Transelliptical Component Analysis

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Abstract

We propose a high dimensional semiparametric scale-invariant principle component analysis, named TCA, by utilize the natural connection between the elliptical distribution family and the principal component analysis. Elliptical distribution family includes many well-known multivariate distributions like multivariate Gaussian, t and logistic and it is extended to the meta-elliptical by Fang et.al (2002) using the copula techniques. In this paper we extend the meta-elliptical distribution family to a even larger family, called *transelliptical*. We prove that TCA can obtain a near-optimal $s\sqrt{\log d/n}$ estimation consistency rate in recovering the leading eigenvector of the latent generalized correlation matrix under the transelliptical distribution family, even if the distributions are very heavy-tailed, have infinite second moments, do not have densities and possess arbitrarily continuous marginal distributions. A feature selection result with explicit rate is also provided. TCA is further implemented in both numerical simulations and large-scale stock data to illustrate its empirical usefulness. Both theories and experiments confirm that TCA can achieve model flexibility, estimation accuracy and robustness at almost no cost.

1 Introduction

Given $x_1, \ldots, x_n \in \mathbb{R}^d$ as n i.i.d realizations of a random vector $X \in \mathbb{R}^d$ with population covariance matrix Σ and correlation matrix Σ^0, the Principal Component Analysis (PCA) aims at recovering the top m leading eigenvectors $u_1, \ldots, u_m$ of Σ. In practice, Σ is unknown and the top m leading eigenvectors $\widehat{u}_1, \ldots, \widehat{u}_m$ of the Pearson sample covariance matrix are obtained as the estimators. However, because the PCA is well-known to be scale-variant, meaning that changing the measurement scale of variables will make the estimators different, the PCA conducted on the sample correlation matrix is also regular in literatures [2]. It aims at recovering the top m leading eigenvectors $\theta_1, \ldots, \theta_m$ of Σ^0 using the top m leading eigenvectors $\widehat{\theta}_1, \ldots, \widehat{\theta}_m$ of the Pearson sample correlation matrix. Because Σ^0 is scale-invariant, we call the PCA aiming at recovering the eigenvectors of Σ^0 the scale-invariant PCA.

In high dimensional settings, when d scales with n, it has been discussed in [14] that $\widehat{u}_1$ and $\widehat{\theta}_1$ are generally not consistent estimators of u_1 and θ_1. For any two vectors $v_1, v_2 \in \mathbb{R}^d$, denote the angle between v_1 and v_2 by $\angle(v_1, v_2)$. [14] proved that $\angle(u_1, \widehat{u}_1)$ and $\angle(\theta_1, \widehat{\theta}_1)$ do not converge to zero. Therefore, it is commonly assumed that $\theta_1 = (\theta_{11}, \ldots, \theta_{1d})^T$ is sparse, meaning that $\text{card}(\text{supp}(\theta_1)) := \text{card}(\{\theta_{1j} : \theta_{1j} \neq 0\}) = s < n$. This results in a variety of sparse PCA procedures. Here we note that $\text{supp}(u_j) = \text{supp}(\theta_j)$, for $j = 1, \ldots, d$.

The *elliptical distributions* are of special interest in Principal Component Analysis. The study of elliptical distributions and their extensions have been launched in statistics recently by [4]. The elliptical distributions can be characterized by their stochastic representations [5]. A random vector $Z = (Z_1, \ldots, Z_d)^T$ is said to follow an elliptical distribution or be elliptically distributed with parameters μ, $\Sigma \succeq 0$, and $\text{rank}(\Sigma) = q$, if it admits the stochastic representation: $Z = \mu + \xi AU$, where $\mu \in \mathbb{R}^d$, $\xi \in \mathbb{R}$ and $U \in \mathbb{R}^q$ are independent random variables, $\xi \geq 0$, U is uniformly distributed on the unit sphere in $\mathbb{R}^q$, and $A \in \mathbb{R}^{d\times q}$ is a fixed matrix such that $AA^T = \Sigma$. We call

ξ the *generating variable*. The density of Z does not necessarily exist. Elliptical distribution family includes a variety of famous multivariate distributions: multivariate Gaussian, multivariate Cauchy, Student's t, logistic, Kotz, symmetric Pearson type-II and type-VII distributions. We refer to [3, 5] and [4] for more details.

