A Proof of Theorem 1

Proof. As discussed above, we have two cases as follows.

Case 1: First we consider the case that w, u, v are from left to right, with their stationary points $x_k^{(w)*} < x_k^{(u)*} < x_k^{(v)*}$. A standard gradient descent update is $\tilde{x} \leftarrow \tilde{x} - \eta \nabla_{x_k} f_y^{(u)}|_{x_k=\tilde{x}}$. To ensure one step update could make $\tilde{x}$ jump from u to v, we obtain:

$$\begin{aligned} & \tilde{x} - \eta \nabla_{x_k} f_y^{(u)}|_{x_k=\tilde{x}} \geq x_k^{(v)*} \\ & \Rightarrow \tilde{x} - \eta \sum_{j=p}^{p+q} 2(\tilde{x} - y_j) \geq x_k^{(v)*} \quad \text{(use Equation 5)} \\ & \Rightarrow (1 - 2\eta(q+1))\tilde{x} + 2\eta(q+1) \frac{1}{q+1} \sum_{j=p}^{p+q} y_j \geq x_k^{(v)*} \\ & \Rightarrow (1 - 2\eta(q+1))\tilde{x} \geq x_k^{(v)*} - 2\eta(q+1) x_k^{(u)*} \quad \text{(use Equation 7)} \end{aligned} \tag{11}$$

Now we have two possibilities:

Possibility (a): $1 - 2\eta(q+1) > 0$**:**

$$\text{Inequality 11} \Rightarrow \ \tilde{x} \geq \frac{x_k^{(v)*} - 2\eta(q+1) x_k^{(u)*}}{1 - 2\eta(q+1)}$$

To guarantee the gradient direction points to neighbor v, the starting point $\tilde{x}$ has to be to the left of $x_k^{(u)*}$, thus:

$$\begin{aligned} & \frac{x_k^{(v)*} - 2\eta(q+1) x_k^{(u)*}}{1 - 2\eta(q+1)} \leq \tilde{x} < x_k^{(u)*} \\ & \Rightarrow x_k^{(v)*} - 2\eta(q+1) x_k^{(u)*} < (1 - 2\eta(q+1)) x_k^{(u)*} \\ & \Rightarrow x_k^{(v)*} < x_k^{(u)*} \end{aligned} \tag{12}$$

This is contradictory to the assumption that $x_k^{(w)*} < x_k^{(u)*} < x_k^{(v)*}$, and thus is not valid.

Possibility (b): $1 - 2\eta(q+1) < 0$**:**

$$\text{Inequality 11} \Rightarrow \ \tilde{x} \leq \frac{x_k^{(v)*} - 2\eta(q+1) x_k^{(u)*}}{1 - 2\eta(q+1)}$$

Due to the fact that $\tilde{x}$ is inside region u, which must be somewhere to the right of $x_k^{(w)*}$ (with the assumption that w has a bowl-shape), we have:

$$\begin{aligned} & \frac{x_k^{(v)*} - 2\eta(q+1) x_k^{(u)*}}{1 - 2\eta(q+1)} \geq \tilde{x} > x_k^{(w)*} \\ & \Rightarrow x_k^{(v)*} - 2\eta(q+1) x_k^{(u)*} < (1 - 2\eta(q+1)) x_k^{(w)*} \\ & \Rightarrow x_k^{(v)*} - x_k^{(w)*} < 2\eta(q+1)(x_k^{(u)*} - x_k^{(w)*}) \\ & \Rightarrow \eta > \frac{1}{2(q+1)} \left(\frac{x_k^{(v)*} - x_k^{(w)*}}{x_k^{(u)*} - x_k^{(w)*}} \right) \end{aligned} \tag{13}$$

Recall that q is an integer and $q \geq 0$, thus

$$1 - 2\eta(q+1) < 0 \ \Rightarrow \ \eta > \frac{1}{2(q+1)} \tag{14}$$

Also notice that

$$x_k^{(w)*} < x_k^{(u)*} < x_k^{(v)*} \ \Rightarrow \ \frac{x_k^{(v)*} - x_k^{(w)*}}{x_k^{(u)*} - x_k^{(w)*}} > 1 \tag{15}$$

Putting Inequalities 13, 14 and 15 together, we obtain $\eta > \frac{1}{2(q+1)}$.

