
Spectral Norm Regularization of Orthonormal Representations for Graph Transduction

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A Preliminaries and Definitions

For completeness, we state/prove some of the non-trivial results used in the paper.

Notations. Let $\|\cdot\|_0$ and $\|\cdot\|_\infty$ be the L0 and L-infinity norms respectively. For $y \in \mathcal{Y}$ and $\hat{y} \in \widehat{\mathcal{Y}}$; let $\ell^{hng}(y, \hat{y}) = (1 - y\hat{y})_+^1$, $\ell^r(y, \hat{y}) = \min\{1, (1 - y\hat{y})_+\}$ and $\ell^{0-1}(y, \hat{y}) = \mathbb{1}[y\hat{y} < 0]$ denote the hinge, ramp and 0-1 loss respectively. Note that these loss functions upper bound the other in same order, $\ell^{hng} \geq \ell^r \geq \ell^{0-1}$.

Asymptotic Notations [1]. For non-negative functions $f_1(n)$ and $f_2(n)$

- $f_1(n) = O(f_2(n)) \implies \exists n_0$ and a constant $c > 0$ such that $\forall n > n_0, f_1(n) \leq cf_2(n)$.
- $f_1(n) = \Omega(f_2(n)) \implies \exists n_0$ and a constant $c > 0$ such that $\forall n > n_0, f_1(n) \geq cf_2(n)$.
- $f_1(n) = \Theta(f_2(n))$ iff $f_1(n) = O(f_2(n))$ and $f_1(n) = \Omega(f_2(n))$.

B Generalization Bound

Let $\mathbf{U} \in \text{Lab}(G)$ be the orthonormal embedding corresponding to the graph kernel $\mathbf{K} \in \mathcal{K}(G)$. Note that the classifier learnt by the SVM formulation $\omega_C(\mathbf{K}, \mathbf{y}_S)$ as in (3) (paper), is of the form $h = \mathbf{U}\alpha$, where α is in the feasible set. In general, we define the following function class associated with the orthonormal embedding

$$\tilde{\mathcal{H}}_{\mathbf{U}} = \{h \mid h = \mathbf{U}\alpha, \alpha \in \mathbb{R}^n, \|\alpha\|_\infty \leq C, \|\alpha\|_0 \leq m\} \quad (1)$$

We follow a similar proof technique as in [3], however specialize to the class of orthonormal representation and SPORE formulation.

Theorem 1. (Paper) Let $G = (V, E)$ be a simple graph with unknown binary labels $\mathbf{y} \in \mathcal{Y}^n$ on the vertices V . Let $\mathbf{K} \in \mathcal{K}(G)$. Given G , and labels of a randomly drawn subgraph S , let $\hat{\mathbf{y}} \in \widehat{\mathcal{Y}}^n$ be the predictions learnt by $\omega_C(\mathbf{K}, \mathbf{y}_S)$ as in (3) (paper). Then, for $m \leq n/2$, with probability $\geq 1 - \delta$ over the choice of $S \subset V$, such that $|S| = m$

$$er_S^{0-1}[\hat{\mathbf{y}}] \leq \frac{1}{m} \sum_{i \in S} \ell^{hng}(y_i, \hat{y}_i) + 2C\sqrt{2\lambda_1(\mathbf{K})} + O\left(\sqrt{\frac{1}{m} \log \frac{1}{\delta}}\right) \quad (2)$$

¹ $(a)_+ = \max(a, 0)$.

Proof. Let $\mathbf{U} \in \text{Lab}(G)$ be the orthonormal representation associated with \mathbf{K} . Let $\pi = [\pi_1, \dots, \pi_n]$ denote a permutation on $[n]$. For any π , without loss of generality, let the first $m \in [n]$ nodes be labelled. Let $er_S^{r,\pi}[\hat{\mathbf{y}}] = \frac{1}{m} \sum_{i=1}^m \ell^r(y_{\pi_i}, \hat{y}_{\pi_i})$, where ℓ^r is the ramp loss as in Section A; and similarly for $er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}]$. Let $\tilde{\pi} = [1, 2, \dots, n]$ denote the trivial permutation. Introduce a ghost permutation –

$$\begin{aligned} er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] &\leq er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] + er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] - er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] \\ &\leq er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] + \sup_{\tilde{\mathbf{y}} \in \tilde{\mathcal{Y}}_{\mathbf{U}}} \left[er_S^{r,\tilde{\pi}}[\tilde{\mathbf{y}}] - er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] \right] \leq er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] + \Phi_{\mathbf{U}}^{\tilde{\pi}}[\hat{\mathbf{y}}] \end{aligned} \quad (3)$$

where

$$\Phi_{\mathbf{U}}^{\tilde{\pi}}[\hat{\mathbf{y}}] = \mathbb{E}_{\pi} \sup_{\tilde{\mathbf{y}} \in \tilde{\mathcal{Y}}_{\mathbf{U}}} \left[er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}] - er_S^{r,\tilde{\pi}}[\tilde{\mathbf{y}}] + er_S^{r,\pi}[\tilde{\mathbf{y}}] - er_S^{r,\pi}[\hat{\mathbf{y}}] \right]$$

where $\tilde{\mathcal{Y}}_{\mathbf{U}} = \{\tilde{\mathbf{y}} | \tilde{\mathbf{y}} = \mathbf{U}^{\top} h, h \in \tilde{\mathcal{H}}_{\mathbf{U}}\}$, for $\tilde{\mathcal{H}}_{\mathbf{U}}$ as in (1). (3) follows by adding and subtracting $\mathbb{E}_{\pi} [er_S^{r,\pi}[\hat{\mathbf{y}}]] = \mathbb{E}_{\pi} [er_S^{r,\tilde{\pi}}[\hat{\mathbf{y}}]]$ inside the supremum and applying Jensen's inequality to bring the expectation of out the supremum.

We use Doob's martingale process (see [2]) to bound the function $\Phi_{\mathbf{U}}^{\tilde{\pi}}$ by its expectation –

Lemma 1. *Given $n \in \mathbb{N}$ and $m \in [n]$, let π be a random permutation vector over $[n]$. Let π^{ij} to denote perturbed permutation, where i^{th} and j^{th} elements are exchanged. Let $f(\pi)$ be an π -permutation symmetric function satisfying $|f(\pi) - f(\pi^{ij})| \leq \beta \forall i \in [m], j \notin [m]$. Then*

$$\Pr_{\pi} \{f(\pi) - \mathbb{E}_{\pi}[f(\pi)] \geq \epsilon\} \leq \exp\left(-\frac{\epsilon^2}{\beta^2 m}\right)$$

Using the above for $\beta = \frac{2}{m}$, we get for $\delta > 0$, w.p. $\geq 1 - \delta$ over the permutation $\tilde{\pi}$

$$\Phi_{\mathbf{U}}^{\tilde{\pi}}[\hat{\mathbf{y}}] \leq \mathbb{E}_{\pi} [\Phi_{\mathbf{U}}^{\tilde{\pi}}[\hat{\mathbf{y}}]] + 2\sqrt{\frac{1}{m} \log \frac{1}{\delta}} \quad (4)$$