[4] introduce the term *meta-elliptical distribution* in extending the continuous elliptical distributions whose densities exist to a wider class of distributions with densities existing. The construction of the meta-elliptical distributions is based on the *copula* technique and it was initially introduced by [25]. In particular, when the latent elliptical distribution is the multivariate Gaussian, we have the *meta-Gaussian* or the *nonparanormal* distributions introduced by [16] and [19].

The elliptical distribution is of special interest in Principal Component Analysis (PCA). It has been shown in a variety of literatures [27, 11, 22, 12, 24] that the PCA conducted on elliptical distributions shares a number of good properties enjoyed by the PCA conducted on the Gaussian distribution. In particular, [11] show that with regard to a range of hypothesis relevant to PCA, tests based on a multivariate Gaussian assumption have the identical power for all elliptical distributions even without second moments. We will utilize this connection to construct a new model in this paper.

In this paper, a new high dimensional scale-invariant principle component analysis approach is proposed, named Transelliptical Component Analysis (TCA). Firstly, to achieve both the estimation accuracy and model flexibility, we build the model of TCA on the *transelliptical distributions*. A random vector $X = (X_1, \ldots, X_d)^T$ is said to follow a transelliptical distribution if there exists a set of univariate strictly monotone functions $f = \{f_j\}_{j=1}^d$ such that $f(X) := (f_1(X_1), \ldots, f_d(X_d))^T$ follows a continuous elliptical distribution with parameters $\mu = 0$ and $\Sigma^0 = [\Sigma^0_{jk}] \succeq 0$. Here $\mathrm{diag}(\Sigma^0) = \mathbf{1}$. Transelliptical distributions do not necessarily possess densities and are strict extensions to the meta-elliptical distributions defined in [4]. TCA aims at recovering the top m leading eigenvectors $\theta_1, \ldots, \theta_m$ of Σ^0.

Secondly, to estimate Σ^0 robustly and efficiently, instead of estimating the transformation functions $\{\widehat{f}_j\}_{j=1}^d$ of $\{f_j\}_{j=1}^d$ as [19] did, realizing that $\{f_j\}_{j=1}^d$ preserve the ranks of the data, we utilize the nonparametric rank-based correlation coefficient estimator, Kendall's tau, to estimate Σ^0. We prove that even though the generating variable ξ is changing and marginal distributions are arbitrarily continuous, Kendall's tau correlation matrix approximates Σ^0 in a parametric rate $O_P(\sqrt{\log d/n})$. This key observation makes Kendall's tau a better estimator than Pearson sample correlation matrix with regard to a much larger distribution family than the Gaussian.

Thirdly, in terms of methodology and theory, we analyze the general case that X follows a transelliptical distribution and θ_1 is sparse. Here θ_1 is the leading eigenvector of Σ^0. We obtain the TCA estimator $\widetilde{\theta}_1^*$ of θ_1 utilizing the Kendall's tau correlation matrix. We prove that the TCA can obtain a fast convergence rate in terms of parameter estimation and is of the rate $\sin\angle(\theta_1, \widetilde{\theta}_\infty) = O_P(s\sqrt{\log d/n})$, where $\widetilde{\theta}_\infty$ is the estimator TCA obtains. A feature selection consistency result with explicit rate is also provided.

2 Background

We start with notations: Let $M = [M_{jk}] \in \mathbb{R}^{d\times d}$ and $v = (v_1, ..., v_d)^T \in \mathbb{R}^d$. Let v's subvector with entries indexed by I be denoted by v_I, M's submatrix with rows indexed by I and columns indexed by J be denoted by M_{IJ}. Let $M_{I\cdot}$ and $M_{\cdot J}$ be the submatrix of M with rows in I and all columns, and the submatrix of M with columns in J and all rows. For $0 < q < \infty$, we define the ℓ_0, ℓ_q and ℓ_∞ vector norm as

$$\|v\|_0 := \mathrm{card}(\mathrm{supp}(v)), \quad \|v\|_q := (\sum_{i=1}^d |v_i|^q)^{1/q} \text{ and } \|v\|_\infty := \max_{1\leq i\leq d} |v_i|.$$

