

Push-pull Feedback Implements Hierarchical Information Processing Efficiently: Supplementary Information

1 The correlation structure of hierarchical memory patterns

For the hierarchical memory patterns defined in the main text, the following relationships hold:

- The correlation between grandparent patterns, $1/N \sum_i \xi_i^\alpha \xi_i^{\alpha'} = 0$, for $\alpha \neq \alpha'$;
- The correlation between grandparent and its descending parent patterns, $1/N \sum_i \xi_i^\alpha \xi_i^{\alpha, \beta} = b_2$;
- The correlation between parent patterns from the same grandparent, $1/N \sum_i \xi_i^{\alpha, \beta} \xi_i^{\alpha, \beta'} = b_2^2$, for $\beta \neq \beta'$;
- The correlation between parent patterns from different grandparents, $1/N \sum_i \xi_i^{\alpha, \beta} \xi_i^{\alpha', \beta'} = 0$, for $\alpha \neq \alpha'$;
- The correlation between parent pattern and its descending child patterns, $1/N \sum_i \xi_i^{\alpha, \beta} \xi_i^{\alpha, \beta, \gamma} = b_1$;
- The correlation between child patterns from the same parent pattern, $1/N \sum_i \xi_i^{\alpha, \beta, \gamma} \xi_i^{\alpha, \beta, \gamma'} = b_1^2$, for $\gamma \neq \gamma'$;
- The correlation between child patterns from different parents, $1/N \sum_i \xi_i^{\alpha, \beta, \gamma} \xi_i^{\alpha, \beta', \gamma'} = b_2^2 b_1^2$, for $\beta \neq \beta'$.

Moreover, it can be checked that when the number of parent patterns P_β is sufficiently large, the average of parent patterns approaches to their ascending grandparent pattern, i.e., $1/P_\beta \sum_{\beta=1}^{P_\beta} \xi_i^{\alpha, \beta} \approx b_2 \xi_i^\alpha$.

Similarly, when the number of child patterns P_γ is sufficiently large, the average of child patterns approaches to their ascending parent pattern, i.e., $1/P_\gamma \sum_{\gamma=1}^{P_\gamma} \xi_i^{\alpha, \beta, \gamma} \approx b_1 \xi_i^{\alpha, \beta}$.

2 Memory retrieval without feedback

2.1 The neuronal input at layer 1 without feedback

According to Eq.(2) in the main text, without feedback, we have

$$\begin{aligned}
h_i^1(0) &= \frac{1}{N} \sum_j W_{ij}^1 x_j^1(0), \\
&= \frac{1}{N} \sum_{\alpha, \beta, \gamma} \xi_i^{\alpha, \beta, \gamma} \sum_j \xi_j^{\alpha, \beta, \gamma} \xi_j^{\alpha_0, \beta_0, \gamma_0}, \\
&= \xi_i^{\alpha_0, \beta_0, \gamma_0} + A_i + \widetilde{A}_i + \widetilde{\widetilde{A}}_i,
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
A_i &= \frac{1}{N} \sum_{\gamma \neq \gamma_0} \xi_i^{\alpha_0, \beta_0, \gamma} \sum_j \xi_j^{\alpha_0, \beta_0, \gamma} \xi_j^{\alpha_0, \beta_0, \gamma_0}, \\
&= b_1^2 \sum_{\gamma \neq \gamma_0} \xi_i^{\alpha_0, \beta_0, \gamma},
\end{aligned} \tag{2}$$

$$\begin{aligned}
\widetilde{A}_i &= \frac{1}{N} \sum_{\beta \neq \beta_0} \sum_{\gamma} \xi_i^{\alpha_0, \beta, \gamma} \sum_j \xi_j^{\alpha_0, \beta, \gamma} \xi_j^{\alpha_0, \beta_0, \gamma_0}, \\
&= b_1^2 b_2^2 \sum_{\beta \neq \beta_0, \gamma} \xi_i^{\alpha_0, \beta, \gamma},
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
\widetilde{\widetilde{A}}_i &= \frac{1}{N} \sum_{\alpha \neq \alpha_0} \sum_{\beta, \gamma} \xi_i^{\alpha, \beta, \gamma} \sum_j \xi_j^{\alpha, \beta, \gamma} \xi_j^{\alpha_0, \beta_0, \gamma_0}, \\
&= 0.
\end{aligned} \tag{4}$$