Now we bound $\mathbb{E}_{\pi} [\Phi_{\mathbf{U}}^{\tilde{\pi}}[\hat{\mathbf{y}}]]$ using results of [3] as follows

$$\mathbb{E}_{\pi} [\Phi_{\mathbf{U}}^{\tilde{\pi}}[\hat{\mathbf{y}}]] \leq R\left(\mathcal{L}_{\mathbf{U}}^r, m, \frac{mu}{n^2}\right) + O\left(\frac{1}{\sqrt{m}}\right) \quad (5)$$

where $\mathcal{L}_{\mathbf{U}}^r = \{\ell^r(y_1, \tilde{y}_1), \dots, \ell^r(y_n, \tilde{y}_n) | \tilde{\mathbf{y}} \in \tilde{\mathcal{Y}}_{\mathbf{U}}\}$, and $R(\mathcal{V}, m) := \frac{2}{m} \mathbb{E}_{\sigma} \left[\sup_{\mathbf{v} \in \mathcal{V}} \mathbf{v}^{\top} \sigma \right]$ is the Rademacher average of the vector space $\mathcal{V} \subseteq \mathbb{R}^n$, where σ is an *i.i.d.* random vector, each entry taking values $+1, -1, 0$ with probability $p, p, 1 - p$ respectively; $p := m/n$. We recall the following interesting property of Rademacher averages [3] –

Lemma 2. *For any $\mathcal{V} \subseteq \mathbb{R}^n$, $m \in [n]$ and $0 \leq p_1 \leq p_2 \leq \frac{1}{2}$, $R(\mathcal{V}, m, p_1) \leq R(\mathcal{V}, m, p_2)$.*

Using the above, we get

$$R\left(\mathcal{L}_{\mathbf{U}}^r, m, \frac{mu}{n^2}\right) \leq R\left(\mathcal{L}_{\mathbf{U}}^r, m, \frac{m}{n}\right) \leq R\left(\tilde{\mathcal{Y}}_{\mathbf{U}}, m, \frac{m}{n}\right) \quad (6)$$

where the last inequality follows from the contraction property [3] of the class complexity. Note that (6) relates to the function class, for which we can derive a tight estimate as follows – We derive the following tight Rademacher complexity estimate for the class of orthonormal embeddings –

Lemma 3. $R\left(\tilde{\mathcal{Y}}_{\mathbf{U}}, m, \frac{m}{n}\right) \leq 2C\sqrt{2\lambda_1(\mathbf{K})}$

For any random vector σ , the supremum is given by

$$\sup_{\tilde{\mathbf{y}} \in \tilde{\mathcal{Y}}_{\mathbf{U}}} \sum_{i \in [n]} \sigma_i \tilde{y}_i = \sup_{h \in \tilde{\mathcal{H}}_{\mathbf{U}}} \sum_{i \in [n]} \sigma_i \langle h, \mathbf{u}_i \rangle = \sup_{h \in \tilde{\mathcal{H}}_{\mathbf{U}}} \left\langle h, \sum_{i \in [n]} \sigma_i \mathbf{u}_i \right\rangle \leq C\sqrt{m\lambda_1(\mathbf{K})} \left\| \sum_{i \in [n]} \sigma_i \mathbf{u}_i \right\|$$

The last equality from optimality over supremum and the norm constraint

$$\max_{h \in \mathcal{H}_{\mathbf{U}}} \|h\| = \max_{\|\alpha\|_{\infty} \leq C, \|\alpha\|_0 \leq m} \sqrt{\alpha^{\top} \mathbf{K} \alpha} \leq C \sqrt{m \lambda_1(\mathbf{K})}$$

Now, taking expectation over σ , one obtains

$$R\left(\tilde{\mathcal{Y}}_{\mathbf{U}}, m, \frac{m}{n}\right) = 2C \sqrt{\frac{\lambda_1(\mathbf{K})}{m}} \mathbb{E}_{\sigma} \left[\sqrt{\sigma^{\top} \mathbf{K} \sigma} \right] \quad (7)$$

Using Jensen's inequality, the expectation term can be upper bounded by $\sqrt{\mathbb{E}_{\sigma}[\sigma^{\top} \mathbf{K} \sigma]}$. Further, using *i.i.d.* assumption of σ , the expectation evaluates to $(2m/n) \sum_{i \in [n]} K_{ii} = 2m$. Plugging back in (7) proves the lemma. Finally, (2) is immediate by combining Lemma 3, (6), (5), (4) and (3). \square

C SPORE formulation and PAC analysis

We show that the spectral norm of the kernel relates to the structural properties of the graph

Lemma 2. (*Paper*) Given a simple, undirected graph $G = (V, E)$, $\max_{\mathbf{K} \in \mathcal{K}(G)} \lambda_1(\mathbf{K}) = \vartheta(\bar{G})$.

Proof. We recall another definition of the ϑ function [6]

$$\vartheta(\bar{G}) = \max_{\mathbf{U} \in \text{Lab}(G)} \max_{\mathbf{c} \in \mathcal{S}_2^{d-1}} \sum_{i \in [n]} (\mathbf{u}_i^{\top} \mathbf{c})^2 \quad (8)$$

For a fixed $\mathbf{U} \in \text{Lab}(G)$ and $\mathbf{c} \in \mathcal{S}_2^{d-1}$, the summation evaluates to $\mathbf{c}^{\top} \mathbf{U} \mathbf{U}^{\top} \mathbf{c}$. Thus, for any fixed $\mathbf{U} \in \text{Lab}(G)$, $\max_{\mathbf{c} \in \mathcal{S}_2^{d-1}} \sum_{i \in [n]} (\mathbf{u}_i^{\top} \mathbf{c})^2 = \lambda_1(\mathbf{U} \mathbf{U}^{\top})$. From first principles, $\lambda_1(\mathbf{U} \mathbf{U}^{\top}) = \lambda_1(\mathbf{U}^{\top} \mathbf{U}) = \lambda_1(\mathbf{K})$, where $\mathbf{K} = \mathbf{U}^{\top} \mathbf{U} \in \mathcal{K}(G)$ (Section 1, paper). As there is correspondence between the two sets $\text{Lab}(G)$ and $\mathcal{K}(G)$ (Section 1), (8) evaluates to $\vartheta(\bar{G}) = \max_{\mathbf{K} \in \mathcal{K}(G)} \lambda_1(\mathbf{K})$, thus proving the claim. \square

Following the proof technique of [7] and [5], we prove –

Lemma 3. (*paper*) Given G and \mathbf{y} , for any $S \subseteq V$ and $C > 0$

$$\min_{\mathbf{K} \in \mathcal{K}(G)} \omega_C(\mathbf{K}^S, \mathbf{y}_S) \leq \vartheta(G)/2$$

Proof. We recall another definition of the ϑ function [6]

$$\vartheta(G) = \min_{\mathbf{U} \in \text{Lab}(G)} \min_{\mathbf{c} \in \mathcal{S}_2^{d-1}} \max_{i \in [n]} \frac{1}{(\mathbf{c}^{\top} \mathbf{u}_i)^2} \quad (9)$$