We define the matrix $\ell_{\max}$ norm as the elementwise maximum value: $\|M\|_{\max} := \max\{|M_{ij}|\}$ and the ℓ_∞ norm as $\|M\|_\infty := \max_{1\leq i\leq m}\sum_{j=1}^n |M_{ij}|$. Let $\Lambda_j(M)$ be the toppest $j-$th eigenvalue of M. In special, $\Lambda_{\min}(M) := \Lambda_d(M)$ and $\Lambda_{\max}(M) := \Lambda_1(M)$ are the smallest and largest eigenvalues of M. The vectorized matrix of M, denoted by $\mathrm{vec}(M)$, is defined as: $\mathrm{vec}(M) := (M_{\cdot 1}^T, \ldots, M_{\cdot d}^T)^T$. Let $\mathbb{S}^{d-1} := \{v \in \mathbb{R}^d : \|v\|_2 = 1\}$ be the d-dimensional unit sphere. The sign $=^d$ denotes that the two sides of the equality have the same distributions. For any two vectors $a, b \in \mathbb{R}^d$ and any two squared matrices $A, B \in \mathbb{R}^{d\times d}$, denote the inner product of a and b, A and

B by

$$\langle a, b \rangle := a^T b \quad \text{and} \quad \langle A, B \rangle := \text{Tr}(A^T B).$$

2.1 Elliptical and Transelliptical Distributions

This section is devoted to a brief discussion of elliptical and transelliptical distributions. In the sequel, to be clear, a random vector $X = (X_1, \ldots, X_d)^T$ is said to be *continuous* if the marginal distribution functions are all continuous.

2.1.1 Elliptical Distributions

In this section we shall firstly provide a definition of the elliptical distributions following [5].

Definition 2.1. Given $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$, where $\text{rank}(\Sigma) = q \le d$, a random vector $Z = (Z_1, \ldots, Z_d)^T$ is said to have an elliptical distribution or is elliptically distributed with parameters μ and Σ, if and only if Z has a stochastic representation: $Z =^d \mu + \xi AU$, where $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times q}$, $AA^T = \Sigma$, $\xi \ge 0$ is a random variable independent of U, $U \in \mathbb{S}^{q-1}$ is uniformly distributed in the unit sphere in $\mathbb{R}^q$. In this setting we denote by $Z \sim EC_d(\mu, \Sigma, \xi)$.

A random variable in $\mathbb{R}$ with continuous marginal distribution function does not necessarily possess density. A well-known set of examples is the *cantor distribution*, whose support set is the cantor set. We refer to [7] for more discussions on this phenomenon. Σ is symmetric and positive semi-definite, but not necessarily to be positive definite.

Proposition 2.1. *A random vector $Z = (Z_1, \ldots, Z_d)^T$ has the stochastic representation $Z \sim EC_d(\mu, \Sigma, \xi)$, if and only if Z has the characteristic function $\exp(it'\mu)\phi(t'\Sigma t)$, where ϕ is a properly-defined characteristic function. We denote by $X \sim EC_d(\mu, \Sigma, \phi)$. If ξ is absolutely continuous and Σ is non-singular, then the density of Z exists and is of the form: $p_Z(z) = |\Sigma|^{-1/2} g\left((z-\mu)^T \Sigma^{-1}(z-\mu)\right)$, where $g : [0, \infty) \to [0, \infty)$. We denote by $Z \sim EC_d(\mu, \Sigma, g)$.*

A proof can be found in page 42 of [5]. When the density exists, ξ, ϕ and g are uniquely determined by one of the other. The relationship among ξ, ϕ and g are described in Theorem 2.2 and Theorem 2.9 of [5]. The next proposition states that Σ, ϕ, ξ and A are not unique.

