to an $O(\gamma)$ error. Our proof consists of three steps: The first step is to show that with high probability, there must be some sample with quantile q_i that is very close to q^* so its revenue $R(q_i) \approx R(q^*) \approx \frac{i}{N} v_i$. The second step is to argue that all samples with quantile $q_j < \frac{1}{2e}$ are unlikely to be chosen by ERM^c because q_j is too small and the gap between q^* and $\frac{1}{2e}$ leads to a large loss in revenue, roughly speaking, $\frac{j}{N} v_j \approx R(q_j) < (1 - \frac{1}{4}(\frac{1}{2e})^2) R(q^*) \approx (1 - \Omega(1)) \frac{i}{N} v_i$. The final step is to show that if a quantile $q_j > \frac{1}{2e}$ is to be chosen by ERM^c, then it must have equally good revenue as q_i.

Formally:

1. Firstly, consider the quantile interval $[q^* - \gamma, q^*]$. Each random draw q_i, if falling into this interval, will satisfy:

$$\frac{i}{N} v_i \geq (q_i - \gamma) v_i \geq (q^* - 2\gamma) v_i \geq (q^* - 2\gamma) v^* \geq (1 - 2e\gamma) q^* v^*, \tag{15}$$

where the last but one inequality is because $q_i \leq q^*$ and the last one follows from $q^* \geq \frac{1}{e}$. The probability that no quantile falls into $[q^* - \gamma, q^*]$ is at most

$$(1 - \gamma)^{N-m} = \left(1 - O\left(\sqrt{\frac{\log N}{N}}\right)\right)^{N-m} = o(\frac{1}{N}).$$

2. For the second step, first note that the $q_i \in [q^* - \gamma, q^*]$ in the first step will be considered by ERM^c since $i \geq (q_i - \gamma)N \geq (q^* - 2\gamma)N \geq (\frac{1}{e} - 2\gamma)N > cN$. Then suppose ERM^c chooses another quantile q_j instead of q_i, we must have

$$\frac{j}{N} v_j \geq \frac{i}{N} v_i. \tag{16}$$

We will show that such probability is small if $q_j < \frac{1}{2e} + \gamma$. Pick a threshold quantile $\frac{1}{T}$ where $T = N^{1/4}$. Consider two cases:

- If $0 \leq q_j < \frac{1}{T}$. We argue that ERM^c picks q_j with probability at most $o(\frac{1}{N})$. Note that

$$\frac{j}{N} v_j \leq (q_j + \gamma) v_j \leq (\frac{1}{T} + \gamma) v_j, \tag{17}$$

together with (16) and (15), we obtain $(\frac{1}{T} + \gamma) v_j \geq (1 - 2e\gamma) q^* v^*$, implying

$$v_j \geq \frac{1 - 2e\gamma}{e} \frac{T v^*}{1 + T\gamma} = \Omega(T v^*),$$

since $T\gamma = O\left(\frac{\sqrt{\log N}}{N^{1/4}}\right) \to 0$. According to Lemma D.3, the probability that there exists $v_j > \Omega(T v^*)$ is at most

$$N \exp\left\{-\frac{\Omega(T v^*)}{v^*} + 1\right\} = o(\frac{1}{N}).$$

- If $\frac{1}{T} \leq q_j < \frac{1}{2e} + \gamma$. We argue that ERM^c will never choose such q_j. Note that

$$\frac{j}{N} v_j \leq (q_j + \gamma) v_j \leq (1 + T\gamma) q_j v_j, \tag{18}$$

together with (16) and (15), we obtain $(1 + T\gamma) q_j v_j > (1 - 2e\gamma) q^* v^*$. Then by Lemma D.2,

$$\frac{1 - 2e\gamma}{1 + T\gamma} \leq \frac{q_j v_j}{q^* v^*} \leq 1 - \frac{1}{4}(q_j - q^*)^2 \leq 1 - \frac{1}{4}(\frac{1}{2e} - \gamma)^2. \tag{19}$$

However, the left hand side of (19) approaches to 1 since γ and $T\gamma$ approach 0 while the right hand side is strictly less than 1, a contradiction. So this case never happens.

3. Finally, if $q_j \geq \frac{1}{2e} + \gamma$. We argue that if ERM^c picks q_j instead of q_i, then $R(q_j)$ approximates $R(q_i)$ well, satisfying the conclusion in the lemma. This is because

$$\begin{aligned} R(q_j) = q_j v_j &\geq (\frac{j}{N} - \gamma) v_j \\ &\geq (1 - 2e\gamma) \frac{j}{N} v_j && \frac{j}{N} \geq q_j - \gamma \geq \frac{1}{2e} \\ &\geq (1 - 2e\gamma) \frac{i}{N} v_i && \text{Eq. (16)} \\ &\geq (1 - 2e\gamma)(1 - 2e\gamma) q^* v^* && \text{Eq. (15)} \\ &= (1 - O(\gamma)) R(q^*). \end{aligned}$$

Combining above three steps and the event in the beginning of the proof, we have $R(q_{k^*}) \geq (1 - O(\sqrt{\frac{\log N}{N}})) R(q^*)$ except with probability at most

$$\frac{1}{N - m} + o(\frac{1}{N}) + o(\frac{1}{N}) = O(\frac{1}{N}).$$

□

Lemma D.5. *Let F be an MHR distribution. Suppose $m = o(\sqrt{N})$. Fix m values v_I to be $+\infty$, and randomly draw $N - m$ values v_{-I} from F. Let $k^* = \arg\max_{i > cN} \{i v_i\}$, i.e., the index selected by ERM^c, where $\frac{m}{N} \leq c \leq \frac{1}{2e}$. Let $\epsilon = \sqrt[4]{\frac{\log N}{N}}$. Then with probability at least $1 - O\left(\frac{1}{N}\right)$, the following inequalities hold:*

1. *$q_{k^*} \geq q^* - O(\epsilon)$;*
2. *$k^* \geq [q^* - O(\epsilon)]N > \frac{1}{2e} N$;*
3. *$v_{k^*} \leq [1 + O(\epsilon)] v^*$.*

Proof. For inequality (1), by Lemma D.2 and Lemma D.4, with probability at least $1 - O(\frac{1}{N})$, we have

$$\frac{1}{4}(q_{k^*} - q^*)^2 \le \frac{R(q^*) - R(q_{k^*})}{R(q^*)} \le O\left(\sqrt{\frac{\log N}{N}}\right).$$

Taking the square root, we obtain $q_{k^*} \ge q^* - O\left(\sqrt[4]{\frac{\log N}{N}}\right)$.

Assume that (1) holds. To prove (2), note that by Claim B.2, we have $\frac{k^*}{N} \ge q_{k^*} - O\left(\sqrt{\frac{\log N}{N}}\right) \ge q^* - O(\epsilon)$ except with probability at most $O(\frac{1}{N})$, and $q^* > \frac{1}{e}$.

Finally, inequality (3) follows from

$$\frac{v_{k^*}}{v^*} = \frac{R(q_{k^*})}{q_{k^*}} \frac{q^*}{R(q^*)} \le 1 \cdot \frac{q^*}{q_{k^*}} \le \frac{q^*}{q^* - O(\epsilon)} = 1 + \frac{O(\epsilon)}{q^* - O(\epsilon)} \le 1 + O(e\epsilon).$$

□

D.2 Detailed Proof of Theorem 2.1 for MHR Distributions

Let $\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) = U_i^{\mathrm{TP}}(\boldsymbol{v}_i, b_I, v_{-I}) - U_i^{\mathrm{TP}}(\boldsymbol{v}_i, v_I, v_{-I})$. Similar to the proof for bounded distributions, we have for any $\boldsymbol{v}_i, b_I, v_{-I}$,

$$\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \le m_2 \cdot \left(\mathrm{ERM}^c(v_I, v_{-I}) - \mathrm{ERM}^c(b_I, v_{-I})\right) \le m_2 \cdot \mathrm{ERM}^c(v_I, v_{-I}) \cdot \delta_{m_1}^{\mathrm{worst}}(v_{-I}).$$

By Claim A.2, we have $\mathrm{ERM}^c(v_I, v_{-I}) \le \mathrm{ERM}^c(\overline{v}_I, v_{-I})$ where $\overline{v}_I$ can be any m_1 values (e.g., $+\infty$) that are greater than the maximal value in v_{-I}, when $c \ge \frac{m_1}{T_1 K_1}$.