For a fixed \mathbf{K} , from the primal of SVM formulation, it follows that

$$\omega_{\infty}(\mathbf{K}^S, \mathbf{y}_S) = \min_{\mathbf{w} \in \mathbb{R}^d} \frac{1}{2} \|\mathbf{w}\|^2 \quad \text{s.t.} \quad y_i \mathbf{w}^{\top} \mathbf{u}_i \geq 1 \quad \forall i \in S$$

where \mathbf{U} is the orthonormal representation corresponding to \mathbf{K} (Section 1). Now, writing $\mathbf{w} = t\mathbf{c}$, where $\mathbf{c} \in \mathcal{S}_2^{d-1}$

$$\begin{aligned} &= \min_{t, \mathbf{c} \in \mathcal{S}_2^{d-1}} t^2 \quad \text{s.t.} \quad t \geq \frac{1}{y_i \mathbf{c}^{\top} \mathbf{u}_i} \quad \forall i \in S \\ &= \min_{\mathbf{c} \in \mathcal{S}_2^{d-1}} \max_{i \in S} \frac{1}{(y_i \mathbf{c}^{\top} \mathbf{u}_i)^2} \\ &= \min_{\mathbf{c} \in \mathcal{S}_2^{d-1}} \max_{i \in S} \frac{1}{(\mathbf{c}^{\top} \mathbf{u}_i)^2} \leq \min_{\mathbf{c} \in \mathcal{S}_2^{d-1}} \max_{i \in [n]} \frac{1}{(\mathbf{c}^{\top} \mathbf{u}_i)^2} \end{aligned}$$

Thus, the proof follows from (9), using a trivial bound $\omega_C(\mathbf{K}^S, \mathbf{y}_S) \leq \omega_{\infty}(\mathbf{K}^S, \mathbf{y}_S)$ and noting that the sets $\text{Lab}(G)$ and $\mathcal{K}(G)$ are equivalent Section 1. \square

Theorem 4. (paper) Let $G = (V, E)$, $V = [n]$ be a simple graph with unknown binary labels $\mathbf{y} \in \mathcal{Y}^n$ on the vertices V . Given G , and labels of a randomly drawn subgraph $S \subset V$, $m = |S|$; let $\hat{\mathbf{y}}$ be the predictions learnt by SPORE (5), for parameters $C = \left(\frac{\vartheta(G)}{m\sqrt{\vartheta(\bar{G})}}\right)^{\frac{1}{2}}$ and $\beta = \frac{\vartheta(G)}{\vartheta(\bar{G})}$. Then, for $m \leq n/2$, with probability $\geq 1 - \delta$ over the choice of $S \subset V$, such that $|S| = m$

$$er_S^{0.1}[\hat{\mathbf{y}}] = O\left(\frac{1}{m}\left(\sqrt{n\vartheta(G)} + \log\frac{1}{\delta}\right)\right)^{\frac{1}{2}}$$

Proof. Let \mathbf{K}^* be the kernel learnt by SPORE (5). Using Theorem 1 (paper) for final predictions $\hat{\mathbf{y}}$, we obtain

$$er_S^{0.1}[\hat{\mathbf{y}}] \leq \frac{1}{m} \sum_{i \in S} \ell^{hng}(y_i, \hat{y}_i) + 2C\sqrt{2\lambda(\mathbf{K}^*)} + O\left(\sqrt{\frac{1}{m} \log\frac{1}{\delta}}\right)$$

Using Lemma 2 (paper) $\lambda(\mathbf{K}^*) \leq \vartheta(\bar{G})$, where \bar{G} is complement graph of G

$$\leq \frac{1}{m} \sum_{i \in S} \ell^{hng}(y_i, \hat{y}_i) + 2C\sqrt{2\vartheta(\bar{G})} + O\left(\sqrt{\frac{1}{m} \log\frac{1}{\delta}}\right) \quad (10)$$

Recall the primal formulation of (3) (paper) for $\mathbf{K}^* = \mathbf{U}^{*\top} \mathbf{U}^*$, $\mathbf{U}^* \in \text{Lab}(G)$ (Section 1)

$$\omega_C(\mathbf{K}^*, \mathbf{y}_S) = \min_{\mathbf{w} \in \mathbb{R}^d} \frac{1}{2} \|\mathbf{w}\|_2^2 + C \sum_{i \in S} \ell^{hng}(y_i, \mathbf{w}^\top \mathbf{u}_i^*)$$

Let \mathbf{w}^* be the solution at optimal, then note that $\hat{y}_i = \mathbf{w}^{*\top} \mathbf{u}_i^*$, $\forall i \in [n]$. Thus, we bound the empirical error as follows

$$\begin{aligned} C \sum_{i \in S} \ell^{hng}(y_i, \hat{y}_i) &= \omega_C(\mathbf{K}^*, \mathbf{y}_S) - \frac{1}{2} \|\mathbf{w}^*\|_2^2 \leq \omega_C(\mathbf{K}^*, \mathbf{y}_S) \\ &\leq \Psi_{C, \beta}(G, \mathbf{y}_S) = \min_{\mathbf{K} \in \mathcal{K}(G)} \omega_C(\mathbf{K}^S, \mathbf{y}_S) + \beta \lambda_1(\mathbf{K}) \\ &\leq \min_{\mathbf{K} \in \mathcal{K}(G)} \omega_C(\mathbf{K}^S, \mathbf{y}_S) + \beta \max_{\mathbf{K} \in \mathcal{K}(G)} \lambda_1(\mathbf{K}) \leq \frac{\vartheta(G)}{2} + \beta \vartheta(\bar{G}) \end{aligned}$$

The last inequality follows from Lemma 2 and 3 (paper). Plugging back in (10), we get

$$er_S^{0.1}[\hat{\mathbf{y}}] \leq \frac{1}{2Cm} \vartheta(G) + \frac{\beta}{Cm} \vartheta(\bar{G}) + 2C\sqrt{2\vartheta(\bar{G})} + O\left(\sqrt{\frac{1}{m} \log\frac{1}{\delta}}\right) \quad (11)$$

Choosing β such that $\frac{\beta}{Cm} \vartheta(\bar{G}) = 2C\sqrt{2\vartheta(\bar{G})}$ and optimizing for C gives us the choice of parameters as in the statement of the theorem. Plugging back in (11), we get

$$= O\left(\frac{1}{\sqrt{m}} \left(\sqrt{\vartheta(G)} \sqrt{\vartheta(\bar{G})} + \sqrt{\log\frac{1}{\delta}}\right)\right)$$

Finally, using $\vartheta(G)\vartheta(\bar{G}) = n$ [6], and concavity $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$ proves the result. \square

D Proposed Algorithms

D.1 The Projection Algorithm

Algorithm 1 lists the steps of the accelerated gradient descent algorithm FISTA applied to 16. The objective $p_{\sigma_{\mathcal{X}}}$ is composed of the smooth term $p_{\sigma_{\mathcal{X}}}^s(a, b) = \frac{1}{2} \|v - a - b\|^2$ with the Lipschitz continuous gradients with constant $L_p = 1$, and the non smooth term $p_{\sigma_{\mathcal{X}}}^n(a, b) = \sigma_A(a) + \sigma_B(b)$. Step 6 executes a gradient descent on a with respect to $p_{\sigma_{\mathcal{X}}}$ followed by a proximal mapping with σ_A . The gradient of $p_{\sigma_{\mathcal{X}}}$ equals $a + b - v$, and the stepsize chosen to be $\frac{1}{L_p} = 1$. Similarly step 6 perform the

gradient descent on b followed by the proximal mapping with σ_B , The definitions for the support functions and its associated operators are provided in Section D.3.

The algorithm requires as input the current iterate $v_k = x_k - \alpha_k g_k$ to project into the set \mathcal{X} , and the number of iterations S and returns an approximate projection $x_{k+1} = v_k - a_S - b_S$.