Case 2: here w, u, v are from right to left, and $x_k^{(w)*} > x_k^{(u)*} > x_k^{(v)*}$. This is very similar to Case 1, so we omit the details and provide the final result as

$$\eta > \frac{1}{2(q+1)}\left(\frac{x_k^{(w)*} - x_k^{(v)*}}{x_k^{(u)*} - x_k^{(v)*}}\right) \quad \text{and} \quad \frac{x_k^{(w)*} - x_k^{(v)*}}{x_k^{(u)*} - x_k^{(v)*}} > 1 \tag{16}$$

We will arrive at the same result: $\eta > \frac{1}{2(q+1)}$.

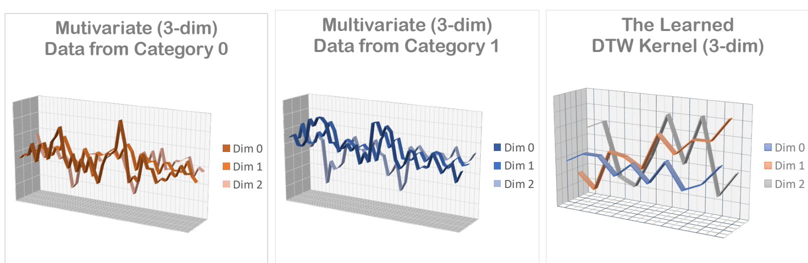
Note that the length of the pattern x is l and the length of input y is n. As a result, the expected number of elements in y aligned to a single x_i, $i \in [0, l-1]$ should be n/l, i.e. $\mathbb{E}[q] = n/l - 1$. Taking expectation on both sides of the above inequality, we obtain $\mathbb{E}[\eta] > \frac{1}{2(n/l-1+1)} = \frac{l}{2n}$. $\square$

Corollary 1. *With the same assumption of Theorem 1, let pattern x and input y have the same length, i.e., $n = l$. In order to make one step jumping out of local region u, we should have $\mathbb{E}[\eta] > \frac{1}{2}$.*

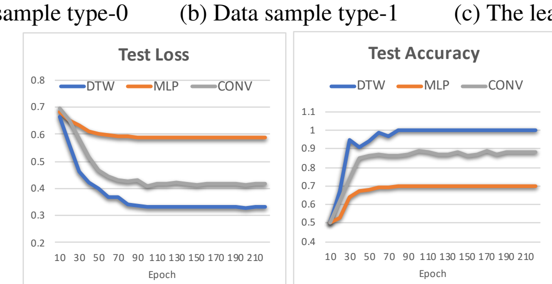
B Detailed Experimental Results for Multivariate DTWNet

Multivariate DTWs are often computed in two forms: MDTW-I and MDTW-D [21]. MDTW-I treats each dimension independently, so it is simply a stack of multiple univariate DTWs, thus directly applies to our method. MDTW-D needs to compute multivariate distance $\text{mdtw}^2 = \sum ||\mathbf{x_i} - \mathbf{y_j}||^\mathbf{2}, \mathbf{x_i} \in \mathbf{R^m}, \mathbf{y_j} \in \mathbf{R^m}$ in the Dynamic Programming step, instead of the scaler version $\sum ||x_i - y_j||^2$. As long as the norm is well defined, e.g., Euclidean distance, the forward pass and the backpropagation are performed in the same manner. We can even define other distances, as long as their gradients w.r.t. to the vector $\mathbf{x}$ can be computed.

We run a 3-dim multivariate time series classification task here, using MDTW-D and Euclidean distance in our approach. The experiment settings follow Section 6.1 in the paper. The following figures show: one sample data (3-variate series) from each category, the learned kernel, test loss and test acc comparison. Our method (DTW) outperforms others.