Let $N = T_1 K_1$, define two threshold prices $T_1 = \sqrt{N} v^*$ and $T_2 = [1 + O(\epsilon)]v^*$ where $\epsilon = \sqrt[4]{\frac{\log N}{N}}$ as in Lemma D.5. Note that for sufficiently large N, $T_1 > T_2$. With the random draw of v_{-I} from F, denote the random variable $\mathrm{ERM}^c(\overline{v}_I, v_{-I})$ by P, we have:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{v}_{-i}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I})\right] &= \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P \le T_2\right] \cdot \Pr[P \le T_2] \\
&\quad + \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid T_2 < P \le T_1\right] \cdot \Pr[T_2 < P \le T_1] \\
&\quad + \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P > T_1\right] \cdot \Pr[P > T_1] \\
&\stackrel{\text{def}}{=} \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3.
\end{aligned} \tag{20}$$

1. For the first term $\mathbb{E}_1$,

$$\begin{aligned}
\mathbb{E}_1 &= \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P \le T_2\right] \cdot \Pr[P \le T_2] \\
&\le \mathbb{E}_{v_{-I}}\left[m_2 \cdot P \cdot \delta_{m_1}^{\mathrm{worst}}(v_{-I}) \mid P \le T_2\right] \cdot \Pr[P \le T_2] \\
&\le m_2 \cdot T_2 \cdot \mathbb{E}_{v_{-I}}\left[\delta_{m_1}^{\mathrm{worst}}(v_{-I}) \mid P \le T_2\right] \cdot \Pr[P \le T_2] \\
&\le m_2 \cdot [1 + O(\epsilon)]v^* \cdot \mathbb{E}_{v_{-I}}\left[\delta_{m_1}^{\mathrm{worst}}(v_{-I})\right] \\
&= O\left(m_2 \cdot v^* \cdot \Delta_{N,m_1}^{\mathrm{worst}}\right).
\end{aligned}$$

2. For the second term, we claim that $\mathbb{E}_2 = O(\frac{m_2 v^*}{\sqrt{N}})$.

   By Lemma D.5, we have $\Pr[P > [1 + O(\epsilon)]v^*] \le O(\frac{1}{N})$. Therefore,

$$\begin{aligned}
\mathbb{E}_2 &= \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid T_2 < P \le T_1\right] \cdot \Pr[T_2 < P \le T_1] \\
&\le \mathbb{E}_{v_{-I}}\left[m_2 \cdot P \cdot 1 \mid T_2 < P \le T_1\right] \cdot \Pr[T_2 < P \le T_1] \\
&\le m_2 \cdot T_1 \cdot \Pr[P > T_2] \\
&\le m_2 \cdot \sqrt{N} v^* \cdot O\left(\frac{1}{N}\right) \\
&= O\left(\frac{m_2 v^*}{\sqrt{N}}\right).
\end{aligned}$$

3. For the third term, we claim that $\mathbb{E}_3 = o(\frac{m_2 v^*}{N})$.
Let B be the upper bound on the support of F (B can be $+\infty$). Let $F_P(x)$ be the distribution of P. For convenience, suppose it is continuous and has density $f_P(x)$. We have:

$$\begin{aligned}\mathbb{E}_3 &= \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P > T_1\right] \cdot \Pr[P > T_1] \\ &\le \mathbb{E}_{v_{-I}}\left[m_2 \cdot P \cdot 1 \mid P > T_1\right] \cdot \Pr[P > T_1] \\ &= m_2 \cdot \mathbb{E}_{v_{-I}}\left[P \mid P > T_1\right] \cdot \Pr[P > T_1] \\ &= m_2 \cdot \int_{T_1}^{B} x f_P(x) \mathrm{d}x \\ &= m_2 \cdot \left(\int_{T_1}^{B} [1 - F_P(x)] \mathrm{d}x + T_1[1 - F_P(T_1)] \right).\end{aligned}$$

Let $\max\{v_{-I}\}$ denote the maximum value in the $N - m_1$ samples v_{-I}. By Lemma D.3, we have for any $x \ge v^*$,

$$1 - F_P(x) = \Pr[P > x] \le \Pr[\max\{v_{-I}\} > x] \le N e^{-\frac{x}{v^*} + 1}.$$

Thus,

$$\begin{aligned}\int_{T_1}^{B} [1 - F_P(x)] \mathrm{d}x + T_1[1 - F_P(T_1)] &\le v^* N e^{-\frac{T_1}{v^*} + 1} + T_1 N e^{-\frac{T_1}{v^*} + 1} \\ &= v^* N (1 + \sqrt{N}) e^{-\sqrt{N} + 1} \\ &= o(\frac{v^*}{N}),\end{aligned}$$

as desired.

Combining the three items,

$$\mathbb{E}_{\boldsymbol{v}_{-i}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I})\right] = O\left(m_2 v^* \Delta^{\mathrm{worst}}_{N, m_1}\right) + O\left(\frac{m_2 v^*}{\sqrt{N}}\right).$$

D.3 An Improved Bound on Incentive-Awareness Measure for MHR Distributions

Here we improve the upper bound on $\Delta^{\mathrm{worst}}_{N,m}$ for MHR distributions by proving:

Lemma D.6 (Tigher bound for MHR distributions)**.** *Moreover, if F is MHR, let $d = \frac{1}{2e}$, and suppose $\frac{m}{N} \le c \le \frac{1}{4e}$, we have*

$$\Delta^{\mathrm{worst}}_{N,m} \le O\left(\frac{m}{d^{7/2}} \frac{\log^3 N}{\sqrt{N}}\right) + \Pr[\mathrm{E}].$$

The main idea is to limit the range of the quantile q_j of the "bad value" v_j in $\mathrm{Bad}(\eta_t, \theta_t)$ in Lemma B.5. Recall that in the proof of Lemma B.5 we assume q_j can take any value in $[0, 1]$, divide $[0, 1]$ into $O(1/h)$ intervals (as in (11)), and take a union bound to upper-bound the probability that a bad q_j exists. For MHR distributions, however, we will show that $q_j v_j$ is a $(1 - O(\sqrt{\frac{\log N}{N}}))$ approximation to $R(q^*)$, thus we can use Lemma D.2 to reduce the possible range of q_j from 1 to $O(\sqrt[4]{\frac{\log N}{N}})$.

D.3.1 Proof of Lemma D.6

We repeat the argument for Lemma B.1 until Claim B.2, before which we have:

$$\Delta^{\mathrm{worst}}_{N,m} \le \int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}] \mathrm{d}\eta \; + \; \Pr[\mathrm{E}]. \tag{21}$$

Let $\gamma = O(\sqrt{\frac{\log N}{N}})$ be the upper bound on $|q_j - \frac{j}{N}|$ in Conc. With the random draw of $N - m$ samples v_{-I} from F (and assume other m samples $\overline{v}_I$ are equal to $\max\{v_{-I}\}$), we have $|q_j - \frac{j}{N}| < \gamma$ for any $j > m$ with probability at least $1 - \frac{1}{N-m}$. Moreover, by Lemma D.4 and Lemma D.5, with probability at least $1 - O(\frac{1}{N})$ we have $R(q_{k^*}) = q_{k^*} v_{k^*} \ge (1 - \epsilon) R(q^*)$ and $q_{k^*} \ge q^* - O(\sqrt{\epsilon}) > \frac{1}{2e}$, where $\epsilon = O(\sqrt{\frac{\log N}{N}})$. Combine the above two inequalities with Conc and denote the combined event by Conc$'$, i.e.,

$$\mathrm{Conc}' \stackrel{\mathrm{def}}{=} \mathrm{Conc} \wedge [R(q_{k^*}) \ge (1 - \epsilon) R(q^*)] \wedge \left[q_{k^*} \ge q^* - O(\sqrt{\epsilon}) > \frac{1}{2e} \right].$$

We have $\Pr[\overline{\text{Conc}'}] \le O(\frac{1}{N})$. Re-define $G(\eta) = \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\text{E}} \wedge \text{Conc}']$, and re-write (4):

$$\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\text{E}}] \le G(\eta) + O\left(\frac{1}{N}\right). \tag{22}$$

The following steps of bounding $G(\eta) = \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\text{E}} \wedge \text{Conc}']$ are the same as before (in particular, Lemma B.4 in Lemma B.3), until upper-bounding $\Pr[\text{Bad}(\eta,\theta) \wedge \overline{\text{E}} \wedge \text{Conc}']$ (Lemma B.5), where we improve the bound by a factor of $\sqrt[4]{\frac{\log N}{N}}$.

Lemma D.7 (Improved Lemma B.5 for MHR distributions)**.** *Let* $d = \frac{1}{2e}$. *If* η *and* θ *are at least* $\Omega\left(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}}\right)$, *then* $\Pr[\text{Bad}(\eta,\theta) \wedge \overline{\text{E}} \wedge \text{Conc}'] = O\left(\sqrt[4]{\frac{\log N}{N}}\frac{m\log^2 N}{d^4\theta\sqrt{\eta^3 N}}\right)$.

Proof. The proof is the same as that of Lemma B.5 (in Appendix B.5), except that before dividing the quantile space $[0,1]$, we argue that the space to be divided can be shortened to $[q^* - O(\sqrt{\epsilon}), q^* + O(\sqrt{\epsilon})]$.