For a graph $G = (V, E)$, the set $\mathcal{X} = A \cap B$ is defined with

$$A = \{\mathbf{K} \in \mathbb{R}^{n \times n} | \mathbf{K} \succeq 0, \text{tr}(\mathbf{K}) = n\} \text{ and } B = \{\mathbf{K} \in \mathbb{R}^{n \times n} | \mathbf{K}_{ii} = 1, i \in [n], \mathbf{K}_{ij} = 0, (i, j) \notin E\} \quad (12)$$

Algorithm 1 Accelerated Gradient Descent (FISTA) to solve 11

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1: function IIP_FISTA( $v, S$ )
2:   Initialize  $a_0 = 0, b_0 = 0$ .
3:   Initialize  $(\hat{a}_0, \hat{b}_0) = (a_0, b_0)$ .
4:   Initialize  $t_0 = 1$ .
5:   for  $t = 1, \dots, S$  do
6:      $a_t = \text{prox}_{\sigma_A}(v - b_t)$ . ▷ Use (14) for  $\text{prox}_{\sigma_A}$ 
7:      $b_t = \text{prox}_{\sigma_B}(v - a_t)$ . ▷ Use (21) for  $\text{prox}_{\sigma_B}$ 
8:      $\beta_t = \frac{1 + \sqrt{1 + 4\beta_{t-1}^2}}{2}$ 
9:      $\hat{a}_t = a_t + \frac{\beta_{t-1} - 1}{\beta_t} (a_t - a_{t-1})$ 
10:     $\hat{b}_t = b_t + \frac{\beta_{t-1} - 1}{\beta_t} (b_t - b_{t-1})$ 
11:   end for
12:   return  $z = v - a_S - b_S$ 
13: end function

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D.2 Subgradient Descent Algorithm With Approximate Projection

We now use the approximate projection on \mathcal{X} computed in Algorithm 1 to solve (8). In particular we analyze the following algorithm