Thus, according to Eq.(7,8) in the main text, we have

$$C_i = b_1^2 \xi_i^{\alpha_0, \beta_0, \gamma_0} \sum_{\gamma \neq \gamma_0} \xi_i^{\alpha_0, \beta_0, \gamma}, \tag{5}$$

$$\widetilde{C}_i = b_1^2 b_2^2 \xi_i^{\alpha_0, \beta_0, \gamma_0} \sum_{\beta \neq \beta_0} \sum_{\gamma} \xi_i^{\alpha_0, \beta, \gamma}, \tag{6}$$

$$\widetilde{\widetilde{C}}_i = 0. \tag{7}$$

2.2 The distribution of the intra-class noise C_i

Define the sets $\mathcal{C}_+ \equiv \{\gamma \mid \xi_i^{\alpha_0, \beta_0, \gamma} = \xi_i^{\alpha_0, \beta_0}, \gamma \neq \gamma_0\}$, and $\mathcal{C}_- \equiv \{\gamma \mid \xi_i^{\alpha_0, \beta_0, \gamma} = -\xi_i^{\alpha_0, \beta_0}, \gamma \neq \gamma_0\}$, and their numbers are denoted as $\text{card}(\mathcal{C}_+)$ and $\text{card}(\mathcal{C}_-)$, respectively. Note $\text{card}(\mathcal{C}_-) = P_\gamma - 1 - \text{card}(\mathcal{C}_+)$.

According to Eq.(5) in the main text,

$$P(\xi_i^{\alpha,\beta,\gamma}) = \left(\frac{1+b_1}{2}\right)\delta(\xi_i^{\alpha,\beta,\gamma} - \xi_i^{\alpha,\beta}) + \left(\frac{1-b_1}{2}\right)\delta(\xi_i^{\alpha,\beta,\gamma} + \xi_i^{\alpha,\beta}).$$

Since the generalizations of memory patterns are independent to each other, $\text{card}(\mathcal{C}_+)$ follows a binomial distribution, $\text{card}(\mathcal{C}_+) \sim \mathcal{B}(P_\gamma - 1, P_+)$, with $P_+ = P(\xi_i^{\alpha_0,\beta_0,\gamma} = \xi_i^{\alpha_0,\beta_0}) = (1+b_1)/2$. Here, $\mathcal{B}(n, P)$ denotes a binomial distribution, with n the total number of experiments and P the probability of each experiment yielding a successful result.

In the large P_γ limit, a binomial distribution can be approximated as a normal one, which is written as

$$\text{card}(\mathcal{C}_+) \sim \mathcal{N}((P_\gamma - 1)P_+, (P_\gamma - 1)P_+(1 - P_+)). \quad (8)$$

Thus, we have

$$\begin{aligned} \sum_{\gamma \neq \gamma_0} \xi_i^{\alpha_0,\beta_0,\gamma} &= \text{card}(\mathcal{C}_+) \xi_i^{\alpha_0,\beta_0} - \text{card}(\mathcal{C}_-) \xi_i^{\alpha_0,\beta_0}, \\ &= 2\text{card}(\mathcal{C}_+) \xi_i^{\alpha_0,\beta_0} - (P_\gamma - 1) \xi_i^{\alpha_0,\beta_0}, \end{aligned} \quad (9)$$

and hence

$$C_i = b_1^2 [2\text{card}(\mathcal{C}_+) - (P_\gamma - 1)] \xi_i^{\alpha_0,\beta_0,\gamma_0} \xi_i^{\alpha_0,\beta_0}. \quad (10)$$

Since the product $\xi_i^{\alpha_0,\beta_0,\gamma_0} \xi_i^{\alpha_0,\beta_0} = 1$ with the probability $(1+b_1)/2$ and $\xi_i^{\alpha_0,\beta_0,\gamma_0} \xi_i^{\alpha_0,\beta_0} = -1$ with the probability $(1-b_1)/2$, the distribution of C_i can be written as superposition of two normal distributions, i.e.,

$$P(C_i) = \frac{1+b_1}{2} \mathcal{N}(E_C, V_C) + \frac{1-b_1}{2} \mathcal{N}(-E_C, V_C), \quad (11)$$

where $E_C = b_1^3(P_\gamma - 1)$ and $V_C = b_1^4(P_\gamma - 1)(1 - b_1^2)$.