Consider the index j that is promised to exist in $\text{Bad}(\eta,\theta)$,

$$\begin{aligned}
R(q_j) &= q_j v_j \\
&\ge (\frac{j}{N} - \gamma)v_j && \text{Conc}' \\
&\ge \frac{k^*}{N}v_{k^*} - \frac{m}{N\theta}v_{k^*} - \gamma v_{k^*} && jv_j \ge k^* v_{k^*} - \frac{m}{\theta}v_{k^*} \text{ and } v_{j^*} \le v_{k^*} \\
&\ge (q_{k^*} - \gamma)v_{k^*} - \frac{m}{N\theta}v_{k^*} - \gamma v_{k^*} && \text{Conc}' \\
&= \left[q_{k^*} - (2\gamma + \frac{m}{N\theta})\right] v_{k^*} \\
&= \left[1 - \frac{2\gamma + \frac{m}{N\theta}}{q_{k^*}}\right] R(q_k^*) \\
&\ge \left[1 - 2e(2\gamma + \frac{m}{N\theta})\right](1-\epsilon)R(q^*) && q_{k^*} \ge \frac{1}{2e} \text{ and } R(q_{k^*}) \ge (1-\epsilon)R(q^*) \text{ in Conc}' \\
&= \left[1 - O\left(\sqrt{\frac{\log N}{N}}\right)\right] R(q^*) && \text{Definition of } \gamma \text{ and } \epsilon\text{, and } \theta = \Omega\left(m\sqrt{\frac{\log N}{N}}\right).
\end{aligned}$$

By Lemma D.2, we have

$$|q_j - q^*| \le 2\sqrt{O\left(\sqrt{\frac{\log N}{N}}\right)} = O\left(\sqrt[4]{\frac{\log N}{N}}\right).$$

Now we modify the analysis after Claim B.8. Consider those intervals I_l's with length h in (11) that intersect with

$$I_{q_j} = \left[q^* - O\left(\sqrt[4]{\frac{\log N}{N}}\right), q^* + O\left(\sqrt[4]{\frac{\log N}{N}}\right)\right].$$

There are at most $O\left(\frac{1}{h}\sqrt[4]{\frac{\log N}{N}}\right)$ such intervals and we denote the set of (indices of) those intervals by $\mathcal{L}$. The definitions of i_l^*, $i_{<(l+1)}^*$, A_l, $A_{<(l+1)}$, and W_l remain unchanged. By choosing the index l such that $q_j \in I_{l+2}$, we know that if the event $[\text{Bad}(\eta,\theta) \wedge \overline{\text{E}} \wedge \text{Conc}']$ holds then W_l must hold for some l such that $l+2 \in \mathcal{L}$. By Lemma B.9 we have $\Pr[W_l] \le O(\frac{H\log^2 N}{\sqrt{hd^3N}})$. Taking a union bound over l, we obtain

$$\Pr[\text{Bad}(\eta,\theta) \wedge \overline{\text{E}} \wedge \text{Conc}'] \le O\left(\frac{1}{h}\sqrt[4]{\frac{\log N}{N}} \cdot \frac{H\log^2 N}{\sqrt{hd^3N}}\right) = O\left(\sqrt[4]{\frac{\log N}{N}}\frac{m\log^2 N}{d\theta\sqrt{(d\eta)^3d^3N}}\right),$$

where the last equality is because $H = O(\frac{m}{d\theta})$ and $h = \Omega(d\eta)$ under the assumption that η and θ are at least $\Omega(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}})$. $\square$

Then we improve Lemma B.3 based on Lemma D.7.

Lemma D.8 (Improved Lemma B.3)**.** *If* η *is at least* $\Omega\left(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}}\right)$, *then* $G(\eta) = O\left(\frac{m\log^3 N}{d^4N^{3/4}}\frac{1}{\eta^{3/2}}\right)$.

Proof. Modify the end of Appendix B.3,

$$\begin{aligned}
\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}'] &\le \sum_{t=0}^{M} \Pr[\mathrm{Bad}(\eta_t, \theta_t) \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}'] && \text{Lemma B.4} \\
&= \sum_{t=0}^{M} O\left(\sqrt[4]{\frac{\log N}{N}} \frac{m \log^2 N}{d^4 \theta_t \sqrt{\eta_t^3 N}}\right) && \text{Lemma D.7} \\
&= \sum_{t=0}^{M} O\left(\sqrt[4]{\frac{\log N}{N}} \frac{m \log^2 N}{d^4 \sqrt{N}} \frac{\eta_{t+1}}{\eta \sqrt{\eta_t^3}}\right) && \text{Definition of } \theta_t \\
&= O\left(\sqrt[4]{\frac{\log N}{N}} \frac{m \log^2 N}{d^4 \sqrt{N}} \cdot \sum_{t=0}^{M} \frac{\eta_{t+1}}{\eta} \frac{1}{\eta_t^{3/2}}\right)
\end{aligned}$$

Note that because $\eta_t, \theta_t \ge \frac{\eta}{2}$, the condition of Lemma D.7 is satisfied when $\eta = \Omega(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}})$.

By Claim B.6, we can choose a sequence of $\{\eta_t\}$ such that

$$\sum_{t=0}^{M} \frac{\eta_{t+1}}{\eta} \frac{1}{\eta_t^{3/2}} = O\left(\frac{\log\log(N-m)}{\eta^{3/2}}\right).$$

Therefore,

$$\Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}} \wedge \mathrm{Conc}'] \le O\left(\frac{m \log^{2+1/4} N}{d^4 N^{1/2+1/4}} \cdot \frac{\log\log(N-m))}{\eta^{3/2}}\right) = O\left(\frac{m \log^3 N}{d^4 N^{3/4}} \frac{1}{\eta^{3/2}}\right).$$

□

We finish the proof of Lemma D.6 by computing the integral in (21). Let $C = \Theta\left(\frac{m \log^3 N}{d^4 N^{3/4}}\right)$ be the bound on $G(\eta)$ in Lemma D.8, and let $A = \Theta(\frac{m}{d}\sqrt{\frac{\log(N-m)}{N-m}}) = \Theta(\frac{m}{d}\sqrt{\frac{\log N}{N}})$ be the condition on the lower bound on η in Lemma D.8. Then $G(\eta) < \frac{C}{\eta^{3/2}}$ when $\eta > \max\{C^{2/3}, A\}$. If $C^{2/3} > A$, then we have

$$\begin{aligned}
\int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}] \mathrm{d}\eta &\le \int_0^1 \left(G(x) + O\left(\frac{1}{N}\right)\right) \mathrm{d}x && \text{by (22)} \\
&\le \int_0^{C^{\frac{2}{3}}} 1 \mathrm{d}x + \int_{C^{\frac{2}{3}}}^1 \frac{C}{x^{\frac{3}{2}}} \mathrm{d}x + O\left(\frac{1}{N}\right) \\
&= C^{\frac{2}{3}} + \frac{C}{-\frac{1}{2}} - \frac{C}{-\frac{1}{2}} C^{-\frac{1}{3}} + O\left(\frac{1}{N}\right) \\
&\le 3C^{\frac{2}{3}} + O\left(\frac{1}{N}\right) \\
&= O\left(\frac{m^{2/3}}{d^{8/3}} \frac{\log^2 N}{\sqrt{N}}\right) + O\left(\frac{1}{N}\right) \\
&= O\left(\frac{m^{2/3}}{d^{8/3}} \frac{\log^2 N}{\sqrt{N}}\right).
\end{aligned}$$

If $A > C^{2/3}$, then we have:

$$\begin{aligned}
\int_0^1 \Pr[\delta_I(\overline{v}_I, v_{-I}) > \eta \wedge \overline{\mathrm{E}}] \mathrm{d}\eta &\le \int_0^1 \left(G(x) + O\left(\frac{1}{N}\right)\right) \mathrm{d}x && \text{by (22)} \\
&\le \int_0^A 1 \mathrm{d}x + \int_A^1 \frac{C}{x^{\frac{3}{2}}} \mathrm{d}x + O\left(\frac{1}{N}\right) \\
&= A + \frac{C}{-\frac{1}{2}} - \frac{C}{-\frac{1}{2}} A^{-\frac{1}{2}} + O\left(\frac{1}{N}\right) \\
&\le \Theta\left(\frac{m}{d}\sqrt{\frac{\log N}{N}}\right) + \Theta\left(\frac{m^{1-1/2} \log^{3-1/4} N}{d^{4-1/2} N^{3/4-1/4}}\right) + O\left(\frac{1}{N}\right) \\
&= O\left(\frac{m}{d^{7/2}} \frac{\log^3 N}{\sqrt{N}}\right),
\end{aligned}$$

which, together with (21), concludes the proof of Lemma D.6.

E Analysis for α-Strongly Regular Distributions

E.1 Useful Lemmas

Lemma E.1 (Cole and Roughgarden [11]). *Any α-strongly regular distribution has a unique q^*, and $q^* \geq \alpha^{\frac{1}{1-\alpha}}$.*

Lemma E.2 (Huang et al. [23], Lemma 3.5). *For any α-strongly regular distribution F, for any $0 \leq q \leq 1$, $R(q^*) - R(q) \geq \frac{\alpha}{3}(q^* - q)^2 R(q^*)$.*

Lemma E.3. *Let F be an α-strongly regular distribution. Let $X = \max\{v_1, \ldots, v_N\}$ where $v_1, \ldots, v_N$ are N i.i.d. samples from F. For any $x \geq v^*$, we have $\Pr[X > x] \leq N\left(\frac{v^*}{(1-\alpha)x + \alpha v^*}\right)^{\frac{1}{1-\alpha}}$.*

Proof. Note that $1 - F(x) = \exp\left\{-\int_0^x \frac{f(v)}{1-F(v)}\mathrm{d}v\right\} \leq \exp\left\{-\int_{v^*}^x \frac{f(v)}{1-F(v)}\mathrm{d}v\right\}$. By the definition of v^* we know $(v^*[1 - F(v^*)])' = 0$, or $\frac{f(v^*)}{1-F(v^*)} = \frac{1}{v^*}$. By the definition of α-strong regularity, we have

$$\left(\frac{1 - F(x)}{f(x)}\right)' = 1 - \frac{d\phi}{dx} \leq 1 - \alpha$$

and

$$\frac{1 - F(x)}{f(x)} \leq \frac{1 - F(v^*)}{f(v^*)} + (1 - \alpha)(x - v^*).$$

Thus

$$\int_{v^*}^x \frac{f(v)}{1 - F(v)} \geq \int_{v^*}^x \frac{1}{\frac{1-F(v^*)}{f(v^*)} + (1 - \alpha)(v - v^*)}\mathrm{d}v = \frac{1}{1 - \alpha}\left[\ln\left(v^* + (1 - \alpha)(x - v^*)\right) - \ln v^*\right]$$

and

$$1 - F(x) \leq \exp\left\{-\frac{1}{1 - \alpha}\ln\frac{v^* + (1 - \alpha)(x - v^*)}{v^*}\right\} = \left(\frac{v^*}{(1 - \alpha)x + \alpha v^*}\right)^{\frac{1}{1-\alpha}}.$$

Then the lemma follows from a simple union bound:

$$\Pr[X > x] = \Pr[\exists i, v_i > x] \leq N[1 - F(x)] \leq N\left(\frac{v^*}{(1 - \alpha)x + \alpha v^*}\right)^{\frac{1}{1-\alpha}}.$$

□

Claim E.4. *(Improved Claim B.2) Define event* Conc*:*

$$\mathrm{Conc} = \left[\forall j > m, \left|q_j - \frac{j}{N}\right| \leq 2\sqrt{\frac{3}{\alpha}\frac{\ln(2(N - m))}{N - m}} + \frac{m}{N}\right],$$

then $\Pr[\overline{\mathrm{Conc}}] \leq \frac{1}{(N-m)^{\frac{3-2\alpha}{2\alpha}}}$, where the probability is over the random draw of the $N - m$ samples v_{-I}.