Proposition 2.2 (Theorem 2.15 of [5]). *(i) If $Z = \mu + \xi AU$ and $Z = \mu^* + \xi^* A^* U^*$, where $A \in \mathbb{R}^{d\times q}$ and $A^* \in \mathbb{R}^{d\times q}$, Z is continuous, then there exists a constant $c > 0$ such that $\mu^* = \mu, \quad A^* A^{*T} = cAA^T, \quad \xi^* = c^{-1/2}\xi$. (ii) If $Z \sim EC_d(\mu, \Sigma, \phi)$ and $Z \sim EC_d(\mu^*, \Sigma^*, \phi^*)$, Z is continuous, then there exists a constant $c > 0$ such that $\mu^* = \mu, \quad \Sigma^* = c\Sigma, \quad \phi^*(\cdot) = \phi(c^{-1}\cdot)$.*

The next proposition discusses the cases where (μ, Σ, ξ) is identifiable for Z.

Proposition 2.3. *If $Z \sim EC_d(\mu, \Sigma, \xi)$ is continuous with $\text{rank}(\Sigma) = q$, then (1) $\mathbb{P}(\xi = 0) = 0$; (2)$\Sigma_{ii} > 0$ for $i \in \{1, \ldots, d\}$; (3)(μ, Σ, ξ) is identifiable for Z under the constraint that $\max(\text{diag}(\Sigma)) = 1$.*

We define $\Sigma^0 = [\Sigma^0_{jk}]$ with $\Sigma^0_{jk} = \Sigma_{jk}/\sqrt{\Sigma_{jj}\Sigma_{kk}}$ to be the *generalized correlation matrix* of Z. Σ^0 is the correlation matrix of Z when Z's second moment exists and still reflects the rank dependency even when Z has infinite second moment [13].

2.1.2 Transelliptical Distributions

To extend the elliptical distribution, we firstly define two sets of symmetric matrices: $\mathcal{R}^+_d = \{\Sigma \in \mathbb{R}^{d\times d} : \Sigma^T = \Sigma, \text{diag}(\Sigma) = \mathbf{1}, \Sigma \succ 0\}$; $\mathcal{R}_d = \{\Sigma \in \mathbb{R}^{d\times d} : \Sigma^T = \Sigma, \text{diag}(\Sigma) = \mathbf{1}, \Sigma \succeq 0\}$.

Definition 2.2. A random vector $X = (X_1, \ldots, X_d)^T$ with continuous marginal distribution functions $F_1, \ldots, F_d$ and density existing is said to follow a meta-elliptical distribution if and only if there exists a continuous elliptically distributed random vector $Z \sim EC_d(0, \Sigma^0, g)$ with the marginal distribution function Q_g and $\Sigma^0 \in \mathcal{R}^+_d$, such that $(Q_g^{-1}(F_1(X_1)), \ldots, Q_g^{-1}(F_d(X_d)))^T =^d Z$.

In this paper, we generalize the meta-elliptical distribution family to a broader class, named the transelliptical. The transelliptical distributions do not assume that densities exist for both X and Z and are therefore strict extensions to meta-elliptical distributions.

Definition 2.3. A random vector $X = (X_1, \ldots, X_d)^T$ is said to follow a transelliptical distribution if and only if there exists a set of strictly monotone functions $f = \{f_j\}_{j=1}^d$ and a latent continuous elliptically distributed random vector $Z \sim EC_d(0, \Sigma^0, \xi)$ with $\Sigma^0 \in \mathcal{R}_d$, such that $(f_1(X_1), \ldots, f_d(X_d))^T =^d Z$. We call such $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$ and Σ^0 the latent generalized correlation matrix.

Proposition 2.4. *If X follows a meta-elliptical distribution, in other words, X possesses density and has continuous marginal distributions $F_1, \ldots, F_d$ of X and a continuous random vector $Z \sim EC_d(0, \Sigma^0, g)$ such that $(Q_g^{-1}(F_1(X_1)), \ldots, Q_g^{-1}(F_d(X_d)))^T =^d Z$, then we have $X \sim TE_d(\Sigma^0, \xi; Q_g^{-1}(F_1), \ldots, Q_g^{-1}(F_d))$.*

To be more clear, the transelliptical distribution family is strictly larger than the meta-elliptical distribution family in three senses: (i) the generating variable ξ of the latent elliptical distribution is not necessarily absolute continuous in transelliptical distributions; (ii) the parameter Σ^0 is strictly enlarged from $\mathcal{R}_d^+$ to $\mathcal{R}_d$; (iii) the marginal distributions of X do not necessarily possess densities.