(a) Data sample type-0 (b) Data sample type-1 (c) The learned kernel



(d) Test Loss (e) Test Acc

Figure 6: Multivariate DTWNet Experiment

C Detailed Experimental Results for Barycenter Experiment

Table 2: Barycenter Experiment, Average DTW Loss on Training Set

	SoftDTW $\gamma = 1$	SoftDTW $\gamma = 0.1$	SoftDTW $\gamma = 0.01$	SoftDTW $\gamma = 0.001$	SSG	DBA	**Ours**
50words	7.337	5.294	5.375	5.478	4.904	4.604	**4.415**
Adiac	0.246	0.218	0.219	0.663	**0.133**	0.136	0.461
ArrowHead	2.713	2.223	1.690	1.979	1.856	**1.555**	1.717
Beef	17.753	9.260	7.889	8.617	17.056	8.618	**7.487**
BeetleFly	34.656	23.368	22.839	23.729	24.771	22.341	**20.544**
BirdChicken	21.017	10.711	9.459	11.982	11.115	12.768	**9.136**
CBF	22.421	14.292	12.891	14.062	11.086	**11.039**	11.592
Car	2.003	1.252	1.063	1.386	**0.940**	1.089	1.230
ChlorineConcentration	24.167	14.339	16.069	15.773	**12.871**	13.227	13.829
CinC_ECG_torso	137.172	99.014	80.749	91.645	80.430	79.076	**69.681**
Coffee	1.056	0.740	1.228	2.038	**0.451**	0.478	0.945
Computers	192.741	198.368	162.193	163.533	**155.350**	164.252	158.226
Cricket_X	45.766	33.312	32.477	33.539	33.537	32.728	**30.582**
Cricket_Y	43.959	31.407	31.865	30.490	31.170	32.640	**29.766**
Cricket_Z	48.030	34.976	34.390	35.690	33.936	34.711	**31.129**
DiatomSizeReduction	0.161	0.146	0.139	0.576	0.061	**0.055**	0.760
DistalPhalanxOutlineAgeGroup	1.791	1.366	1.573	2.406	**1.088**	1.141	1.332
DistalPhalanxOutlineCorrect	2.643	2.081	2.227	2.453	1.896	**1.842**	1.874
DistalPhalanxTW	1.358	1.010	1.031	1.399	**0.691**	0.736	0.937
ECG200	7.365	5.677	7.488	6.931	6.171	6.262	**5.638**
ECG5000	12.282	11.623	13.441	11.390	12.471	11.427	**10.567**
ECGFiveDays	9.987	8.418	7.821	7.001	7.750	**6.927**	11.559
Earthquakes	150.271	**87.765**	91.644	88.571	88.961	88.290	89.370
ElectricDevices	31.243	28.281	28.409	27.694	27.151	27.649	**27.033**
FISH	0.913	0.777	0.639	0.692	**0.487**	0.497	0.789
FaceAll	18.250	15.273	17.053	15.958	13.898	14.416	**13.678**
FaceFour	28.604	24.447	26.018	27.347	30.171	29.341	**22.983**
FacesUCR	16.913	14.094	15.200	15.657	12.979	**12.656**	13.025
FordA	63.812	53.893	55.744	55.351	51.829	53.025	**51.076**
FordB	66.695	56.032	56.154	55.071	52.866	53.447	**51.686**
Gun_Point	7.586	2.525	2.208	2.354	3.393	**2.113**	2.259
Ham	25.753	21.942	19.811	20.530	19.482	20.669	**19.319**
HandOutlines	-	-	-	-	2.094	1.975	2.859
Haptics	19.904	15.475	15.414	16.633	**12.714**	14.710	14.638
Herring	1.778	1.245	1.520	1.526	**0.873**	1.118	1.172
InlineSkate	91.103	34.482	25.691	27.585	30.498	22.163	**21.671**
InsectWingbeatSound	14.798	13.418	13.024	12.506	12.148	12.407	**11.909**
ItalyPowerDemand	2.748	2.317	3.343	2.810	2.222	**2.161**	2.302
LargeKitchenAppliances	125.113	**99.668**	104.094	100.850	109.575	112.017	102.100
Lighting2	84.514	73.412	75.298	74.958	72.800	**71.848**	72.641
Lighting7	32.775	27.390	25.715	26.188	25.786	25.216	**24.230**