2.3 The distribution of the inter-class noise \tilde{C}_i

Similarly, it can be checked that in the large P_γ, P_β limit, \tilde{C}_i satisfies superposition of two normal distributions,

$$P(\tilde{C}_i) = \frac{1+b_1b_2}{2} \mathcal{N}(E_{\tilde{C}}, V_{\tilde{C}}) + \frac{1-b_1b_2}{2} \mathcal{N}(-E_{\tilde{C}}, V_{\tilde{C}}), \quad (12)$$

where $E_{\tilde{C}} = b_1^3b_2^3P_\gamma(P_\beta - 1)$ and $V_{\tilde{C}} = b_1^4b_2^4P_\gamma(P_\beta - 1)(1 - b_1^2)(1 - b_2^2)$.

2.4 Retrieval error without feedback

We first calculate retrieval error without the pull-feedback. The probabilities $P(C_i)$ and $P(\tilde{C}_i)$ can be calculated under four different conditions, which are summarized below:

- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = \xi_i^{\alpha_0, \beta_0} = \xi_i^{\alpha_0}$, which has the probability $P = (1 + b_1)(1 + b_2)/4$, $P(C_i) = \mathcal{N}(E_C, V_C)$ and $P(\tilde{C}_i) = \mathcal{N}(E_{\tilde{C}}, V_{\tilde{C}})$.
- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = \xi_i^{\alpha_0, \beta_0} = -\xi_i^{\alpha_0}$, which has the probability $P = (1 + b_1)(1 - b_2)/4$, $P(C_i) = \mathcal{N}(E_C, V_C)$ and $P(\tilde{C}_i) = \mathcal{N}(-E_{\tilde{C}}, V_{\tilde{C}})$.
- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = -\xi_i^{\alpha_0, \beta_0} = \xi_i^{\alpha_0}$, which has the probability $P = (1 - b_1)(1 + b_2)/4$, $P(C_i) = \mathcal{N}(-E_C, V_C)$ and $P(\tilde{C}_i) = \mathcal{N}(-E_{\tilde{C}}, V_{\tilde{C}})$.
- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = -\xi_i^{\alpha_0, \beta_0} = -\xi_i^{\alpha_0}$, which has the probability $P = (1 - b_1)(1 - b_2)/4$, $P(C_i) = \mathcal{N}(-E_C, V_C)$ and $P(\tilde{C}_i) = \mathcal{N}(E_{\tilde{C}}, V_{\tilde{C}})$.

The above four conditions can be described by two auxiliary variables, which are $s_1 \equiv \xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0, \beta_0} = \pm 1$ and $s_2 \equiv \xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0} = \pm 1$.

The probability of retrieval error without the pull-feedback is calculated to be,

$$\begin{aligned}
p(C_i + \tilde{C}_i < -1) &= \int_{-\infty}^{+\infty} p(C_i = x) p(\tilde{C}_i < -1 - x) dx \\
&= \sum_{s_1, s_2 = \pm 1} \frac{1}{4} (1 + s_1 b_1) (1 + s_1 s_2 b_2) \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi V_C}} \exp\left(-\frac{x - s_1 E_C}{2V_C}\right) \\
&\quad \int_{-\infty}^{\frac{-1 - x - s_2 E_{\tilde{C}}}{\sqrt{2V_{\tilde{C}}}}} \frac{dy}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right), \\
&= \sum_{s_1, s_2 = \pm 1} \frac{1}{4} (1 + s_1 b_1) (1 + s_1 s_2 b_2) \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp\left(-\frac{x\sigma_C - s_1 E_C}{2}\right) \\
&\quad \int_{-\infty}^{\frac{-1 - x\sigma_{\tilde{C}} - s_2 E_{\tilde{C}}}{\sigma_{\tilde{C}}}} \frac{dy}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right), \\
&\quad (\sigma_C = \sqrt{V_C}; \sigma_{\tilde{C}} = \sqrt{V_{\tilde{C}}}), \\
&= \sum_{s_1, s_2 = \pm 1} \frac{1}{4} (1 + s_1 b_1) (1 + s_1 s_2 b_2) \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right), \\
&\quad (\sigma_C x + \sigma_{\tilde{C}} y < -1 - s_1 E_C - s_2 E_{\tilde{C}}) \\
&= \sum_{s_1, s_2 = \pm 1} \frac{1}{8} (1 + s_1 b_1) (1 + s_1 s_2 b_2) \left[1 + \operatorname{erf}\left(\frac{-1 - s_1 E_C - s_2 E_{\tilde{C}}}{\sqrt{2(\sigma_C^2 + \sigma_{\tilde{C}}^2)}}\right) \right]. \quad (13)
\end{aligned}$$