The term *meta-Gaussian* (or the nonparanormal) is introduced by [16, 19]. The term *meta-elliptical copula* is introduced in [6]. This is actually an alternative definition of the meta-elliptical distribution. The term *elliptical copula* is introduced in [18]. In summary,

$$\textsf{transelliptical} \supset \textsf{meta-elliptical} = \textsf{meta-elliptical copula} \supset \textsf{elliptical*} \supset \textsf{elliptical copula},$$

$$\textsf{transelliptical} \supset \textsf{meta-Gaussian} = \textsf{nonparanormal}.$$

Here elliptical* represents the elliptical distributions which are continuous and possess densities.

2.2 Latent Correlation Matrix Estimation for Transelliptical Distributions

We firstly study the correlation and covariance matrices of elliptical distributions. Given $Z \sim EC_d(\mu, \Sigma, \xi)$, we first explore the relationship between the moments of Z and μ and Σ.

Proposition 2.5. *Given $Z \sim EC_d(\mu, \Sigma, \xi)$ with $\mathrm{rank}(\Sigma) = q$ and finite second moments and Σ^0 the generalized correlation matrix of Z, we have $\mathbb{E}(Z) = \mu$, $\mathrm{Var}(Z) = \frac{\mathbb{E}(\xi^2)}{q}\Sigma$, and $\mathrm{Cor}(Z) = \Sigma^0$.*

When the random vector is elliptically distributed with second moment finite, the sample mean and correlation matrices are element-wise consistent estimators of μ and Σ^0. However, the elliptical distributions are generally very heavy-tailed (multivariate t or Cauchy distributions for example), making Pearson sample correlation matrix a bad estimator. When the distribution family is extended to the transelliptical, the Pearson sample correlation matrix is generally no longer a element-wise consistent estimator of Σ^0. A similar "plug-in" idea as [6] works when ξ is known. In the general case when ξ is unknown, the "plug-in" idea itself is unavailable.

3 The TCA

In this section we propose the TCA approach. TCA is a two-stage method in estimating the leading eigenvectors of Σ^0. Firstly, we estimate the Kendall's tau correlation matrix $\widehat{R}$. Secondly, we plug $\widehat{R}$ into a sparse PCA algorithm.

3.1 Rank-based Measures of Associations

The main idea of the TCA is to exploit the Kendall's tau statistic to estimate the generalized correlation matrix Σ^0 efficiently and robustly. In detail, let $X = (X_1, \ldots, X_d)^T$ be a $d-$dimensional random vector with marginal distributions $F_1, \ldots, F_d$ and the joint distributions F_{jk} for the pair (X_j, X_k). The population Spearman's rho and Kendall's tau correlation coefficients are given by

$$\rho(X_j, X_k) = \mathrm{Corr(F_j(X_j), F_k(X_k))},$$
$$\tau(X_j, X_k) = \mathbb{P}((X_j - \widetilde{X}_j)(X_k - \widetilde{X}_k) > 0) - \mathbb{P}((X_j - \widetilde{X}_j)(X_k - \widetilde{X}_k) < 0),$$

where $(\widetilde{X}_j, \widetilde{X}_k)$ is a independent copy of (X_j, X_k). In particular, for Kendall's tau, we have the following theorem, which states an explicit relationship between τ_{jk} and Σ^0_{jk} given $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$, no matter what the generating variable ξ is. This is a strict extension to [4]'s result on the meta-elliptical distribution family.

Theorem 3.1. *Given $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$ transelliptically distributed, we have*

$$\Sigma^0_{jk} = \sin\left(\frac{\pi}{2}\tau(X_j, X_k)\right). \tag{3.1}$$

Remark 3.1. Although the conclusion in Theorem 3.1 of [4] is correct, the proof provided is wrong or at least very ambiguous. Theorem 2.22 in [5] builds the result only for one sample statistic and cannot be generalized to the statistic of multiple samples, like the Kendall's tau or Spearman's rho. Therefore, we provide a new and clear version here. Detailed proofs can be found in the long version of this paper [8].