MALLAT	4.744	5.130	3.332	4.122	2.091	**1.949**	3.461
Meat	0.820	0.492	1.046	1.122	0.040	**0.039**	2.381
MedicalImages	8.110	8.041	8.662	9.552	**6.188**	6.900	6.477
MiddlePhalanxOutlineAgeGroup	0.853	0.748	0.914	1.097	**0.511**	0.511	0.766
MiddlePhalanxOutlineCorrect	0.825	0.652	0.720	1.181	**0.507**	0.508	0.551
MiddlePhalanxTW	0.752	0.584	0.949	1.449	0.416	**0.404**	0.692
MoteStrain	24.706	22.632	**19.159**	20.079	21.629	20.790	20.155
NonInvasiveFatalECG_Thorax1	2.418	2.607	6.163	3.169	**1.139**	1.149	2.852
NonInvasiveFatalECG_Thorax2	2.311	2.169	2.536	2.858	1.088	**1.081**	2.318
OSULeaf	32.206	21.743	21.641	20.844	20.371	19.971	**19.120**
OliveOil	1.217	1.180	1.362	4.419	0.018	**0.017**	1.187
PhalangesOutlinesCorrect	1.681	1.312	1.946	1.684	**1.132**	1.146	1.277
Phoneme	181.389	134.930	135.640	136.674	133.475	121.774	**118.926**
Plane	1.058	0.783	1.202	1.809	0.444	**0.416**	1.075
ProximalPhalanxOutlineAgeGroup	0.620	0.530	0.807	0.946	0.352	**0.336**	0.431
ProximalPhalanxOutlineCorrect	0.859	0.707	1.053	1.164	**0.512**	0.514	0.537
ProximalPhalanxTW	0.630	0.513	0.645	1.089	**0.247**	0.261	0.800
RefrigerationDevices	182.659	156.169	151.285	149.829	152.254	155.795	**137.791**
ScreenType	187.460	156.534	155.496	155.359	**151.867**	158.273	153.550
ShapeletSim	236.166	123.055	124.282	127.652	122.746	123.699	**111.356**
ShapesAll	15.045	8.828	7.745	8.333	8.755	8.929	**7.448**
SmallKitchenAppliances	184.888	177.515	181.719	**177.369**	177.863	181.845	179.879
SonyAIBORobotSurface	8.765	6.722	7.511	7.685	**5.876**	6.252	6.663
SonyAIBORobotSurfaceII	11.896	10.897	12.207	12.315	**9.463**	9.754	11.239
StarLightCurves	16.522	9.449	6.834	6.645	6.557	**6.156**	6.448
Strawberry	2.034	1.642	1.315	1.704	1.225	**1.223**	1.489
SwedishLeaf	2.891	2.124	2.419	2.400	**1.887**	1.936	2.149
Symbols	2.140	1.012	**0.798**	1.212	1.133	0.882	1.573
ToeSegmentation1	35.839	29.387	27.285	26.861	27.618	30.725	**26.201**
ToeSegmentation2	36.316	26.012	23.562	22.410	24.837	24.226	**21.764**
Trace	2.169	2.206	1.249	1.374	**0.767**	0.964	0.981
TwoLeadECG	1.616	1.354	1.900	1.349	**1.015**	1.118	1.084
Two_Patterns	12.811	10.047	8.079	**7.850**	8.528	9.718	8.010
UWaveGestureLibraryAll	77.858	46.754	43.451	42.883	42.470	43.537	**37.055**
Wine	0.738	0.517	0.633	1.109	**0.112**	0.113	0.435
WordsSynonyms	17.235	11.125	10.347	10.832	12.036	11.489	**8.428**
Worms	107.089	73.280	68.799	74.002	75.779	80.812	**60.592**
WormsTwoClass	128.791	86.165	82.996	82.916	83.880	90.286	**77.192**
synthetic_control	16.805	8.880	8.886	9.295	8.924	8.926	**8.328**
uWaveGestureLibrary_X	33.867	20.419	19.155	19.131	18.673	19.393	**17.997**
uWaveGestureLibrary_Y	35.155	19.121	16.982	17.484	18.505	16.598	**15.786**
uWaveGestureLibrary_Z	33.574	19.668	18.401	18.701	18.693	18.248	**16.934**
wafer	30.883	**21.369**	24.101	25.974	23.579	31.298	24.725
yoga	33.428	14.453	14.055	11.822	11.343	12.424	**10.882**