3 The effect of pull-feedback

3.1 The distribution of the noise C_i^* after applying the pull-feedback

According to the definition, $C_i^* = C_i - b_1 \xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0, \beta_0}$. It can be checked that for $\xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0, \beta_0} = 1$, the mean $E(C_i^*) = E_C - b_1$, and for $\xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0, \beta_0} = -1$, the mean $E(C_i^*) = -E_C + b_1$. Therefore, we have

$$P(C_i^*) = \frac{1+b_1}{2} \mathcal{N}(E_C - b_1, V_C^*) + \frac{1-b_1}{2} \mathcal{N}(-E_C + b_1, V_C^*), \quad (14)$$

where $E_C = b_1^3(P_\gamma - 1)$ and $V_C^* = V_C = b_1^4(P_\gamma - 1)(1 - b_1^2)$.

3.2 The retrieval error with the pull-feedback

We now calculate retrieval error with the pull-feedback. Given $W_{ij}^{1,2} = -b_1 \delta_{ij}$ and that layer 2 is at the parent pattern $\mathbf{x}^2(0) = \xi^{\alpha_0, \beta_0}$, the alignment between the neuronal input and the child pattern at layer 1 is written as,

$$\xi^{\alpha_0, \beta_0, \gamma_0} h_i^1(0) = 1 + C_i + \tilde{C}_i - b_1 \xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0, \beta_0}. \quad (15)$$

We obtain

$$\begin{aligned} m^{\alpha_0, \beta_0, \gamma_0}(1) &= \frac{1}{N} \sum_{i=1}^N \text{sign}[1 + C_i + \tilde{C}_i - b_1 \xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0, \beta_0}], \\ &= \frac{1}{N} \sum_{i=1}^N \text{sign}[1 + C_i^* + \tilde{C}_i] \end{aligned} \quad (16)$$

where $C_i^* = C_i - b_1 \xi_i^{\alpha_0, \beta_0, \gamma_0} \xi_i^{\alpha_0, \beta_0}$.

The probability $P(C_i^*)$ can be calculated under four different conditions, which are,

- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = \xi_i^{\alpha_0, \beta_0} = \xi_i^{\alpha_0}$, which has the probability $P = \frac{(1+b_1)(1+b_2)}{4}$, $P(C_i^*) = \mathcal{N}[(E_C - b_1), V_C]$;
- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = \xi_i^{\alpha_0, \beta_0} = -\xi_i^{\alpha_0}$, which has the probability $P = \frac{(1+b_1)(1-b_2)}{4}$, $P(C_i^*) = \mathcal{N}[(E_C - b_1), V_C]$;
- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = -\xi_i^{\alpha_0, \beta_0} = \xi_i^{\alpha_0}$, which has the probability $P = \frac{(1-b_1)(1+b_2)}{4}$, $P(C_i^*) = \mathcal{N}[-(E_C - b_1), V_C]$;
- When $\xi_i^{\alpha_0, \beta_0, \gamma_0} = -\xi_i^{\alpha_0, \beta_0} = -\xi_i^{\alpha_0}$, which has the probability $P = \frac{(1-b_1)(1-b_2)}{4}$, $P(C_i^*) = \mathcal{N}[-(E_C - b_1), V_C]$.