Spearman's rho depends not only on Σ but also on the generating variable ξ. When X follows multivariate Gaussian, [17] proves that: $\rho(X_j, X_k) = \frac{6}{\pi}\arcsin(\Sigma^0_{jk}/2)$. On the other hand, when $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$ with $\xi =^d 1$, [10] proves that: $\rho(X_j, X_k) = 3(\frac{\arcsin \Sigma^0_{jk}}{\pi}) - 4(\frac{\arcsin \Sigma^0_{jk}}{\pi})^3$.

In estimating $\tau(X_j, X_k)$, let $x_1, \ldots, x_n$ be n independent realizations of X, where $x_i = (x_{i1}, \ldots, x_{id})^T$. We consider the following rank-based statistic:

$$\begin{cases} \widehat{\tau}_{jk} = \dfrac{2}{n(n-1)} \displaystyle\sum_{1 \leq i < i' \leq n} \mathrm{sign}\,(x_{ij} - x_{i'j})\,(x_{ik} - x_{i'k})\,, & \text{if } j \neq k \\ \widehat{\tau}_{jk} = 1, \quad \text{if } j = k. \end{cases} \tag{3.2}$$

to approximate $\tau(X_j, X_k)$ and measure the association between X_j and X_k. We define the *Kendall's tau correlation matrix* $\widehat{R} = [\widehat{R}_{jk}]$ such that $\widehat{R}_{jk} = \sin\left(\frac{\pi}{2}\widehat{\tau}_{jk}\right)$.

3.2 Methods

The elliptical distribution is of special interest in Principal Component Analysis (PCA). It has been shown in a variety of literatures [27, 11, 22, 12, 24] that the PCA conducted on elliptical distributions share a number of good properties enjoyed by the PCA conducted on the Gaussian distribution. We will utilize this connection to construct a new model in this paper.

3.2.1 TCA Model

Utilizing the natural relationship between elliptical distributions and the PCA, we propose the model of Transelliptical Component Analysis (TCA). Here ideas of transelliptical distribution family and scale-invariant PCA are exploited. We wish to estimate the leading eigenvector of the latent generalized correlation matrix. In particular, the following model $\mathcal{M}_d(\Sigma^0, \xi, s; f)$ with $f = \{f_j\}_{j=1}^d$ is considered:

$$\mathcal{M}_d(\Sigma^0, \xi, s; f): \quad \begin{cases} X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d), \\ \|\theta_1\|_0 = s, \end{cases} \tag{3.3}$$

where θ_1 is the leading eigenvectors of the latent generalized correlation matrix Σ^0 we are interested in estimating. By spectral decomposition, we write: $\Sigma^0 = \sum_{j=1}^d \lambda_d \theta_d \theta_d^T$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \geq 0$ and $\lambda_1 > 0$ to make Σ^0 non-degenerate. $\theta_1, \ldots, \theta_d \in \mathbb{S}^{d-1}$ are the corresponding eigenvectors of $\lambda_1, \ldots, \lambda_d$. Inspired by the model $\mathcal{M}_d(\Sigma^0, \xi, s; f)$, it is natural to consider the following optimization problem:

$$\begin{aligned} \widetilde{\theta}_1^* = \mathop{\arg\max}_{v \in \mathbb{R}^d} \; & v^T \widehat{R} v, \\ \text{subject to} \;\; & v \in \mathbb{S}^{d-1} \cap \mathbb{B}_0(s), \end{aligned} \tag{3.4}$$

where $\mathbb{B}_0(s) := \{v \in \mathbb{R}^d : \|v\|_0 \leq s\}$ and $\widehat{R}$ is the estimated Kendall's tau correlation matrix. The corresponding global optimum is denoted by $\widetilde{\theta}_1^*$.