Proof. Set $\delta = \frac{1}{(N-m)^{\frac{3-2\alpha}{2\alpha}}}$ in Lemma A.3. □

Lemma E.5. *Let F be an α-strongly regular distribution. Suppose $m = o(\sqrt{N})$. Fix m values v_I to be $+\infty$, and randomly draw $N - m$ values v_{-I} from F. Let $k^* = \arg\max_{i > cN}\{iv_i\}$, i.e., the index selected by ERM^c, where $\left(\frac{\log N}{N}\right)^{\frac{1}{3}} \leq c \leq \frac{\alpha^{1/(1-\alpha)}}{2}$. Then we have*

$$R(q_{k^*}) \geq \left(1 - O\left(\sqrt{\frac{\log N}{N}}\right)\right) R(q^*),$$

with probability at least $1 - O\left(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}}\right)$. The constants in the big O's depend on α.

Proof. Let $\gamma \stackrel{\text{def}}{=} 2\sqrt{\frac{3}{\alpha}\frac{\log(2(N-m))}{N-m}} + \frac{m}{N} = O\left(\sqrt{\frac{\log N}{N}}\right)$ as in Claim E.4. We have $|q_j - \frac{j}{N}| \leq \gamma$ for any $j > m$ with probability at least $1 - \frac{1}{(N-m)^{\frac{3-2\alpha}{2\alpha}}}$. We thus assume $|q_j - \frac{j}{N}| \leq \gamma$. For simplicity, we define $e(\alpha) = \alpha^{1/(1-\alpha)}$, and Lemma E.1 implies $q^* \geq e(\alpha)$.

The intuition is follows: The product jv_j divided by N approximates $R(q_j) = q_j v_j$ up to an $O(\gamma)$ error. Our proof consists of three steps: The first step is to show that with high probability, there must be some sample with quantile q_i that is very close to q^* so its revenue $R(q_i) \approx R(q^*) \approx \frac{i}{N} v_i$. The second step is to argue that all samples with quantile $q_j < \frac{e(\alpha)}{2}$ are unlikely to be chosen by ERM^c because q_j is too small and the gap between q^* and $\frac{e(\alpha)}{2}$ leads to a large loss in revenue, roughly speaking, $\frac{j}{N} v_j \approx R(q_j) < (1 - \frac{\alpha}{3}(\frac{e(\alpha)}{2})^2) R(q^*) \approx (1 - \Omega(1)) \frac{i}{N} v_i$. The final step is to show that if a quantile $q_j > \frac{e(\alpha)}{2}$ is to be chosen by ERM^c, then it must have equally good revenue as q_i.

Formally:

1. Firstly, consider the quantile interval $[q^* - \gamma, q^*]$. Each random draw q_i, if falling into this interval, will satisfy:

$$\frac{i}{N} v_i \geq (q_i - \gamma) v_i \geq (q^* - 2\gamma) v_i \geq (q^* - 2\gamma) v^* \geq (1 - 2\gamma / e(\alpha)) q^* v^*, \tag{23}$$

   where the last but one inequality is because $q_i \leq q^*$ and the last one follows from $q^* \geq e(\alpha)$. The probability that no quantile falls into $[q^* - \gamma, q^*]$ is at most

$$(1-\gamma)^{N-m} = \left(1 - O\left(\sqrt{\frac{\log N}{N}}\right)\right)^{N-m} = o(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}}).$$

2. For the second step, first note that the $q_i \in [q^* - \gamma, q^*]$ in the first step will be considered by ERM^c since $i \geq (q_i - \gamma)N \geq (q^* - 2\gamma)N \geq (e(\alpha) - 2\gamma)N > cN$. Then suppose ERM^c chooses another quantile q_j instead of q_i, we must have

$$\frac{j}{N} v_j \geq \frac{i}{N} v_i. \tag{24}$$

   We will show that such q_j does not exist.

   Suppose ERM^c chooses q_j, then j must satisfy $j/N > c > \left(\frac{\log N}{N}\right)^{\frac{1}{3}}$, and as a result, $q_j > \left(\frac{\log N}{N}\right)^{\frac{1}{3}} - \gamma$.

   If $\left(\frac{\log N}{N}\right)^{\frac{1}{3}} - \gamma < q_j < \frac{e(\alpha)}{2} + \gamma$, note that

$$\frac{j}{N} v_j \leq (q_j + \gamma) v_j \leq \left(1 + \frac{\gamma}{\left(\frac{\log N}{N}\right)^{\frac{1}{3}} - \gamma}\right) q_j v_j, \tag{25}$$

   together with (24) and (23), we obtain $\left(1 + \frac{\gamma}{\left(\frac{\log N}{N}\right)^{\frac{1}{3}} - \gamma}\right) q_j v_j > (1 - 2\gamma/e(\alpha)) q^* v^*$. Then by Lemma E.2,

$$\frac{1 - 2\gamma/e(\alpha)}{1 + \frac{\gamma}{\left(\frac{\log N}{N}\right)^{\frac{1}{3}} - \gamma}} \leq \frac{q_j v_j}{q^* v^*} \leq 1 - \frac{\alpha}{3}(q_j - q^*)^2 \leq 1 - \frac{\alpha}{3}(\frac{e(\alpha)}{2} - \gamma)^2. \tag{26}$$

   However, the left hand side of (26) approaches to 1 while the right hand side is strictly less than 1, a contradiction. So this case never happens.

3. Finally, if $q_j \geq \frac{e(\alpha)}{2} + \gamma$. We argue that if ERM^c picks q_j instead of q_i, then $R(q_j)$ approximates $R(q_i)$ well, satisfying the conclusion in the lemma. This is because

$$\begin{aligned}
R(q_j) = q_j v_j &\geq (\frac{j}{N} - \gamma) v_j \\
&\geq (1 - \frac{2\gamma}{e(\alpha)}) \frac{j}{N} v_j && \frac{j}{N} \geq q_j - \gamma \geq \frac{e(\alpha)}{2} \\
&\geq (1 - \frac{2\gamma}{e(\alpha)}) \frac{i}{N} v_i && \text{Eq. (24)} \\
&\geq (1 - \frac{2\gamma}{e(\alpha)})(1 - \frac{2\gamma}{e(\alpha)}) q^* v^* && \text{Eq. (23)} \\
&= (1 - O(\gamma)) R(q^*).
\end{aligned}$$

Combining above three steps and the event in the beginning of the proof, we have $R(q_{k^*}) \geq (1 - O(\sqrt{\frac{\log N}{N}}))R(q^*)$ except with probability at most

$$\frac{1}{(N-m)^{\frac{3-2\alpha}{2\alpha}}} + o(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}}) = O(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}}).$$

□

Lemma E.6. *Let F be an α-strongly regular distribution. Suppose $m = o(\sqrt{N})$. Fix m values v_I to be $+\infty$, and randomly draw $N-m$ values v_{-I} from F. Let $k^* = \arg\max_{i>cN}\{iv_i\}$, i.e., the index selected by ERM^c, where $\left(\frac{\log N}{N}\right)^{\frac{1}{3}} \leq c \leq \frac{\alpha^{1/(1-\alpha)}}{2}$. Let $\epsilon = \sqrt[4]{\frac{\log N}{N}}$. Then with probability at least $1 - O\left(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}}\right)$, the following three inequalities hold:*

1. *$q_{k^*} \geq q^* - O(\epsilon)$;*
2. *$k^* \geq [q^* - O(\epsilon)]N > \frac{\alpha^{1/(1-\alpha)}}{2}N$;*
3. *$v_{k^*} \leq [1 + O(\epsilon)]v^*$.*

The constants in the big O's depend on α.

Proof. Define $e(\alpha) = \alpha^{1/(1-\alpha)}$. We have $q^* \geq e(\alpha)$.

For inequality (1), by Lemma E.2 and Lemma E.5, with probability at least $1 - O(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}})$, we have

$$\frac{\alpha}{3}(q_{k^*} - q^*)^2 \leq \frac{R(q^*) - R(q_{k^*})}{R(q^*)} \leq O\left(\sqrt{\frac{\log N}{N}}\right).$$

Taking the square root, we obtain $q_{k^*} \geq q^* - O\left(\sqrt[4]{\frac{\log N}{N}}\right)$.