Table 3: Barycenter Experiment, Average DTW Loss on Testing Set

	SoftDTW $\gamma = 1$	SoftDTW $\gamma = 0.1$	SoftDTW $\gamma = 0.01$	SoftDTW $\gamma = 0.001$	SSG	DBA	**Ours**
50words	13.770	11.642	11.031	10.913	11.162	11.195	**10.709**
Adiac	0.838	0.305	0.285	0.585	**0.245**	0.256	0.564
ArrowHead	5.427	4.402	3.690	**3.478**	3.766	3.688	4.009
Beef	14.946	14.178	9.814	11.599	14.822	12.016	**8.813**
BeetleFly	53.030	38.986	**36.598**	38.443	41.506	40.907	36.863
BirdChicken	43.657	**19.897**	22.499	20.594	36.865	30.522	22.585
CBF	24.343	15.210	17.237	15.566	14.194	14.596	**13.755**
Car	4.355	3.072	2.723	2.753	2.654	**2.462**	2.573
ChlorineConcentration	25.474	17.345	18.659	17.919	**16.459**	17.341	16.492
CinC_ECG_torso	180.663	141.548	141.753	132.553	166.838	136.126	**128.023**
Coffee	1.226	0.826	1.293	1.577	0.658	**0.654**	1.045
Computers	176.868	157.962	148.396	153.658	**146.668**	153.739	159.314
Cricket_X	51.207	37.780	38.551	37.916	36.245	37.398	**36.214**
Cricket_Y	43.451	34.059	33.094	33.679	**32.414**	33.642	32.629
Cricket_Z	48.466	36.712	36.351	37.211	36.525	36.586	**35.056**
DiatomSizeReduction	4.010	4.024	3.902	**3.896**	3.899	3.907	4.246
DistalPhalanxOutlineAgeGroup	1.523	1.304	1.526	2.036	**1.061**	1.065	1.841
DistalPhalanxOutlineCorrect	2.785	2.351	2.750	2.557	1.958	**1.956**	1.979
DistalPhalanxTW	1.405	0.997	1.977	1.705	0.806	**0.806**	1.179
ECG200	9.344	7.555	8.911	8.784	7.780	8.422	**6.959**
ECG5000	26.295	22.558	22.592	24.241	25.302	27.389	**22.372**
ECGFiveDays	8.767	10.175	10.589	10.965	7.422	**7.336**	8.647
Earthquakes	156.759	110.492	108.953	109.385	106.678	**102.084**	107.288
ElectricDevices	43.261	37.808	37.130	37.421	36.852	36.939	**35.069**
FISH	1.808	1.661	1.535	1.682	1.464	**1.422**	1.652
FaceAll	20.162	18.590	19.029	20.625	17.536	**17.428**	19.027
FaceFour	38.929	38.416	40.709	40.024	**36.045**	39.897	39.898
FacesUCR	20.339	19.555	20.671	20.296	17.635	**17.090**	18.105
FordA	65.742	56.997	55.973	55.459	53.513	53.965	**52.667**
FordB	69.145	60.406	61.231	59.489	57.921	58.954	**56.693**
Gun_Point	8.030	2.816	2.478	2.428	3.172	2.796	**2.315**
Ham	31.478	28.953	26.317	30.149	26.639	26.455	**25.510**
HandOutlines	-	-	-	-	10.864	10.794	7.733
Haptics	25.431	22.077	20.684	21.966	**17.196**	17.926	24.014
Herring	1.465	1.385	1.331	1.462	**0.915**	0.948	1.372
InlineSkate	127.648	65.537	48.215	45.301	58.181	55.270	**40.608**
InsectWingbeatSound	17.576	16.680	15.102	**15.010**	16.210	15.079	15.493
ItalyPowerDemand	2.523	2.578	2.523	2.820	**2.147**	2.705	2.369
LargeKitchenAppliances	118.456	114.174	**107.120**	107.325	122.776	131.369	110.499
Lighting2	86.450	75.184	**72.139**	81.725	78.196	78.350	73.070
Lighting7	48.326	37.158	**36.557**	38.376	37.673	37.747	41.663
MALLAT	6.002	4.692	4.717	6.280	3.668	**3.379**	4.938
Meat	0.593	0.259	0.438	1.569	0.041	**0.041**	0.470