3.2.2 TCA Algorithm

Generally we can plug in the Kendall's tau correlation matrix $\widehat{R}$ to any sparse PCA algorithm listed above. In this paper, to approximate θ_1, we consider using the Truncated Power method (TPower) proposed by [28] and [20]. The main idea of the TPower is to utilize the power method, but truncate the vector to a ℓ_0 ball with radius k in each iteration. Detailed algorithms are provided in the long version of this paper [8]. The final estimator is denoted by $\widetilde{\theta}_\infty$ with $\|\widetilde{\theta}_\infty\|_0 = k$. It will be shown in Section 4 and Section 5 that the Kendall's tau correlation matrix is a better statistic in estimating the correlation matrix than the Pearson sample correlation matrix in the sense that (i) it enjoys the Gaussian parametric rate in a much larger distribution family, including many distributions with heavy tails; (ii) it is a more robust estimator, i.e. resistant to outliers.

We use the iterative deflation method to learn the first k instead of the first one leading eigenvectors, following the discussions of [21, 15, 28, 29]. In detail, a matrix $\widehat{\Gamma} \in \mathbb{R}^{d \times s}$ deflates a vector $v \in \mathbb{R}^d$ and achieves a new matrix $\widehat{\Gamma}'$: $\widehat{\Gamma}' := (I - vv^T)\widehat{\Gamma}(I - vv^T)$. In this way, $\widehat{\Gamma}'$ is orthogonal to v.

4 Theoretical Properties

In this section the theoretical properties of the TCA estimators are provided. Especially, we are interested in the high dimensional case when $d > n$.

4.1 Rank-based Correlation Matrix Estimation

This section is devoted to the concentration result of the Kendall sample correlation matrix $\widehat{R}$ to the Pearson correlation matrix Σ^0. The $\ell_{\max}$ convergence rate of $\widehat{R}$ is provided in the next theorem.

Theorem 4.1. *Given $x_1, \ldots, x_n$ n independent realizations of $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$ and letting $\widehat{R}$ be the Kendall tau correlation matrix, we have with probability at least $1 - d^{-5/2}$,*

$$\|\widehat{R} - \Sigma^0\|_{\max} \leq 3\pi\sqrt{\log d/n}. \tag{4.1}$$

Proof sketch. Theorem 4.1 can be proved by realizing that $\widehat{\tau}_{jk}$ is an unbiased estimator of $\tau(X_j, X_k)$ and is a U-statistic with size 2. Hoeffding's inequality for U-statistic can then be applied to obtain the result. Detailed proofs can be found in the long version of this paper [8]. □

4.2 TCA Estimators

This section is devoted to the statement of our main result on the upper bound of the estimated error of the TCA global optimum $\widetilde{\theta}_1^*$ and TPower solver $\widetilde{\theta}_\infty$. We assume that the Model $\mathcal{M}_d(\Sigma^0, \xi, s; f)$ holds and the next theorem provides an upper bound on the angle between the estimated leading eigenvector $\widetilde{\theta}_1^*$ and true leading eigenvector θ_1.

Theorem 4.2. *Let $\widetilde{\theta}_1^*$ be the global solution to Equation* (3.4) *and the Model $\mathcal{M}_d(\Sigma^0, \xi, s; f)$ holds. For any two vectors $v_1 \in \mathbb{S}^{d-1}$ and $v_2 \in \mathbb{S}^{d-1}$, letting*

$$|\sin\angle(v_1, v_2)| = \sqrt{1 - (v_1^T v_2)^2},$$

then we have

$$\mathbb{P}\left(|\sin\angle(\widetilde{\theta}_1^*, \theta_1)| \leq \frac{6\pi}{\lambda_1 - \lambda_2} \cdot s\sqrt{\frac{\log d}{n}}\right) \geq 1 - d^{-5/2}. \tag{4.2}$$

Proof sketch. The key idea of the proof is to utilize the $\ell_{\max}$ norm convergence result of $\widehat{R}$ to Σ^0. Detailed proofs can be found in the long version of this paper [8]. □

Generally, when s and λ_1, λ_2 do not scale with (n, d), the rate is $O_P(\sqrt{\log d/n})$, which is the parametric rate [20, 26, 23] obtains. When (n, d) goes to infinity, the two leading eigenvalues λ_1 and λ_2 will typically go to infinity and will at least be away from zero. Hence, our rate shown in Theorem 4.2 will be usually better than the seemingly more common rate: $\frac{6\pi\lambda_1}{\lambda_1 - \lambda_2} \cdot s\sqrt{\frac{\log d}{n}}$.