To prove (2), note that by Claim E.4, we have $\frac{k^*}{N} \geq q_{k^*} - O\left(\sqrt{\frac{\log N}{N}}\right) \geq q^* - O(\epsilon)$.

Finally, (3) follows from

$$\frac{v_{k^*}}{v^*} = \frac{R(q_{k^*})}{q_{k^*}}\frac{q^*}{R(q^*)} \leq 1 \cdot \frac{q^*}{q_{k^*}} \leq \frac{q^*}{q^* - O(\epsilon)} = 1 + \frac{O(\epsilon)}{q^* - O(\epsilon)} \leq 1 + O\left(\frac{\epsilon}{e(\alpha)}\right).$$

□

E.2 Detailed Proof of Theorem 2.1 for α-Strongly Regular Distributions

Let $\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) = U_i^{\mathrm{TP}}(\boldsymbol{v}_i, b_I, v_{-I}) - U_i^{\mathrm{TP}}(\boldsymbol{v}_i, v_I, v_{-I})$. Similar to the proof for bounded distributions, we have for any $\boldsymbol{v}_i, b_I, v_{-I}$,

$$\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \leq m_2 \cdot \left(\mathrm{ERM}^c(v_I, v_{-I}) - \mathrm{ERM}^c(b_I, v_{-I})\right) \leq m_2 \cdot \mathrm{ERM}^c(v_I, v_{-I}) \cdot \delta_{m_1}^{\mathrm{worst}}(v_{-I}).$$

By Claim A.2, we have $\mathrm{ERM}^c(v_I, v_{-I}) \leq \mathrm{ERM}^c(\overline{v}_I, v_{-I})$ where $\overline{v}_I$ can be any m_1 values (e.g., $+\infty$) that are greater than the maximal value in v_{-I}, when $c \geq \frac{m_1}{T_1 K_1}$.

Let $N = T_1 K_1$, define two threshold prices $T_1 = N^{\frac{3(1-\alpha)}{2\alpha}} v^*$ and $T_2 = [1 + O(\epsilon)]v^*$ where $\epsilon = \sqrt[4]{\frac{\log N}{N}}$ as in Lemma E.6.

Note that for sufficiently large N, $T_1 > T_2$. With the random draw of v_{-I} from F, denote the random variable $\mathrm{ERM}^c(\overline{v}_I, v_{-I})$ by P, we have:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{v}_{-i}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I})\right] = \; & \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P \leq T_2\right] \cdot \Pr[P \leq T_2] \\
& + \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid T_2 < P \leq T_1\right] \cdot \Pr[T_2 < P \leq T_1] \\
& + \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P > T_1\right] \cdot \Pr[P > T_1] \\
\stackrel{\mathrm{def}}{=} \; & \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3.
\end{aligned} \tag{27}$$

1. For the first term $\mathbb{E}_1$,

$$\begin{aligned}\mathbb{E}_1 &= \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P \le T_2\right] \cdot \Pr[P \le T_2] \\ &\le \mathbb{E}_{v_{-I}}\left[m_2 \cdot P \cdot \delta_{m_1}^{\text{worst}}(v_{-I}) \mid P \le T_2\right] \cdot \Pr[P \le T_2] \\ &\le m_2 \cdot T_2 \cdot \mathbb{E}_{v_{-I}}\left[\delta_{m_1}^{\text{worst}}(v_{-I}) \mid P \le T_2\right] \cdot \Pr[P \le T_2] \\ &\le m_2 \cdot [1 + O(\epsilon)] v^* \cdot \mathbb{E}_{v_{-I}}\left[\delta_{m_1}^{\text{worst}}(v_{-I})\right] \\ &= O\left(m_2 \cdot v^* \cdot \Delta_{N,m_1}^{\text{worst}}\right).\end{aligned}$$

2. For the second term, we claim that $\mathbb{E}_2 = O(\frac{m_2 v^*}{\sqrt{N}})$.
   By Lemma E.6, we have $\Pr[P > [1 + O(\epsilon)]v^*] \le O(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}})$. Therefore,

$$\begin{aligned}\mathbb{E}_2 &= \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid T_2 < P \le T_1\right] \cdot \Pr[T_2 < P \le T_1] \\ &\le \mathbb{E}_{v_{-I}}\left[m_2 \cdot P \cdot 1 \mid T_2 < P \le T_1\right] \cdot \Pr[T_2 < P \le T_1] \\ &\le m_2 \cdot T_1 \cdot \Pr[P > T_2] \\ &\le m_2 \cdot N^{\frac{3(1-\alpha)}{2\alpha}} v^* \cdot O\left(\frac{1}{N^{\frac{3-2\alpha}{2\alpha}}}\right) \\ &= O\left(\frac{m_2 v^*}{\sqrt{N}}\right).\end{aligned}$$

3. For the third term, we claim that $\mathbb{E}_3 = o(\frac{m_2 v^*}{\sqrt{N}})$.
   Let B be the upper bound on the support of F (B can be $+\infty$). Let $F_P(x)$ be the distribution of P. For convenience, suppose it is continuous and has density $f_P(x)$. We have:

$$\begin{aligned}\mathbb{E}_3 &= \mathbb{E}_{v_{-I}}\left[\Delta U(\boldsymbol{v}_i, b_I, v_{-I}) \mid P > T_1\right] \cdot \Pr[P > T_1] \\ &\le \mathbb{E}_{v_{-I}}\left[m_2 \cdot P \cdot 1 \mid P > T_1\right] \cdot \Pr[P > T_1] \\ &= m_2 \cdot \mathbb{E}_{v_{-I}}\left[P \mid P > T_1\right] \cdot \Pr[P > T_1] \\ &= m_2 \cdot \int_{T_1}^{B} x f_P(x) \mathrm{d}x \\ &= m_2 \cdot \left(\int_{T_1}^{B} [1 - F_P(x)]\mathrm{d}x + T_1[1 - F_P(T_1)]\right).\end{aligned}$$

   Let $\max\{v_{-I}\}$ denote the maximum value in the $N - m_1$ samples v_{-I}. By Lemma E.3, we have for any $x \ge v^*$,

$$1 - F_P(x) = \Pr[P > x] \le \Pr[\max\{v_{-I}\} > x] \le N\left(\frac{v^*}{(1-\alpha)x + \alpha v^*}\right)^{\frac{1}{1-\alpha}}.$$

   Thus,

$$\begin{aligned}&\int_{T_1}^{B} [1 - F_P(x)]\mathrm{d}x + T_1[1 - F_P(T_1)] \\ \le\ &\int_{T_1}^{B} N\left(\frac{v^*}{(1-\alpha)x + \alpha v^*}\right)^{\frac{1}{1-\alpha}} \mathrm{d}x + T_1 N\left(\frac{v^*}{(1-\alpha)T_1 + \alpha v^*}\right)^{\frac{1}{1-\alpha}} \\ \le\ &\frac{N}{\alpha} \cdot \frac{(v^*)^{\frac{1}{1-\alpha}}}{[(1-\alpha)T_1 + \alpha v^*]^{\frac{\alpha}{1-\alpha}}} + T_1 N\left(\frac{v^*}{(1-\alpha)T_1 + \alpha v^*}\right)^{\frac{1}{1-\alpha}} \\ =\ &\frac{N}{\alpha} \cdot (T_1 + \alpha v^*)\left(\frac{v^*}{(1-\alpha)T_1 + \alpha v^*}\right)^{\frac{1}{1-\alpha}} \\ =\ &O\left(\frac{v^*}{\sqrt{N}}\right),\end{aligned}$$

   as desired.

Combining the three items,

$$\mathbb{E}_{v_{-I}}\left[\Delta U(v_I, b_I, v_{-I})\right] = O\left(m_2 v^* \Delta_{N,m_1}^{\text{worst}}\right) + O\left(\frac{m_2 v^*}{\sqrt{N}}\right),$$

where the constants in O's depend on α.

F Lower Bounds

F.1 Discussion

A lower bound on $\Delta^{\text{worst}}_{N,m}$. Theorem 1.3 gives an upper bound on $\Delta^{\text{worst}}_{N,m}$ for a specific range of c's. When one considers respective lower bounds, a preliminary question would be: how does the choice of c affect the possible lower bound? The following result shows that it is enough to prove a lower bound for one specific c in the range of allowed c's. The same lower bound will then hold for all c's in that range.