Algorithm 2 Approximate Projected sub-gradient descent

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1: function APPROX-PROJ-SUBG( $\mathbf{K}_0, L, R, \hat{R}, T$ )
2:    $s = \frac{L}{\sqrt{T}} \cdot (\sqrt{R^2} + \hat{R}^2)$  ▷ compute stepsize
3:   Initialize  $t_0 = 1$ .
4:   for  $t = 1, \dots, T$  do
5:     compute  $h_{t-1}$  ▷ subgradient of  $f$  at  $\mathbf{K}_{t-1}$ 
6:      $v_t = \mathbf{K}_{t-1} - \frac{s}{\|h_{t-1}\|} h_{t-1}$ 
7:      $\tilde{\mathbf{K}}_t = \text{IIP\_FISTA}(v_t, \sqrt{T})$  ▷ Use Algorithm 1
8:      $\mathbf{K}_t = \text{Proj}_A(\tilde{\mathbf{K}}_t) = \mathbf{K}_t - \text{prox}_{\sigma_A}(\mathbf{K}_t)$  ▷ Use (14)
9:   end for
10: end function

```

D.3 Support Functions and their Proximal operators

D.3.1 Support functions

The expressions for the support functions σ_A and σ_B are provided below.

Claim 1. $\sigma_A(a) = n \max(\lambda_{\max}(a), 0)$

Proof. $\sigma_A(a) = \max_{\mathbf{K} \in A} \text{tr}(a^\top \mathbf{K})$. The Eigen decomposition of a gives $a = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^\top = \mathbf{U} \Lambda \mathbf{U}^\top$, where the \mathbf{u}_i 's are chosen to be forming an orthogonal basis. The matrix \mathbf{K} can be written

using this basis as \mathbf{UMU}^\top , where \mathbf{M} need not be a diagonal one.

$$\text{tr}(a^\top \mathbf{K}) = \text{tr}(\mathbf{UMU}^\top \mathbf{U}\Lambda\mathbf{U}^\top) = \text{tr}(\mathbf{UM}\Lambda\mathbf{U}^\top) = \text{tr}(\mathbf{M}\Lambda) \leq \sum_{i:\lambda_i>0} M_{ii}\lambda_i, \quad (13)$$

since M_{ii} need to be non-negative. Now,

$$\max_{\mathbf{K} \in \mathcal{S}} \text{tr}(a^\top \mathbf{K}) \leq \max_{M_{ii}>0, \sum_i M_{ii} \leq n} \sum_{i:\lambda_i>0} M_{ii}\lambda_i = n \max(\lambda_{\max}(\mathbf{X}), 0)$$

Hence, by choosing $M_{ii} = n$ for i corresponding to the largest positive eigen value, or $M_{ii} = 0$, if $\lambda_i < 0, \forall i$, we get $\sigma_A(a) = n \max(\lambda_{\max}(a), 0)$. \square

Claim 2. $\sigma_B(b) = \max_{K \in B} \text{tr}(b^\top K) = \begin{cases} \text{tr}(b) & b_{ij} = 0, \forall (i, j) \in E \\ \infty & \text{otherwise.} \end{cases}$

Proof. Trivial by using the definition of the support function. \square

D.3.2 Proximal operators

Claim 3. $\text{prox}_{\sigma_A}^\alpha(\hat{a}) = \mathbf{U}\text{Diag}(\text{prox}_m^\alpha(z))\mathbf{U}^\top$, where $\mathbf{U}\text{Diag}(z)\mathbf{U}^\top$ is the eigen decomposition of \hat{a} and $m(z) = \max([z; 0])$. And $\text{prox}_m^\alpha(z)|_i = \min(z_i, \max(t_{\max}^*, 0))$ where t_{\max}^* is the solution of

$$\sum_{i=1}^n \frac{1}{\alpha} (z_i - t)_+ = 1 \quad (14)$$

Proof.

$$\begin{aligned} \text{prox}_m^\alpha(z) &= \underset{x}{\text{argmin}} \frac{1}{2\alpha} \|x - z\|_2^2 + \max([x; 0]) \\ &= \underset{z; t \geq x_i, t \geq 0}{\text{argmin}} \frac{1}{2\alpha} \|x - z\|_2^2 + t \end{aligned} \quad (15)$$

Let $L(x, t, \mu) = \frac{1}{2\alpha} \|x - z\|_2^2 + t - \xi t + \sum_{i=1}^n \mu_i (x_i - t)$. Equating the gradient of the Lagrangian function to 0 at optimality,

$$\frac{\partial L}{\partial x_i} = \frac{1}{\alpha} (x_i^* - z_i^*) + \mu_i^* = 0. \quad (16)$$

$$\frac{\partial L}{\partial t} = \sum_{i=1}^n \mu_i^* + \xi^* = 1. \quad (17)$$

The KKT optimality conditions provide

$$\xi^* t^* = 0, \mu_i^* (x_i^* - t^*) = 0 \quad (18)$$

$\mu_i^* > 0 \Rightarrow x_i = t^*$. Combining this with the constraint $\mu_i \geq 0$ and (16) gives $\mu_i = \frac{1}{\alpha} (z_i - t^*)_+$

$$\sum_{i=1}^n \frac{1}{\alpha} (z_i - t^*)_+ + \xi^* = 1 \quad (19)$$

$t^* > 0$ solves the above equation if and only if

$$\sum_{i=1}^n \frac{1}{\alpha} (z_i - t)_+ = 1 \quad (20)$$

since $\xi^* = 0$ in that case. Hence $t^* = \max(t_{\max}^*, 0)$, where t_{\max}^* is the solution for (20). And we can recover x from the previous equations. \square

Claim 4. $b^* = \text{prox}_{\sigma_B}^\alpha(\hat{b}) = \underset{b \in B}{\text{argmin}} \frac{1}{2\alpha} \|b - \hat{b}\|^2 + \sigma_B(b)$

$$\Rightarrow b_{i,j}^* = \begin{cases} 0 & (i, j) \in E \\ \hat{b}_{i,j} & i \neq j, (i, j) \notin E \\ \hat{b}_{i,j} - \sigma & i = j \end{cases} \quad (21)$$

D.4 Proof of Theorems

We define

$$\partial_\epsilon p_{\sigma_{\mathcal{X}}} = \left\{ z \left| \frac{1}{2} \|z - v\|^2 + x^\top z \leq \min_{z \in \mathbb{R}^n} p_{\sigma_{\mathcal{X}}}(z; v) + \epsilon, \quad \forall x \in \mathcal{X} \right. \right\} \quad (22)$$

we make the following claims.

Claim 5. *If $z = v - P_{\mathcal{X}}^\epsilon(v) \in \partial_\epsilon p_{\sigma_{\mathcal{X}}}(v)$, defined in (22), then*

$$\|P_A^\epsilon(v) - P_A(v)\| \leq \sqrt{2\epsilon} \quad (23)$$

For any $x \in \mathcal{X}$

$$\|P_A^\epsilon(v) - x\|^2 \leq \|v - x\|^2 + \epsilon \quad (24)$$

Proof. To prove (23), note that $p_{\sigma_{\mathcal{X}}}$ is strongly convex in z , and for any $z \in \partial_\epsilon p_{\sigma_{\mathcal{X}}}(v)$ the following is true

$$\epsilon \geq p_{\sigma_{\mathcal{X}}}(z; v) - p_{\sigma_{\mathcal{X}}}(z^*; v) \geq \frac{1}{2} \|z - z^*\|^2$$

where $z^* = \text{prox}_{\sigma_{\mathcal{X}}}(v) = v - P_{\mathcal{X}}(v)$. Plugging $z = v - P_{\mathcal{X}}^\epsilon(v)$ in the above relation proves (23). To prove (24) we note that for any $z \in \partial_\epsilon p_{\sigma_{\mathcal{X}}}(v)$ and $x \in \mathcal{X}$

$$\frac{1}{2} \|z - v\|^2 + x^\top z \leq \min_{z \in \mathbb{R}^n} p_{\sigma_{\mathcal{X}}}(z; v) + \epsilon \leq p_{\sigma_{\mathcal{X}}}(0; v) + \epsilon = \frac{1}{2} \|v\|^2 + \epsilon$$

Setting $z = v - P_{\mathcal{X}}^\epsilon(v)$, and rearranging terms proves (24). \square

Claim 6.

$$\text{Prox}_{\sigma_{\mathcal{X}}}(v) = \underset{(a,b)}{\text{argmin}} p_{\sigma_{\mathcal{X}}}(a, b; v) \left(= \frac{1}{2} \|(a+b) - v\|^2 + \sigma_A(a) + \sigma_B(b) \right) \quad (25)$$

Proof. Check that $\iota_{\mathcal{X}}(x) = \iota_A(x) + \iota_B(x)$ and the $\sigma_{\mathcal{X}}(z) = \max_x x^\top z - \iota_{\mathcal{X}}(z)$. Following the definition of indicator function of $\sigma_{\mathcal{X}}$, we have

