

A A Supplementary Lemma

Lemma 2. *Let ϵ_t be Rademacher random vectors, and Z_t be non-negative real-valued random variables, such that $\mathbb{E}[Z_t^2] \leq M$. Then:*

$$\mathbb{E}_{Z_{1:T}, \epsilon_{1:T}} \left[\max_{\pi \in \Pi} \sum_{t=1}^T \epsilon_t(\pi(x_t)) \cdot Z_t \right] \leq \sqrt{2TM \log(N)}$$

Proof:

$$\begin{aligned} \mathbb{E}_{Z_{1:T}, \epsilon_{1:T}} \left[\max_{\pi \in \Pi} \sum_{t=1}^T \epsilon_t(\pi(x_t)) \cdot Z_t \right] &= \mathbb{E}_{Z_{1:T}} \left[\frac{1}{\lambda} \mathbb{E}_{\epsilon_{1:T}} \left[\log \left(\max_{\pi \in \Pi} e^{\lambda \sum_{t=1}^T \epsilon_t(\pi(x_t)) \cdot Z_t} \right) \right] \right] \\ &\leq \mathbb{E}_{Z_{1:T}} \left[\frac{1}{\lambda} \log \left(\mathbb{E}_{\epsilon_{1:T}} \left[\max_{\pi \in \Pi} e^{\lambda \sum_{t=1}^T \epsilon_t(\pi(x_t)) \cdot Z_t} \right] \right) \right] \\ &\leq \mathbb{E}_{Z_{1:T}} \left[\frac{1}{\lambda} \log \left(\mathbb{E}_{\epsilon_{1:T}} \left[\sum_{\pi \in \Pi} e^{\lambda \sum_{t=1}^T \epsilon_t(\pi(x_t)) \cdot Z_t} \right] \right) \right] \\ &= \mathbb{E}_{Z_{1:T}} \left[\frac{1}{\lambda} \log \left(\sum_{\pi \in \Pi} \mathbb{E}_{\epsilon_{1:T}} \left[\prod_{t=1}^T e^{\lambda \epsilon_t(\pi(x_t)) \cdot Z_t} \right] \right) \right] \\ &= \mathbb{E}_{Z_{1:T}} \left[\frac{1}{\lambda} \log \left(\sum_{\pi \in \Pi} \prod_{t=1}^T \mathbb{E}_{\epsilon_t} \left[e^{\lambda \epsilon_t(\pi(x_t)) \cdot Z_t} \right] \right) \right]. \end{aligned}$$

Now observe that $\mathbb{E}_{\epsilon_t} \left[e^{\lambda \epsilon_t(\pi(x_t)) \cdot Z_t} \right] = \frac{e^{\lambda \cdot Z_t} + e^{-\lambda \cdot Z_t}}{2} \leq e^{\lambda^2 \cdot Z_t^2 / 2}$. Thus

$$\begin{aligned} \mathbb{E}_{Z_{1:T}, \epsilon_{1:T}} \left[\max_{\pi \in \Pi} \sum_{t=1}^T \epsilon_t(\pi(x_t)) \cdot Z_t \right] &\leq \mathbb{E}_{Z_{1:T}} \left[\frac{1}{\lambda} \log \left(\sum_{\pi \in \Pi} \prod_{t=1}^T e^{\lambda^2 \cdot Z_t^2 / 2} \right) \right] \\ &= \mathbb{E}_{Z_{1:T}} \left[\frac{1}{\lambda} \log \left(N e^{\lambda^2 \sum_{t=1}^T Z_t^2 / 2} \right) \right] \\ &= \frac{1}{\lambda} \log(N) + \lambda \mathbb{E}_{Z_{1:T}} \left[\sum_{t=1}^T Z_t^2 / 2 \right] \\ &\leq \frac{1}{\lambda} \log(N) + \lambda MT / 2. \end{aligned}$$

Optimizing over λ yields the result. ■