Proposition F.1. *Let $\Delta^{\text{worst}}_{N,m}(c)$ denote the worst-case incentive-awareness measure of ERM^c. Suppose $m = o(\sqrt{N})$.*

- *For bounded distributions, $\Delta^{\text{worst}}_{N,m}(c) = \Delta^{\text{worst}}_{N,m}(\frac{m}{N})$, for any $c \in [\frac{m}{N}, \frac{1}{2D}]$.*
- *For MHR distributions, $\Delta^{\text{worst}}_{N,m}(c)$ is bounded by $\Delta^{\text{worst}}_{N,m}(\frac{m}{N}) \pm O(\frac{1}{N})$, for any $c \in [\frac{m}{N}, \frac{1}{4e}]$.*

Proof. Fix any $d \in (0,1)$ and any $c \in [\frac{m}{N}, \frac{d}{2}]$. Recall that $\Delta^{\text{worst}}_{N,m}(c) = \mathbb{E}_{v_{-I}\sim F}[\delta^{\text{worst}}_m(v_{-I}, c)]$, where, letting $\overline{v}_I$ be a vector of m identical values that are equal to $\max v_{-I}$, then by Claim A.2,

$$\delta^{\text{worst}}_m(v_{-I}, c) = 1 - \frac{\inf_{b_I \in \mathbb{R}^m_+} P(b_I, v_{-I}, c)}{P(\overline{v}_I, v_{-I}, c)}.$$

Let $k^*(v, c) = \arg\max_{i>cN}\{iv_{(i)}\}$ where $v = (\overline{v}_I, v_{-I})$, i.e. the index of $P(v, c)$ in v. We show that, if $k^*(v, c) > dN$ for $c = \frac{m}{N}$, then

- $P(\overline{v}_I, v_{-I}, c') = P(\overline{v}_I, v_{-I}, \frac{m}{N})$ for any $c' \in [\frac{m}{N}, d]$. To see this, note that
$$k^*(v, \frac{m}{N}) = \arg\max_{i>\frac{m}{N}N}\{iv_{(i)}\} = \arg\max_{i>dN}\{iv_{(i)}\} = \arg\max_{i>c'N}\{iv_{(i)}\} = k^*(v, c'),$$
where the second equality follow from our assumption that $k^*(v, \frac{m}{N}) > dN > m$ and the third equality is because $\frac{m}{N} \le c' \le d$.
- $\inf_{b_I\in\mathbb{R}^m_+} P(b_I, v_{-I}, c') = \inf_{b_I\in\mathbb{R}^m_+} P(b_I, v_{-I}, \frac{m}{N})$ for any $c' \in [\frac{m}{N}, \frac{d}{2}]$. Fix $c = \frac{m}{N}$ and consider any $c' \in [\frac{m}{N}, \frac{d}{2}]$. Let $b_I \in \mathbb{R}^m_+$ be any bids such that $P(b_I, v_{-I}, c) < P(\overline{v}_I, v_{-I}, c)$. Let $v^b = (b_I, v_{-I})$. Consider $k^*(v^b, c)$, we have $v^b_{(k^*(v^b,c))} < v_{(k^*(v,c))}$, so $k^*(v^b, c)$ must be greater than the index of $v_{(k^*(v,c))}$ in v^b. The index of $v_{(k^*(v,c))}$ in v^b is at least $k^*(v, c) - m$, thus $k^*(v^b, c) > k^*(v, c) - m > dN - m \ge \frac{d}{2}N$. We claim that $P(b_I, v_{-I}, c') = P(b_I, v_{-I}, c)$. To see this, note that
$$k^*(b_I, v_{-I}, \frac{m}{N}) = \arg\max_{i>\frac{m}{N}N}\{iv^b_{(i)}\} = \arg\max_{i>\frac{d}{2}N}\{iv^b_{(i)}\} = \arg\max_{i>c'N}\{iv^b_{(i)}\} = k^*(b_I, v_{-I}, c'),$$
where the second equality is because $k^*(v^b, \frac{m}{N}) > \frac{d}{2}N$, and the third equality follows from $\frac{m}{N} \le c' \le \frac{d}{2}$.

Thus, $\delta^{\text{worst}}_m(v_{-I}, c) = \delta^{\text{worst}}_m(v_{-I}, \frac{m}{N})$ for any $c \in [\frac{m}{N}, \frac{d}{2}]$. Define $\mathrm{E}(c) = [k^* \le dN]$, then

$$\begin{aligned}
\Delta^{\text{worst}}_{N,m}(c) &= \mathbb{E}[\delta^{\text{worst}}_m(v_{-I}, c)] \\
&= \mathbb{E}[\delta^{\text{worst}}_m(v_{-I}, c) \mid \overline{\mathrm{E}(c)}] \cdot \Pr[\overline{\mathrm{E}(c)}] + \mathbb{E}[\delta^{\text{worst}}_m(v_{-I}, c) \mid \mathrm{E}(c)] \cdot \Pr[\mathrm{E}(c)] \\
&= \mathbb{E}[\delta^{\text{worst}}_m(v_{-I}, \frac{m}{N}) \mid \overline{\mathrm{E}(c)}] \cdot \Pr[\overline{\mathrm{E}(c)}] + \mathbb{E}[\delta^{\text{worst}}_m(v_{-I}, c) \mid \mathrm{E}(c)] \cdot \Pr[\mathrm{E}(c)].
\end{aligned}$$

For bounded distributions, consider $d = \frac{1}{D}$ and $c \in [\frac{m}{N}, \frac{1}{2D}]$. We have proved in Theorem 1.3 that $\Pr[\mathrm{E}(c)] = 0$ for any $c \in [\frac{m}{N}, \frac{d}{2}]$. Thus $\Delta^{\text{worst}}_{N,m}(c) = \mathbb{E}[\delta^{\text{worst}}_m(v_{-I}, \frac{m}{N}) \mid \overline{\mathrm{E}(c)}] \cdot \Pr[\overline{\mathrm{E}(c)}] = \Delta^{\text{worst}}_{N,m}(\frac{m}{N})$.

For MHR distributions, let $d = \frac{1}{2e}$, as proved in Lemma D.5, $\Pr[\mathrm{E}] = O(\frac{1}{N})$ for $c \in [\frac{m}{N}, \frac{1}{4e}]$. Then $0 \le \mathbb{E}[\delta^{\text{worst}}_m(v_{-I}, c) \mid \mathrm{E}(c)] \cdot \Pr[\mathrm{E}(c)] \le 1 \cdot \Pr[\mathrm{E}(c)] = O(\frac{1}{N})$. Thus $\left|\Delta^{\text{worst}}_{N,m}(c) - \Delta^{\text{worst}}_{N,m}(\frac{m}{N})\right| \le O(\frac{1}{N})$ for any $c \in [\frac{m}{N}, \frac{1}{4e}]$.

□

Lavi et al. [26] show a lower bound that can be compared to our upper bounds in Theorem 1.3. Specifically, they show that for the two-point distribution $v = 1$ and $v = 2$, each w.p. 0.5, $\Delta^{\text{worst}}_{N,1} = \Omega(1/\sqrt{N})$. It is easy to adopt their analysis to any two-point distribution with $v_1 = 1$ and $v_2 > 1$. Since this is a bounded distribution, we obtain the following corollary:

Corollary F.2. *For the class of bounded distribution with support in* $[1, D]$*, and any choice of* $\frac{m}{N} \le c \le \frac{1}{2D}$*,* ERM^c *gives* $\Delta^{\text{worst}}_{N,m}(c) = \Omega(\frac{1}{\sqrt{N}})$ *where the constant in* Ω *depends on* D*.*

It remains open to prove other lower bounds on $\Delta^{\text{worst}}_{N,m}$, especially for MHR distributions.

A lower bound on the approximate truthfulness parameter, ϵ_1**.** Since $\Delta^{\text{worst}}_{N,m}$ only upper bounds the approximate truthfulness parameter ϵ_1, a lower bound on $\Delta^{\text{worst}}_{N,m}$ does not immediately implies a lower bound on ϵ_1. However, an argument similar to above shows the same lower bound directly on ϵ_1. Consider the two-point distribution F where for $X \sim F$, $\Pr[X = 1] = 1 - \frac{1}{D}$ and $\Pr[X = D] = \frac{1}{D}$. For simplicity let $K_1 = K_2 = 2$ and suppose bidder i participates in m_1 and m_2 auctions in the two phases, respectively. Let $N = T_1 K_1$ and assume $m_1 = o(\sqrt{N})$. Suppose the first-phase mechanism $\mathcal{M}$ is the second price auction with no reserve price. Then,

Proposition F.3. *In the above setting,* $\epsilon_1 = \Omega\left(\frac{m_2}{\sqrt{N}}\right)$ *for any* $c \in [\frac{m_1}{N}, \frac{1}{2D}]$*, where the constant in* Ω *depends on* D*.*

The proof is in Appendix F.2. Once again, it remains open (and interesting, we believe) to prove a lower bound for MHR distributions, and to close the gap between our upper bound for bounded distributions which is $O(N^{-1/3} \log^2 N)$.

F.2 Proof of Proposition F.3: Lower Bound for the Two-Phase Model

Consider the two-point distribution F where for $X \sim F$, $\Pr[X = 1] = 1 - \frac{1}{D}$ and $\Pr[X = D] = \frac{1}{D}$. For simplicity let $K_1 = K_2 = 2$, and suppose bidder i participates in m_1 and m_2 auctions in the two phases, respectively. Let $N = T_1 K_1$ and assume $m_1 = o(\sqrt{N})$. Suppose the first-phase mechanism $\mathcal{M}$ is the second price auction with no reserve price. We argue that to satisfy ϵ_1-approximate truthfulness, ϵ_1 must be $\Omega\left(\frac{m_2(D-1)^2}{\sqrt{(D-1)N}}\right)$, for $c \in [\frac{m}{N}, \frac{1}{2D}]$.