$$\min_z \frac{1}{2} \|z - v\|^2 + \sigma_{\mathcal{X}}(z) = \min_z \frac{1}{2} \|z - v\|^2 + \max_x \{x^\top z - \iota_A(x) - \iota_B(x)\}$$

Introducing the support functions σ_A and σ_B

$$= \min_z \frac{1}{2} \|z - v\|^2 + \max_x \left\{ x^\top z - \max_a (x^\top a - \sigma_A(a)) - \max_b (x^\top b - \sigma_B(b)) \right\}$$

The maximization over a, b can be posed as a minimization because of the negative sign. Using strong duality, we get

$$= \min_z \frac{1}{2} \|z - v\|^2 + \max_x \min_{a,b} \{x^\top (z - a - b) + \sigma_A(a) + \sigma_B(b)\}$$

To be dual feasible the coefficient of x must be zero, leading to $z = a + b$, which is used to eliminate z and we prove the claim. \square

D.5 Proof of Theorem 5 (paper)

Proof. Starting from $x_0 \in \mathbb{R}^{n \times n}$, let $x_k, k \geq 1$ be defined in (12). Let $y_k = x_k - \alpha_k h_k$, $r_k = \|x_k - x^*\|_F$. Then, it follows that

$$\begin{aligned} r_{k+1}^2 &= \|P_X^\epsilon(y_k) - x^*\|_F^2 \leq \|y_k - x^*\|_F^2 + \epsilon \\ &= \|x_k - \alpha_k h_k - x^*\|^2 + \epsilon \\ &= \epsilon + r_k^2 - 2\alpha_k h_k^\top (x_k - x^*) + \alpha_k^2 \|h_k\|^2 \\ &\leq \epsilon + r_k^2 - 2\alpha_k (f(x_k) - f^*) + \alpha_k^2 \|h_k\|^2 \end{aligned} \quad (26)$$

The first inequality is a consequence of (24), and the last inequality is true because of convexity of f . If we define $f_T^* = \min \{f(x_i) \mid i \in \{0, \dots, T\}\}$, then

$$f_T^* - f^* \leq \frac{1}{2 \sum_{k=0}^{T-1} \alpha_k} \left(r_0^2 + \sum_{k=0}^{T-1} (\epsilon + \alpha_k^2 \|h_k\|^2) \right)$$

Under the choice of $\alpha_k = \frac{s}{\|h_k\|}$ we have

$$f_T^* - f^* \leq \frac{L}{2sT} (R^2 + T(\epsilon + s^2))$$

Minimizing RHS as a function of s yields $s = \sqrt{\frac{R^2}{T} + \epsilon}$ and thus proves Theorem 5. \square

D.6 Proof of Theorem 6 (paper)

Proof. Check that for every $t = 1, \dots, T$, Algorithm 1 computes \hat{a}_t, \hat{b}_t such that

$$p_{\sigma_{\mathcal{X}}}(\hat{a}_t, \hat{b}_t; v_t) \leq \min_{a, b \in S_n} p_{\sigma_{\mathcal{X}}}(a, b; v_t) + \frac{\hat{R}^2}{T}$$

and $\mathbf{K}_t = Proj_A(v_t - \hat{a}_t - \hat{b}_t)$. Using $\mathbf{K}^* \in A$ and the non-expansiveness of the projection operator, the following holds

$$\|\mathbf{K}_t - \mathbf{K}^*\|_F^2 \leq \|v_t - \hat{a}_t - \hat{b}_t - \mathbf{K}^*\|_F^2$$

The proof follows by retracing the steps in Theorem 5 with $\epsilon \geq \frac{\hat{R}^2}{T}$. \square

E Computation of Constants

We need to evaluate the constants involved in proposed projection method, which are necessary to compute the step size.

E.1 Computation of \hat{R}

Recall that

$$(a^*(v), b^*(v)) = \operatorname{argmin}_{a, b} \frac{1}{2} \|a + b - v\|^2 + \sigma_A(a) + \sigma_B(b) \quad (27)$$

We define

$$\hat{R}^2 \geq \|a^*(v)\|^2 + \|b^*(v)\|^2 \quad \forall v = \mathbf{K} + h, \mathbf{K} \in A, h \in \partial f(\mathbf{K})$$

where

$$A = \{K \succcurlyeq 0, \operatorname{tr}(K) = n\} \text{ and } B = \{\operatorname{diag}(K) = 1, K_{ij} = 0 \text{ for } (i, j) \notin E\} \quad (28)$$

We have $\sigma_A(M) = \lambda_{\max}(M)$ and $\sigma_B(N) = \operatorname{tr}(M)$ if $M_{ij} = 0$ as soon as $(i, j) \in E$, and infinity otherwise. Given a matrix Z , and its projection \hat{Z} on $A \cap B$, we have $Y = v - \hat{v} = M + N$ with $\sigma_A(M) + \sigma_B(N)$ minimal.

We simply need to exhibit a single pair of optimizers M, N . For this, we may use $\tilde{A} = \{K \succcurlyeq 0\}$, for which $\sigma_{\tilde{A}}(M) = 0$ if $M \preccurlyeq 0$, and infinity otherwise.

We have $Y = M + N$ and thus $Y \preccurlyeq N$. Moreover, $\operatorname{tr}(N) \leq n \lambda_{\max}(Y)$ because the decomposition $Y = 0 + \lambda_{\max}(Y)I$ is feasible.

Thus, $\lambda_{\min}(Y) \leq \lambda_{\min}(N) \leq \lambda_{\max}(N) \leq n \lambda_{\max}(Y)$.

This allows to show that

$$\|N\|_F^2 \leq n \|N\|_{op}^2 \leq n^3 \|Y\|_{op}^2 \quad (29)$$

Now to bound the operator norm of Y we proceed as follows: by choice $v = \mathbf{K} + h$ where $\mathbf{K} \in A$.

Bound on $\|v - \hat{v}\|_F \leq n + L$

Proof:

$$\|Z - \hat{Z}\| \leq \|\mathbf{K} - Proj_{A \cap B}(\mathbf{K})\| + \|h\|$$

We bound the first term as follows. By definition of projection and since \mathbf{I} is feasible for both A and B , we have

$$\|Y\| \leq \|\mathbf{K} - Proj_{A \cap B}(\hat{Z})\| \leq \|\mathbf{K} - \mathbf{I}\|$$

Squaring both sides

$$\|\mathbf{K} - Proj_{A \cap B}(\hat{Z})\|_F^2 \leq \|\mathbf{K} - \mathbf{I}\|^2 = \|\mathbf{K}\|^2 - 2 \operatorname{tr}(\mathbf{K}) + \operatorname{tr}(\mathbf{I}) \leq n$$

Substituting this estimate in $\hat{R}^2 = n^3 \|Y\|_{op}^2 \leq n^5$

E.2 Computation of \mathbf{R}

For the sets A defined in (28), note that $R = \max_{\mathbf{K} \in A} \|\mathbf{K}\|_F$. We derive a bound on the objective function as follows

$$\|\mathbf{K}\|_F^2 = \operatorname{tr}(\mathbf{K}^2) = \sum_{i=1}^n \lambda_i^2(\mathbf{K}) \leq \left(\sum_{i=1}^n \lambda_i(\mathbf{K}) \right)^2 = (\operatorname{Tr}(\mathbf{K}))^2 = n^2$$

and thus the result follows.

E.3 Computation of \mathbf{L}

For SPORE, the subgradient is given by $-\frac{1}{2} \mathbf{Y} \alpha \alpha^\top \mathbf{Y} + \beta \mathbf{v} \mathbf{v}^\top$, which implies that

$$L \leq \max_{0 \leq \alpha_i \leq C, \|v\|_2=1} \left\| \frac{1}{2} \mathbf{Y} \alpha \alpha^\top \mathbf{Y} + \beta \mathbf{v} \mathbf{v}^\top \right\|_F$$

Using the equality $\|\mathbf{M}\|_F^2 = \sum_i \lambda_i^2(\mathbf{M})$ from Section E.2 above, and convexity, we get

$$L \leq \max_{0 \leq \alpha_i \leq C, \|v\|_2=1} \frac{1}{2} \|\alpha\|_2^2 + \beta \|\mathbf{v}\|_2^2$$

The last inequality follows from the fact that for rank 1 matrices $\mathbf{u} \mathbf{u}^\top$, $\lambda_{max} \|\mathbf{u}\|_2^2$. For the chosen constant $\beta = \frac{\vartheta(G)}{\vartheta(\tilde{G})}$ from Theorem 4, note that $\beta \leq 1$, since whenever $\vartheta(G) \geq \sqrt{n}$, we can work on the complement graph and $\vartheta(G)\vartheta(\tilde{G}) = n$ [6]. Thus, from the constraints on α , it follows that $L \leq C^2 \sqrt{n}$.