Analogous to Eq.(13), the probability of retrieval error with the pull-feedback is calculated to be,

$$p(C_i^* + \tilde{C}_i < -1) = \sum_{s_1, s_2 = \pm 1} \frac{1}{8} (1 + s_1 b_1)(1 + s_1 s_2 b_2) \times \left[1 + \text{erf} \left(\frac{-1 - s_1(E_C - b_1) - s_2 E_{\tilde{C}}}{\sqrt{2(\sigma_C^2 + \sigma_{\tilde{C}}^2)}} \right) \right]. \quad (17)$$

4 The effects of push-pull feedback

4.1 Dynamics of the continuous Hopfield model

The dynamics of the continuous Hopfield model are given by

$$\tau \frac{dh_i^1}{dt} = -h_i^1 + \sum_j W_{ij}^1 x_j^1 + \sum_k W_{ik}^{1,2}(t) x_k^2 + I_i^{ext}, \quad (18)$$

$$\tau \frac{dh_i^2}{dt} = -h_i^2 + \sum_j W_{ij}^2 x_j^2 + \sum_k W_{ik}^{2,1}(t) x_k^1 + I_i^{ext}, \quad (19)$$

$$x_i^n = \frac{2}{\pi} \arctan(8\pi h_i^n), \quad \text{for } n = 1, 2, \quad (20)$$

where h_i^n and x_i^n denote the synaptic input and the firing rate of neuron i in layer n . τ is the time constant, I_i^{ext} the external input to layer 1, and $\arctan(x)$ is the inverse of the tangent function.

In the continuous model, each element of a memory pattern (e.g., $\xi_i^{\alpha, \beta, \gamma}$) takes a value of 0 or 1. The recurrent connections between neurons in the same layer are constructed by the Hebbian covariance learning rule, which are given by

$$\tilde{W}_{ij}^1 = \frac{1}{N} \sum_{\alpha, \beta, \gamma} (\xi_i^{\alpha, \beta, \gamma} - \langle \xi \rangle) (\xi_j^{\alpha, \beta, \gamma} - \langle \xi \rangle), \quad (21)$$

$$\tilde{W}_{ij}^2 = \frac{1}{N} \sum_{\alpha, \beta} (\xi_i^{\alpha, \beta} - \langle \xi \rangle) (\xi_j^{\alpha, \beta} - \langle \xi \rangle), \quad (22)$$

$$W_{ij}^n = a_r^n \frac{\tilde{W}_{ij}^n}{|\tilde{\mathbf{W}}^n|}, \quad \text{for } n = 1, 2, \quad (23)$$

where $\langle \xi \rangle$ is the mean activity of all neurons averaged over all memory patterns in the same layer, $|\tilde{\mathbf{W}}^n| = \sqrt{\sum_{ij} (\tilde{W}_{ij}^n)^2 / N^2}$, and a_r^n are positive constants.

We consider the feedback connections from layers 2 to 1, which vary over time. At the early phase, the feedback is positive, which is written as

$$W_{ik}^{1,2} = \frac{a_+}{NP_\gamma} \sum_{\alpha, \beta, \gamma} (\xi_i^{\alpha, \beta, \gamma} - \langle \xi \rangle) (\xi_j^{\alpha, \beta, \gamma} - \langle \xi \rangle), \quad (24)$$

where a_+ is a positive number.

At the later phase, the feedback is negative which is written

$$W_{ik}^{1,2} = -a_- b_1 \delta_{ik}, \quad (25)$$

where $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ otherwise, and a_- is a positive number.

Let us consider the memory pattern to be retrieved at layer 1 is $\xi^{\alpha_0, \beta_0, \gamma_0}$. The external input to the first layer, which conveys the memory pattern information, is set to be

$$I_i^{ext} = a_{ext} \tilde{\xi}_i^{\alpha_0, \beta_0, \gamma_0} + \sigma \eta_i, \quad (26)$$

where η_i is a random number uniformly distributed in the range of $(-1, 1)$, representing the memory-independent noise, and σ controls the noise strength. a_{ext} is a positive number. The pattern $\tilde{\xi}^{\alpha_0, \beta_0, \gamma_0}$ is the noise-corrupted signal, which is constructed as follows: starting from the clean memory pattern $\xi^{\alpha_0, \beta_0, \gamma_0}$, we first randomly select $\lambda_1 N$ number of neurons, with $0 < \lambda_1 < 1$, and change their values to match the pattern $\xi^{\alpha_0, \beta_0, \gamma'}$, with $\gamma' \neq \gamma_0$, which represents the intra-class noise. We then randomly select $\lambda_2 N$ number of neurons, with $0 < \lambda_2 < 1$, and change their values to match the pattern $\xi^{\alpha_0, \beta', \gamma_0}$, with $\beta' \neq \beta_0$, which represents the inter-class noise.