Suppose the values of bidder i across two phases are all D's, i.e.,

$$\boldsymbol{v}_i = (\overbrace{D, \dots, D}^{m_1}, \overbrace{D, \dots, D}^{m_2}),$$

and bidder i bids m_1 $(D - \epsilon)$'s with $\epsilon = \frac{D^2}{N}$ in the first phase (assume $N \gg D^2$),

$$b_I = (\overbrace{D - \epsilon, \dots, D - \epsilon}^{m_1}).$$

Recall the definition of the interim utility of bidder i:

$$\mathbb{E}_{v_{-I}}\left[U_i^{\text{TP}}(\boldsymbol{v}_i, b_I, v_{-I})\right] = \mathbb{E}_{v_{-I}}\left[U_i^{\mathcal{M}}(\boldsymbol{v}_i, b_I, v_{-I}) + m_2 u^{K_2}(D, P(b_I, v_{-I}))\right].$$

- First consider the increase of interim utility in the second phase. If the reserve price is D, then bidder i's utility is

  $$u^{K_2}(D, D) = 0. \tag{28}$$

  If the reserve price is 1, then her utility becomes

  $$u^{K_2}(D, 1) = (1 - \frac{1}{D}) \cdot (D - 1) + \frac{1}{D} \cdot 0 = \frac{(D-1)^2}{D}. \tag{29}$$

  We then consider the probability that the reserve price is decreased from D to 1 because bidder i deviates from D to $D - \epsilon$. This probability is over the random draw of $N - m_1$ values v_{-I}. Suppose there are exactly $(\frac{1}{D}N + 1 - m_1)$ D's in v_{-I}. Then when bidder i bids truthfully, there are $(\frac{1}{D}N + 1)$ D's in (v_I, v_{-I}) in total, which results in $P(v_I, v_{-I}) = D$ because $(\frac{1}{D}N + 1) \cdot D > N \cdot 1$. However, if bidder i deviates to b_I, then $P(b_I, v_{-I})$ becomes 1, because

  $$(\frac{1}{D}N + 1) \cdot (D - \epsilon) = N(1 + \frac{D}{N})(1 - \frac{D}{N}) < N \cdot 1, \quad \text{and} \quad (\frac{1}{D}N + 1 - m_1) \cdot D \le N \cdot 1.$$

  Thus, the reserve price is decreased from D to 1 with probability at least:

  $$\Pr[Bin(N - m_1, \frac{1}{D}) = \frac{1}{D}N + 1 - m_1] = \Omega\left(\frac{1}{\sqrt{\frac{1}{D}(1 - \frac{1}{D})N}}\right). \tag{30}$$

Combining (30), (29) and (28), we obtain

$$\mathbb{E}_{v_{-I}}\left[m_2 u^{K_2}(D, P(b_I, v_{-I})) - m_2 u^{K_2}(D, P(v_I, v_{-I}))\right]$$
$$\geq \Omega\left(\frac{1}{\sqrt{\frac{1}{D}(1-\frac{1}{D})N}}\right) m_2\left(\frac{(D-1)^2}{D} - 0\right) = \Omega\left(\frac{m_2(D-1)^2}{\sqrt{(D-1)N}}\right). \quad (31)$$

- Then we upper bound the utility loss due to non-truthful bidding in the first phase for bidder i. Note that since $\mathcal{M}$ is the second price auction with no reserve price, no matter bidder i bids D or $D-\epsilon > 1$, her interim utility is the same:

$$\mathbb{E}_{v_{-I}}\left[U_i^{\mathcal{M}}(\boldsymbol{v}_i, v_I, v_{-I})\right] = m_1\left((1-\frac{1}{D})(D-1) + \frac{1}{D}\cdot 0\right) = \frac{m_1(D-1)^2}{D},$$
$$\mathbb{E}_{v_{-I}}\left[U_i^{\mathcal{M}}(\boldsymbol{v}_i, b_I, v_{-I})\right] = m_1\left((1-\frac{1}{D})(D-1) + \frac{1}{D}\cdot 0\right) = \frac{m_1(D-1)^2}{D}.$$

Thus
$$\mathbb{E}_{v_{-I}}\left[U_i^{\mathcal{M}}(\boldsymbol{v}_i, b_I, v_{-I}) - U_i^{\mathcal{M}}(\boldsymbol{v}_i, v_I, v_{-I})\right] = 0. \quad (32)$$

Finally, by (31) and (32), we have

$$\mathbb{E}_{v_{-I}}\left[U_i^{\mathrm{TP}}(\boldsymbol{v}_i, b_I, v_{-I}) - U_i^{\mathrm{TP}}(\boldsymbol{v}_i, v_I, v_{-I})\right] \geq \Omega\left(\frac{m_2(D-1)^2}{\sqrt{(D-1)N}}\right),$$

which gives a lower bound on ϵ_1.

G Unbounded Regular Distributions

G.1 Discussion

Theorem 2.2 shows that approximate truthfulness and revenue optimality can be obtained simultaneously for bounded (regular) distributions and for MHR distributions. A natural question would then be: what is the largest class of value distribution that we can consider? If $K_1 > 1$ or $K_2 > 1$ (i.e., each auction includes multiple bidders, at least 2), then running a second price auction with an anonymous reserve price may not be optimal if the distribution is non-regular [30]. Moreover, even in the one-bidder case, the sample complexity literature analyzes the ERM algorithm only for regular or for non-regular and bounded distributions. For other classes of distributions, ERM does not seem to be the correct choice. Thus, the class of general unbounded regular distributions is the largest class we can consider. Still, our results do not cover this entire class since MHR distributions is a strict sub-class of regular distributions and for regular but non-MHR distributions we assume boundedness.

Our results can be generalized to the class of α-strongly regular distributions with $\alpha > 0$. As defined in [11], a distribution F with positive density function f on its support $[A, B]$ where $0 \leq A \leq B \leq +\infty$ is *α-strongly regular* if the virtual value function $\phi(x) = x - \frac{1-F(x)}{f(x)}$ satisfies $\phi(y) - \phi(x) \geq \alpha(y-x)$ whenever $y > x$ (or $\phi'(x) \geq \alpha$ if $\phi(x)$ is differentiable). As special cases, regular and MHR distributions are 0-strongly and 1-strongly regular distributions, respectively. For $\alpha > 0$, we obtain bounds similar to MHR distributions on $\Delta_{N,m}^{\mathrm{worst}}$ and approximate incentive-compatibility in the two-phase model and the uniform-price auction. Specifically, Theorem 1.3 can be extended to any α-strongly regular distribution with $\alpha > 0$ as follows:

Theorem G.1. *If F is α-strongly regular for $0 < \alpha \leq 1$, then $\Delta_{N,m}^{\mathrm{worst}} = O\left(m\frac{\log^3 N}{\sqrt{N}}\right)$, when $m = o(\sqrt{N})$ and $\frac{m}{N} \leq \left(\frac{\log N}{N}\right)^{1/3} \leq c \leq \frac{\alpha^{1/(1-\alpha)}}{4}$. The constants in O and o depend on α.*

Note that this bound holds only for large enough N's since $\alpha^{1/(1-\alpha)} \to 0$ as $\alpha \to 0$. However, for any fixed $\alpha > 0$ there exists a large enough N such that the relevant range for appropriate c's will be non-empty.[5] The proof of the upper bound on $\Delta_{N,m}^{\mathrm{worst}}$ is similar to the proof for MHR distributions (Lemma D.6), except that, we need $c \geq \left(\frac{\log N}{N}\right)^{1/3}$ to guarantee $\Pr[\mathrm{E}] = O(\frac{1}{N})$ (Lemma E.6); thus we omit the proof.

Similarly, both of Theorem 2.1 which says that the two-phase model is $(O(m_2 v^* \Delta_{T_1K_1,m_1}^{\mathrm{worst}}) + O(\frac{m_2 v^*}{\sqrt{T_1K_1}}))$-BIC and Theorem 3.1 which says that the uniform-price auction is $(m, (O(v^*\Delta_{N,m}^{\mathrm{worst}}) + O(\frac{v^*}{\sqrt{N}})))$-group BIC, hold for any α-strongly regular distribution with $\alpha > 0$. The proof of the former is in Appendix E.2 and the latter is omitted.

[5]The constant in the big O depends on α, so it is constant only if the distribution function is fixed. However, it goes to infinity as $\alpha \to 0$.

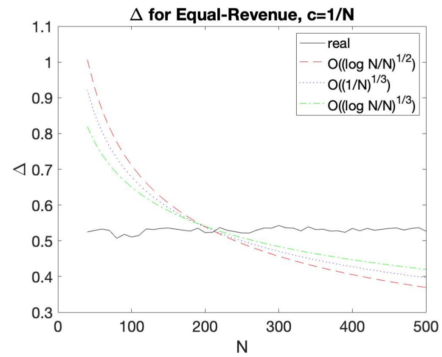

Figure 1: For equal-revenue distribution, $\Delta_{N,1}^{\text{worst}}$ (in black) on the y-axis as a function of N on the x-axis, with $c = 1/N$. Three other functions are plotted for reference.

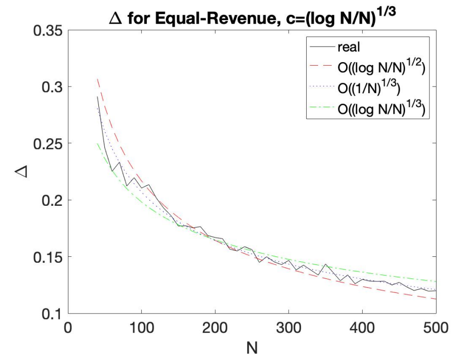

Figure 2: For equal-revenue distribution, $\Delta_{N,1}^{\text{worst}}$ (in black) on the y-axis as a function of N on the x-axis, with $c = \Theta((\log N/N)^{1/3})$. Three other functions are plotted for reference.

It remains an open problem for future research whether ERM^c is incentive-aware in the large for regular distributions that are not α-strongly regular for any $\alpha > 0$. For these distributions additional technical challenges exist since the choice of c in ERM^c creates a clash between approximate truthfulness and approximate revenue optimality. Unlike MHR and bounded regular distributions for which we can fix $c = m/N$ to obtain approximate truthfulness and revenue optimality, for arbitrary unbounded distribution we have to choose c more carefully. If c is too large, for example, a positive constant, then we cannot obtain nearly optimal revenue.