F Multiple Graph Transduction

Recalling the notations in the paper – let $\mathbb{G} = \{G^{(1)}, \dots, G^{(M)}\}$ be a set of simple graphs $G^{(k)} = (V, E^{(k)})$, defined on a common vertex set $V = [n]$. We introduce some more notations – let

$$\mathcal{K}(\mathbb{G}) = \{\mathbb{K} | \mathbb{K} = \{\mathbf{K}^{(1)}, \dots, \mathbf{K}^{(M)}\}, \mathbf{K}^{(k)} \in \mathcal{K}(G^{(k)})\} \quad (30)$$

and let $\tilde{\mathbf{K}}^\eta = \sum_{k \in [M]} \eta_k \mathbf{K}^{(k)}$, for $\eta \in \mathcal{S}^{M-1}$.

At this point, we would like to gather some important results before we prove the improved graph-dependent generalization bound. We define an analog of ϑ for the multiple graphs setting –

$$\vartheta(\mathbb{G}) = \min_{\mathbb{K} \in \mathcal{K}(\mathbb{G})} \min_{\eta \in \mathcal{S}^{M-1}} \bar{\omega}(\tilde{\mathbf{K}}^\eta) \quad (31)$$

where $\bar{\omega}(\cdot)$ is the 1-class SVM dual formulation, defined as –

$$\bar{\omega}(\mathbf{K}) = \max_{\alpha \in \mathbb{R}_+^n} 2\alpha^\top \mathbf{1} - \alpha^\top \mathbf{K} \alpha \quad (32)$$

We also define k^* , for convenience of the subsequent proofs –

$$k^* = \operatorname{argmin}_{k \in [M]} \vartheta(G^{(k)}) = \operatorname{argmax}_{k \in [M]} \vartheta(\bar{G}^{(k)}) \quad (33)$$

where $\bar{G}^{(k)}$ is the complement graph of $G^{(k)}$. The second equivalence follows from the fact that $\vartheta(G)\vartheta(\bar{G}) = n$ [6] for any graph G . We state some important lemmas, used in the generalization analysis. We begin with a trivial bound on the spectral norm of convex combination of graph orthonormal embeddings –

Lemma 4. For any $\mathbb{K} \in \mathcal{K}(\mathbb{G})$ and $\eta \in \mathcal{S}^{M-1}$,

$$\lambda_1(\tilde{\mathbf{K}}^\eta) \leq \lambda_1(\mathbf{K}^{(l^*)}) \quad \text{where} \quad l^* = \operatorname{argmax}_{l \in [M]} \lambda_1(\mathbf{K}^{(l)}) \quad (34)$$

Proof. We use convexity of the spectral norm to prove our result –

$$\lambda_1(\tilde{\mathbf{K}}^\eta) = \lambda_1\left(\sum_{k \in [M]} \eta_k \mathbf{K}^{(k)}\right) \leq \sum_{k \in [M]} \eta_k \lambda_1(\mathbf{K}^{(k)}) \leq \max_{\tau \in \mathcal{S}^{M-1}} \sum_{k \in [M]} \tau_k \lambda_1(\mathbf{K}^{(k)}) = \lambda(\mathbf{K}^{(l^*)})$$

where l^* as in the statement of the Lemma. \square

We bound the spectral norm over all possible orthonormal embeddings of multiple graphs by ϑ –

Lemma 5. Given a set of simple graphs \mathbb{G}

$$\max_{\mathbb{K} \in \mathcal{K}(\mathbb{G})} \max_{k \in [M]} \lambda_1(\mathbf{K}^{(k)}) = \vartheta(\bar{G}^{(k^*)})$$

where $\bar{G}^{(k)}$ is the complement graph of $G^{(k)}$, and k^* as in (33).

Proof. From Lemma 2 (paper), it follows that $\max_{\mathbf{K} \in \mathcal{K}(G)} \lambda_1(\mathbf{K}) = \vartheta(\bar{G})$. Thus, we get

$$\max_{\mathbf{K}^{(k)} \in \mathcal{K}(G^{(k)})} \max_{k \in [M]} \lambda_1(\mathbf{K}^{(k)}) = \max_{k \in [M]} \max_{\mathbf{K} \in \mathcal{K}(G^{(k)})} \lambda_1(\mathbf{K}) = \max_{k \in [M]} \vartheta(\bar{G}^{(k)})$$

The proof follows from the definition of k^* in (33). \square

Similar to Lemma 3 (paper), we relate MKL formulation to ϑ –

Lemma 6. Given a set of simple graphs $\mathbb{G} = \{G^{(1)}, \dots, G^{(M)}\}$ defined on a common vertex set V , let $\mathbf{y} \in \mathcal{Y}^n$ be the unknown binary labels. Then, for any subgraph $S \subseteq V$,

$$\min_{\mathbb{K} \in \mathcal{K}(\mathbb{G})} \min_{\eta \in \mathcal{S}^{M-1}} \omega_C(\tilde{\mathbf{K}}^\eta, \mathbf{y}_S) \leq \vartheta(\mathbb{G})/2$$

where $\vartheta(\mathbb{G})$ as in (31).

Proof. For any simple graph $G = (V, E)$, with unknown binary labels $\mathbf{y} \in \mathcal{Y}^n$ over the set V ; we note an interesting property of orthonormal embedding that if $\mathbf{U} \in \operatorname{Lab}(G)$, then $\tilde{\mathbf{U}} = \mathbf{U}\mathbf{Y} \in \operatorname{Lab}(G)$, where $Y_{ij} = y_i$, for $i = j$; 0 otherwise. Also, before we proceed, we recall a trivial bound $\omega_C(\tilde{\mathbf{K}}^\eta, \mathbf{y}_S) \leq \omega_\infty(\tilde{\mathbf{K}}^\eta, \mathbf{y})$, which follows from the primal formulation of SVM. Now, we bound the quantity of interest as follows –

$$\begin{aligned} \min_{\mathbb{K} \in \mathcal{K}(\mathbb{G})} \min_{\eta \in \mathcal{S}^{M-1}} \omega_C(\tilde{\mathbf{K}}^\eta, \mathbf{y}_S) &\leq \min_{\mathbb{K} \in \mathcal{K}(\mathbb{G})} \min_{\eta \in \mathcal{S}^{M-1}} \omega_\infty(\tilde{\mathbf{K}}^\eta, \mathbf{y}) \\ &= \min_{\mathbf{U}^{(k)} \in \operatorname{Lab}(G^{(k)})} \min_{\eta \in \mathcal{S}^{M-1}} \min_{\mathbf{w}^{(k)} \in \mathbb{R}^{d_k}} \frac{1}{2} \sum_{k \in [M]} \eta_k \|\mathbf{w}^{(k)}\|_2^2 \quad \text{s.t.} \quad y_i \sum_{k \in [M]} \eta_k \mathbf{w}^{(k)\top} \mathbf{u}_i^{(k)} \geq 1, \forall i \in [N] \end{aligned}$$

Note that here $\mathbf{u}_i^{(k)}$ is the i^{th} column of the orthonormal embedding of k^{th} graph. Also, we assume that each orthonormal embedding $\mathbf{U}^{(k)}$ is of the dimension $d_k \times n$. Now, using the property of orthonormal embedding in the beginning of the proof, we get

$$= \min_{\tilde{\mathbf{U}}^{(k)} \in \operatorname{Lab}(G^{(k)})} \min_{\eta \in \mathcal{S}^{M-1}} \min_{\mathbf{w}^{(k)} \in \mathbb{R}^{d_k}} \frac{1}{2} \sum_{k \in [M]} \eta_k \|\mathbf{w}^{(k)}\|_2^2 \quad \text{s.t.} \quad \sum_{k \in [M]} \eta_k \mathbf{w}^{(k)\top} \tilde{\mathbf{u}}_i^{(k)} \geq 1, \forall i \in [N]$$

where $\tilde{\mathbf{U}}^{(k)} = \mathbf{U}^{(k)}\mathbf{Y}$. Finally, using the dual formulation of MKL proves the result. \square

We relate ϑ of the multiple graphs (31) to that of individual graphs –