4.2 Comparing network performances with varying correlation amplitudes

Fig.S1 shows how the retrieval improvement due to the push-pull feedback varies with the correlation b_1 between child patterns. We see that the push-pull feedback works very well for a wide range of correlation amplitudes. We also note that it has little effect when the correlation b_1 is very small, and may worsen retrieval when b_1 is too large. This is understandable, since for a very large b_1 , the strong negative feedback may shut-down the neural activity at layer 1 (see Eq.(10) in the main text).

5 Pre-processing real images with a deep neural network

The dataset, chosen from ImageNet, consists of two types (cat and dog), and each type is made up of 9 sub-types. They are: 1) sub-types of cat: Abyssinian Cat, Angora Cat, Burmese Cat, Egyptian Cat, Manx Cat, Persian Cat, Siamese Cat, Tiger Cat, Tortoiseshell; 2) sub-types of dog: Saint Bernard, Beagle Dog, Border Collie, Cirn terrier, Cardigan Dog, Eskimo Dog, Mastiff Dog, Pug Dog. We chose 100 images per sub-type. We re-scaled each image, with its width and height to be (256, 256), and pixel values in the range of [0, 255]. After that, we presented each image as an input to the VGG16(the VGG16 itself was pre-trained by ImageNet) and took the neural activity before the reading-out lay as the

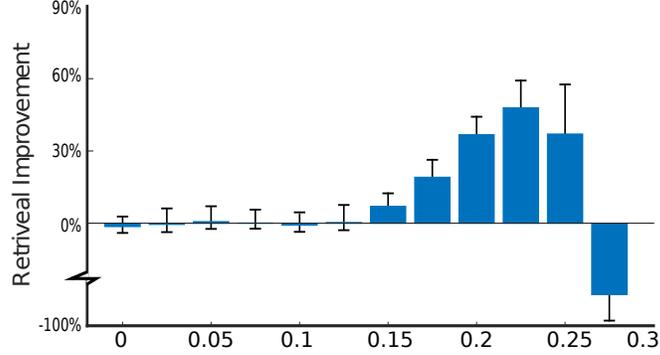


Figure S1: The retrieval improvement due to the push-pull feedback vs. the correlation strength b_1 between child patterns. Both the retrieval accuracies with and without feedback are taken at the moment of 5τ . The results are obtained by averaging over 20 trials. The child pattern is $\xi^{1,1,1}$. Other parameters are the same as in Fig.3 in the main text.

neural representation of the image, referred to as \hat{x} hereafter. Subsequently, we normalized neural activities, i.e.,

$$\tilde{x}_i = \log(1 + \hat{x}_i), \quad (27)$$

$$x_i = \text{ReLu}\left(\frac{\tilde{x}_i - \langle \tilde{x}_i \rangle}{\sigma_{\tilde{x}_i}}\right), \quad (28)$$

where $\text{ReLu}(\cdot) = \max(0, \cdot)$ is the rectified linear function, and $\langle \tilde{x} \rangle$ and $\sigma_{\tilde{x}}$ are the mean and standard deviation of neural activities over all neurons for each image.

Finally, we generated memory patterns for the hierarchical network. For a child pattern, it is given by

$$\xi_i^{\beta_0, \gamma_0} = \text{sign}(\langle x_i \rangle_{\beta_0, \gamma_0}), \quad (29)$$

where the average is over all the images belonging to the same sub-type of animal.

The corresponding parent patterns are constructed to be

$$\xi_i^{\beta_0} = \langle \xi_i^{\beta_0, \gamma} \rangle_{\gamma}, \quad (30)$$

where the average is over all child patterns belonging to the same parent.