Specifically, to obtain close-to-optimal revenue for all bounded distributions on $[1, D]$ it is easy to verify that we need $c \leq 1/D$. Since the class of unbounded regular distributions contains all bounded regular distributions for all $D \in \mathbb{R}_+$, it follows that c cannot be a constant. We therefore need to consider a non-constant $c(N)$. In fact, it has been shown in [23] that if $c(N) \to 0$ as $N \to \infty$ then approximate revenue optimality can be satisfied. However, if c is too small, truthfulness will be violated, as discussed in the following two examples (assume $m = 1$ for simplicity).

Example G.2 (Small c hurts truthfulness)**.** *Suppose we choose $c(N) = \frac{1}{N}$, that is, ERM^c ignores only the largest sample. Consider the equal-revenue distribution $F(v) = 1 - \frac{1}{v}$ for $v \in [1, +\infty)$. Note that this is a 0-strongly regular distribution but not α-strongly regular for any $\alpha > 0$ since for any x, $\phi(x) = 0$. Similarly to Yao [31], we prove in Appendix G.2 that $\Delta_{N,1}^{\text{worst}}$ does not go to 0 as $N \to +\infty$. This is also visible in Fig. 1, which shows in black $\Delta_{N,1}^{\text{worst}}$ (on the y-axis) as a function of N (on the x-axis). This was obtained via simulation, for $c = 1/N$. Three other functions are plotted in other colors, for reference. However, whether $\Delta_{N,1}^{\text{worst}} \to 0$ crucially depends on the choice of c, as can be seen in Fig. 2, where $\Delta_{N,1}^{\text{worst}}$ seems to converge to zero with* $c = \Theta((\log N/N)^{1/3})$.

Example G.3 (Does an intermediate c hurt truthfulness as well?)**.** *Now assume $c = \Theta((\log N/N)^{1/3})$, and consider the "triangular" distribution $F(v) = 1 - \frac{1}{v+1}$ for $v \in [0, +\infty)$. This distribution can be seen as the limit of a series of bounded regular distributions whose upper bounds and optimal reserve prices both tend to $+\infty$. Note that it is a regular distribution but not α-strongly regular for any $\alpha > 0$. We do not know whether $\Delta_{N,1}^{\text{worst}} \to 0$ as $N \to \infty$ (see Fig. 3). In particular, our main lemma (Lemma B.1) may not suffice to analyze this distribution as $\Pr[\mathrm{E}]$ (as defined in that lemma) is unlikely to go to zero as N goes to infinity (see Fig. 4 for simulation results).*

G.2 Proof of Example G.2: $\Delta_{N,1}^{\text{worst}} \not\to 0$ for the Equal-Revenue Distribution and $c = 1/N$

We show that when F is the equal-revenue distribution, $F(v) = 1 - \frac{1}{v}$, ($v \in [1, +\infty)$), and $c = 1/N$, $\Delta_{N,1}^{\text{worst}}$ does not go to 0 as $N \to +\infty$.

Recall Definition 1.2, $\Delta_{N,1}^{\text{worst}} = \mathbb{E}_{v_{-I}}[\delta_{N,1}^{\text{worst}}(v_{-I})]$: we draw $N-1$ i.i.d. values $v_{-I} = \{v_2, \ldots, v_N\}$ from F, and a bidder can change any value $v_I = v_1$ to any non-negative bid b_1. Let the other values $v_{-I} = \{v_2, \ldots, v_N\}$ be sorted as $v_{(2)} \geq \cdots \geq v_{(N)}$. Let $\lambda = 10, \lambda' = 1$, and let T_{N-1} be the event that $v_{(2)} > \lambda(N-1)$ and $v_{(3)} < \lambda'(N-1)$. Let V_{N-1} be the event that $\max_{3 \leq i \leq N}\{i \cdot v_{(i)}\} \leq \lambda(N-1)$.

When v_{-I} satisfies event $T_{N-1} \wedge V_{N-1}$, then

$$\delta_m^{\text{worst}}(v_{-I}) \geq 1 - \frac{\lambda'}{\lambda}. \tag{33}$$

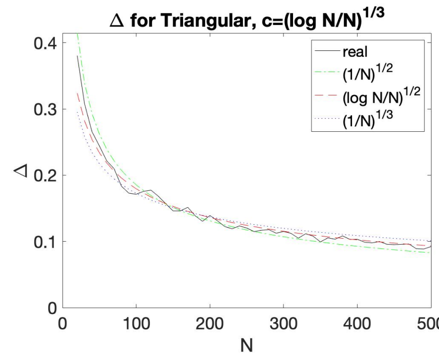

Figure 3: $\Delta^{\text{worst}}_{N,1}$ for the triangular distribution, with $c = \Theta((\log N/N)^{1/3})$

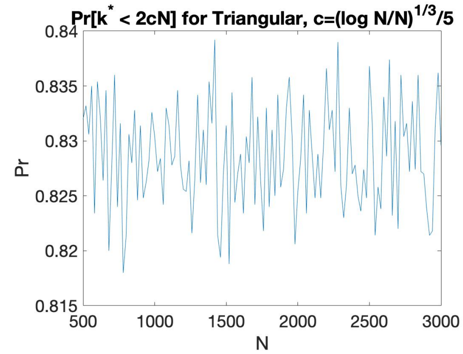

Figure 4: $\Pr[k^* < 2cN]$ as a function of N for the triangular distribution, with $c = (\log N/N)^{1/3}/5$. Our simulations show that $\Pr[\mathrm{E}] \geq \Pr[k^* < 2cN] \geq 0.815$.

This is because when $T_{N-1} \wedge V_{N-1}$ happens, there exists some $v_1 > \lambda(N-1)$, resulting $P(v_I, v_{-I}) > \lambda(N-1)$, while the bidder can strategically bid $b_1 < \lambda'(N-1)$ and change the price to $P(b_I, v_{-I}) < \lambda'(N-1)$.

Moreover, we will show that the probability that the event $T_{N-1} \wedge V_{N-1}$ happens satisfies

$$\Pr[T_{N-1} \wedge V_{N-1}] \geq 0.9 \cdot \frac{1}{\lambda} e^{-\frac{2}{\lambda'}}. \tag{34}$$

Combining (33) and (34), we know

$$\Delta^{\text{worst}}_m \geq (1 - \frac{\lambda'}{\lambda}) \cdot (0.9 \cdot \frac{1}{\lambda} e^{-\frac{2}{\lambda'}}) > 0.$$

The proof of (34) is separated into two parts. Firstly,

$$\Pr[T_{N-1}] = \binom{N-1}{1} \frac{1}{\lambda(N-1)} (1 - \frac{1}{\lambda'(N-1)})^{N-2} \geq \frac{1}{\lambda} e^{-\frac{2}{\lambda'}}.$$

Then we show that

$$\Pr[\overline{V_{N-1}} \mid T_{N-1}] < 0.1.$$

Let $z_3, \ldots, z_N$ be i.i.d. random draws according to F conditioning on $z < \lambda'(N-1)$, i.e., for any $t \in [1, N-1]$, recalling that $\lambda' = 1$,

$$\Pr_{z \sim F}[z > t \mid z < \lambda'(N-1)] = \frac{1}{1 - \frac{1}{N-1}} (\frac{1}{t} - \frac{1}{N-1}).$$

Let $Y^{max}_{N-1} = \max_{3 \leq i \leq N}\{i \cdot z_{(i)}\}$ where $z_{(3)} \geq \cdots \geq z_{(N)}$ is the sorted list of z_i's. Clearly,

$$\Pr[\overline{V_{N-1}} \mid T_{N-1}] = \Pr[Y^{max}_{N-1} \geq \lambda(N-1)].$$

For any $t \geq 1$, let M_t be the number of z_i's ($3 \leq i \leq N$) satisfying $z_i \geq t$, and B_t be the event that $t \cdot (M_t + 2) \geq \frac{\lambda(N-1)}{2}$. Let $t_k = \frac{N-1}{2^k}$ for $1 \leq k \leq \lfloor \log_2(N-1) \rfloor$. As the event $Y^{max}_{N-1} \geq \lambda(N-1)$ implies $\bigvee_{1 \leq k \leq \lfloor \log_2(N-1) \rfloor} B_{t_k}$, we have

$$\Pr[Y^{max}_{N-1} \geq \lambda(N-1)] \leq \sum_{k=1}^{\lfloor \log_2(N-1) \rfloor} \Pr[B_{t_k}]. \tag{35}$$

Note that $\mathbb{E}[M_t] = \frac{N-1}{t} - 1$. Using Chernoff's bound, we have

$$\Pr[B_t] = \Pr[M_t \geq \frac{\lambda(N-1)}{2t} - 2] \leq e^{-\frac{1}{3}(\frac{N-1}{t} - 1)(\frac{\lambda}{2} - 1)^2}. \tag{36}$$

Combining (35) and (36), we know

$$\Pr[\overline{V_{N-1}} \mid T_{N-1}] = \Pr[Y^{max}_{N-1} \geq \lambda(N-1)] \leq \sum_{k=1}^{\lfloor \log_2(N-1) \rfloor} e^{-\frac{1}{3}(2^k - 1)(\frac{\lambda}{2} - 1)^2} < \sum_{k=1}^{+\infty} e^{-\frac{1}{3}k(\frac{\lambda}{2} - 1)^2} < 0.1,$$

and this completes the proof of (34).