Lemma 7. $\vartheta(\mathbb{G}) \leq \vartheta(G^{(k^*)})$, where $\vartheta(\mathbb{G})$ as in (31), and k^* as in (33).

Proof. Proof follows by applying an alternate definition of ϑ as in [4] –

$$\vartheta(G) = \min_{\mathbf{K} \in \mathcal{K}(G)} \bar{\omega}(\mathbf{K})$$

where $\omega(\cdot)$ as in (32). Now, expanding $\vartheta(\mathbb{G})$, we get

$$\begin{aligned} \vartheta(\mathbb{G}) &= \min_{\mathbf{K}^{(k)} \in \mathcal{K}(G^{(k)}), \forall k \in [M]} \min_{\eta \in \mathcal{S}^{M-1}} \bar{\omega}\left(\sum_{k \in [M]} \eta_k \mathbf{K}^{(k)}\right) \\ &\leq \min_{\mathbf{K}^{(k)} \in \mathcal{K}(G^{(k)}), \forall k \in [M]} \min_{k \in [M]} \bar{\omega}(\mathbf{K}^{(k)}) \\ &\leq \min_{k \in [M]} \left(\min_{\mathbf{K} \in \mathcal{K}(G^{(k)})} \bar{\omega}(\mathbf{K}) \right) \end{aligned}$$

Using the definition of ϑ above, and k^* as in (33) proves the result. \square

Now, we prove the main result on multiple graph transduction. Let \mathbb{K}^* be the optimal graph embeddings computed from (17) (paper). Let α^*, η^* be the solution to $\min_{\eta \in \mathcal{S}^{M-1}} \omega_C(\tilde{\mathbf{K}}^{*\eta}, \mathbf{y}_S)$, then the final predictions of (17) (paper) is given by $\hat{y}_i = \sum_{j \in S} \eta_k^* K_{ij}^* \alpha_j^* y_j$, $\forall i \in [n]$. The proposed MKL style solution allows is to extend Theorem 4 (paper) to the multiple graphs setting –

Theorem 8. Given a set of simple graphs \mathbb{G} and labels of a randomly drawn subgraph $S \subset V$, $m = |S|$; let $\hat{\mathbf{y}}$ be the predictions learnt by MKL-SPORE (17) for parameters $C = \left(\frac{\vartheta(\mathbb{G})}{m\sqrt{\vartheta(\bar{G}^{(k^*)})}}\right)^{\frac{1}{2}}$ and $\beta = \frac{\vartheta(\mathbb{G})}{\vartheta(\bar{G}^{(k^*)})}$, where k^* as in (33). Then, for $m \leq n/2$, with probability $\geq 1 - \delta$ over the choice of $S \subseteq V$, such that $|S| = m$

$$er_S^{0.1}[\hat{\mathbf{y}}] = O\left(\frac{1}{m} \left(\sqrt{n\vartheta(\mathbb{G})} + \log \frac{1}{\delta}\right)\right)^{\frac{1}{2}}$$

where $\vartheta(\mathbb{G})$ as in (31).

Proof. Let $\mathbb{K}^* = \{\mathbf{K}^{*(1)}, \dots, \mathbf{K}^{*(M)}\}$, η^* be the kernels learnt by MKL-SPORE (17) (paper). Let $\tilde{\mathbf{K}}^{*\eta^*} = \sum_{k \in [M]} \eta_k^* \mathbf{K}^{*(k)}$. Applying Theorem 1 (paper) for the final predictions $\hat{\mathbf{y}}$, we obtain

$$er_S^{0.1}[\hat{\mathbf{y}}] \leq \frac{1}{m} \sum_{i \in S} \ell^{hng}(y_i, \hat{y}_i) + 2C\sqrt{2\lambda_1(\tilde{\mathbf{K}}^{*\eta^*})} + O\left(\sqrt{\frac{1}{m} \log \frac{1}{\delta}}\right)$$

Using $\lambda_1(\tilde{\mathbf{K}}^{*\eta^*}) \leq \lambda_1(\mathbf{K}^{*(l^*)})$ from Lemma 4, where l^* as in (34) and $\lambda_1(\mathbf{K}^{*(l^*)}) \leq \vartheta(\bar{G}^{(l^*)})$ from Lemma 2 (paper), we get

$$\leq \frac{1}{m} \sum_{i \in S} \ell^{hng}(y_i, \hat{y}_i) + 2C\sqrt{2\vartheta(\bar{G}^{(l^*)})} + O\left(\sqrt{\frac{1}{m} \log \frac{1}{\delta}}\right) \quad (35)$$

where $\bar{G}^{(l^*)}$ is complement graph of $G^{(l^*)}$. Using a similar argument as in the proof of Theorem 4 (paper), using the primal formulation of (3) (paper), we get

$$\begin{aligned} C \sum_{i \in S} \ell^{hng}(y_i, \hat{y}_i) &\leq \omega_C(\tilde{\mathbf{K}}^{*\eta^*}, \mathbf{y}_S) \leq \Phi_{C,\beta}(\mathbb{G}, \mathbf{y}_S) \\ &= \min_{\mathbf{K} \in \mathcal{K}(\mathbb{G})} \left(\min_{\eta \in \mathcal{S}^{M-1}} \omega_C(\tilde{\mathbf{K}}^\eta, \mathbf{y}_S) + \beta \max_{k \in [M]} \lambda_1(\mathbf{K}^{(k)}) \right) \\ &\leq \min_{\mathbf{K} \in \mathcal{K}(\mathbb{G})} \min_{\eta \in \mathcal{S}^{M-1}} \omega_C(\tilde{\mathbf{K}}^\eta, \mathbf{y}_S) + \beta \max_{\mathbf{K} \in \mathcal{K}(\mathbb{G})} \max_{k \in [M]} \lambda_1(\mathbf{K}) \leq \frac{\vartheta(\mathbb{G})}{2} + \beta\vartheta(\bar{G}^{(k^*)}) \end{aligned}$$

The last inequality follows from Lemma 5 and Lemma 6. Plugging back in (35) –

$$er_S^{0.1}[\hat{\mathbf{y}}] \leq \frac{1}{2Cm} \vartheta(\mathbb{G}) + \frac{\beta}{Cm} \vartheta(\bar{G}^{(k^*)}) + 2C \sqrt{2\vartheta(\bar{G}^{(k^*)})} + O\left(\sqrt{\frac{1}{m} \log \frac{1}{\delta}}\right)$$

where the last inequality follows by using $\vartheta(\bar{G}^{(l^*)}) \leq \vartheta(\bar{G}^{(k^*)})$, from the definition of k^* (33). Choosing β such that $\frac{\beta}{Cm} \vartheta(\bar{G}^{(k^*)}) = 2C \sqrt{2\vartheta(\bar{G}^{(k^*)})}$ and optimizing for C gives us the choice of parameters as in the statement of the theorem. Plugging back in (35), we get

$$= O\left(\frac{1}{\sqrt{m}} \left(\sqrt{\vartheta(\mathbb{G}) \vartheta(\bar{G}^{(k^*)})} + \sqrt{\log \frac{1}{\delta}} \right)\right) = O\left(\frac{1}{m} \left(\vartheta(\mathbb{G}) \sqrt{\vartheta(\bar{G}^{(k^*)})} + \log \frac{1}{\delta} \right)\right)^{\frac{1}{2}}$$

where the last inequality follows from concavity $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$. Finally, using $\vartheta(\mathbb{G}) \leq \vartheta(G^{(k^*)})$ from the definition of k^* , and $\vartheta(G^{(k^*)}) \vartheta(\bar{G}^{(k^*)}) = n$ [6] proves the result. \square

We can also apply the proposed algorithm in Section 4 to solve for (17) efficiently. Let \mathbf{Y} be a diagonal matrix such that $Y_{ii} = y_i$, for $i \in S$, and 0 otherwise. The subgradient at t^{th} iteration, for the k^{th} graph is given by $\partial_{\mathbf{K}_t^{(k)}} \bar{g}(\mathbb{K}_t) = -\frac{1}{2} \eta_t \mathbf{Y} \alpha_t \alpha_t^\top \mathbf{Y} + \mathbf{1}[k = l^*] \beta \mathbf{v}_t \mathbf{v}_t^\top$, where η_t, α_t are the solutions returned by MKL $\min_{\eta \in S^{M-1}} \omega_C(\tilde{\mathbf{K}}_t^\eta, \mathbf{y}_S)$, and $\mathbf{v}_t = \operatorname{argmax}_{\mathbf{v} \in \mathbb{R}^n, \|\mathbf{v}\|=1} \mathbf{v}^\top \mathbf{K}_t^{(l^*)} \mathbf{v}$ is maximum Eigen vector of $\mathbf{K}_t^{(l^*)}$, where $l^* = \operatorname{argmax}_{l \in [M]} \lambda_1(\mathbf{K}_t^{(l)})$ as in Lemma 4.

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