MedicalImages	8.252	8.807	9.334	8.741	7.194	**6.916**	7.762
MiddlePhalanxOutlineAgeGroup	0.835	0.770	1.540	1.158	**0.723**	0.733	0.962
MiddlePhalanxOutlineCorrect	1.235	1.223	1.145	2.254	1.187	1.200	**0.998**
MiddlePhalanxTW	0.836	0.729	1.054	1.051	0.592	**0.570**	0.694
MoteStrain	22.970	23.938	22.080	24.643	21.964	**21.094**	25.032
NonInvasiveFatalECG_Thorax1	2.745	2.919	3.560	3.890	1.548	**1.509**	3.561
NonInvasiveFatalECG_Thorax2	2.384	2.888	3.146	3.711	**1.465**	1.521	2.955
OSULeaf	35.193	26.392	23.544	24.177	25.619	**23.188**	23.352
OliveOil	0.961	0.747	2.107	2.002	0.020	**0.020**	1.082
PhalangesOutlinesCorrect	2.067	1.556	1.782	1.487	1.254	**1.239**	1.251
Phoneme	315.058	291.260	**286.486**	286.661	319.069	311.551	289.715
Plane	1.171	0.752	1.247	1.572	**0.534**	0.553	1.109
ProximalPhalanxOutlineAgeGroup	0.595	0.461	0.735	1.230	0.356	**0.351**	0.521
ProximalPhalanxOutlineCorrect	0.704	0.587	0.659	1.020	**0.432**	0.435	0.536
ProximalPhalanxTW	0.775	0.586	0.707	1.699	**0.343**	0.348	0.815
RefrigerationDevices	198.045	167.040	162.204	164.149	164.174	174.493	**156.454**
ScreenType	143.046	**119.853**	126.123	124.484	123.647	136.127	135.216
ShapeletSim	243.103	150.826	151.325	152.915	153.454	153.326	**146.383**
ShapesAll	21.230	15.139	12.812	12.824	14.086	14.084	**12.408**
SmallKitchenAppliances	176.407	173.053	175.462	**171.829**	178.142	173.198	181.316
SonyAIBORobotSurface	8.647	9.489	9.221	10.841	7.459	**7.430**	7.882
SonyAIBORobotSurfaceII	16.137	17.066	16.947	17.336	**14.439**	15.585	15.215
StarLightCurves	17.915	10.925	7.939	7.484	7.457	7.376	**7.316**
Strawberry	2.706	1.921	2.757	2.635	1.609	1.656	**1.599**
SwedishLeaf	2.842	2.442	2.483	2.688	2.087	**2.073**	2.395
Symbols	6.280	5.245	3.973	**3.930**	5.395	4.862	4.498
ToeSegmentation1	43.606	36.174	34.410	35.602	35.703	36.982	**34.158**
ToeSegmentation2	72.831	58.728	**47.650**	51.515	54.558	57.188	53.725
Trace	2.879	1.720	1.374	1.448	**0.818**	0.963	1.048
TwoLeadECG	1.619	1.275	1.441	1.708	**1.185**	1.238	1.383
Two_Patterns	12.546	9.490	8.107	**8.012**	9.502	8.415	8.943
UWaveGestureLibraryAll	78.820	51.054	46.568	47.568	45.660	47.290	**41.052**
Wine	0.807	0.549	2.339	1.202	0.103	**0.101**	0.765
WordsSynonyms	25.751	18.486	17.173	16.387	20.113	19.359	**16.247**
Worms	169.897	111.593	99.132	97.843	123.028	111.755	**94.781**
WormsTwoClass	172.790	119.187	110.483	**105.179**	114.444	113.869	105.696
synthetic_control	17.031	9.794	9.631	9.806	9.620	9.692	**9.126**
uWaveGestureLibrary_X	34.245	21.273	19.248	19.091	20.334	20.586	**18.738**
uWaveGestureLibrary_Y	37.343	20.565	19.212	18.875	19.129	19.481	**17.015**
uWaveGestureLibrary_Z	35.221	22.716	20.444	20.108	20.072	21.296	**18.303**
wafer	30.642	**23.935**	27.221	24.758	30.820	32.328	24.577
yoga	24.982	14.573	15.286	11.849	11.681	11.887	**11.120**