Corollary 4.1 (Feature Selection Consistency of the TCA). *Let $\widetilde{\theta}_1^*$ be the global solution to Equation* (3.4) *and the Model $\mathcal{M}_d(\Sigma^0, \xi, s; f)$ holds. Let*

$$\Theta := \textit{supp}(\theta_1) \quad \text{and} \quad \widehat{\Theta}^* := \textit{supp}(\widetilde{\theta}_1^*).$$

If we further have

$$\min_{j \in \Theta} |\theta_{1j}| \geq \frac{6\sqrt{2}\pi}{\lambda_1 - \lambda_2} \cdot s\sqrt{\frac{\log d}{n}},$$

then we have, $\mathbb{P}(\widehat{\Theta}^* = \Theta) \geq 1 - d^{-5/2}$.

Proof sketch. The key of the proof is to construct a contradiction given Theorem 4.2 and the condition on the minimum value of $|\theta_1|$. Detailed proofs can be found in the long version of this paper [8]. □

5 Experiments

In this section we investigate the empirical performance of the TCA method. We utilize the TPower algorithm proposed by [28] and the following three methods are considered: (1) Pearson: the classic high dimensional scale-invariant PCA using the Pearson sample correlation matrix of the data; (2) Kendall: the TCA using the Kendall correlation matrix; (3) LatPearson: the classic high dimensional scale-invariant PCA using the Pearson sample correlation matrix of the data drawn from the latent elliptical distribution (perfect without data contamination).

5.1 Numerical Simulations

In the simulation study we randomly sample n data points from a certain transelliptical distribution $TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$. Here we consider the set up of $d = 100$. To determine the transelliptical distribution, firstly, we derive Σ^0 in the following way: A covariance matrix Σ is firstly synthesized through the eigenvalue decomposition, where the first two eigenvalues are given and the corresponding eigenvectors are pre-specified to be sparse. In detail, let $\Sigma = \sum_{j=1}^{d} \omega_j u_j u_j^T$, where $\omega_1 = 6, \omega_2 = 3, \omega_3 = \ldots = \omega_d = 1$, and the first two leading eigenvectors of Σ, u_1 and u_2, are sparse with the first $s = 10$ entries of u_1 and the second $s = 10$ entries of u_2 are nonzero, i.e.

$$u_{1j} = \begin{cases} \frac{1}{\sqrt{10}} & 1 \leq j \leq 10 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u_{2j} = \begin{cases} \frac{1}{\sqrt{10}} & 11 \leq j \leq 20 \\ 0 & \text{otherwise} \end{cases}. \tag{5.1}$$

The remaining eigenvectors are chosen arbitrarily. The generalized correlation matrix Σ^0 is generated from Σ, with $\lambda_1 = 4$, $\lambda_2 = 2.5$, $\lambda_3, \ldots, \lambda_d \leq 1$ and the top two leading eigenvectors sparse:

$$\theta_{1j} = \begin{cases} -\frac{1}{\sqrt{10}} & 1 \leq j \leq 10 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \theta_{2j} = \begin{cases} -\frac{1}{\sqrt{10}} & 11 \leq j \leq 20 \\ 0 & \text{otherwise} \end{cases}. \tag{5.2}$$

Secondly, using Σ^0, we consider the following three generating schemes:

[Scheme 1] $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$ with $\xi \sim \chi_d$ and $f_1(x) = \ldots = f_d(x) = x$. Here $\sqrt{Y_1^2 + \ldots + Y_d^2} \sim \chi_d$ with $Y_1, \ldots, Y_d \sim^{i.i.d} N(0, 1)$. In other words, χ_d is the chi-distribution with degree of freedom d. This is equivalent to say that $X \sim N(0, \Sigma^0)$ (Example 2.4 of [5]).

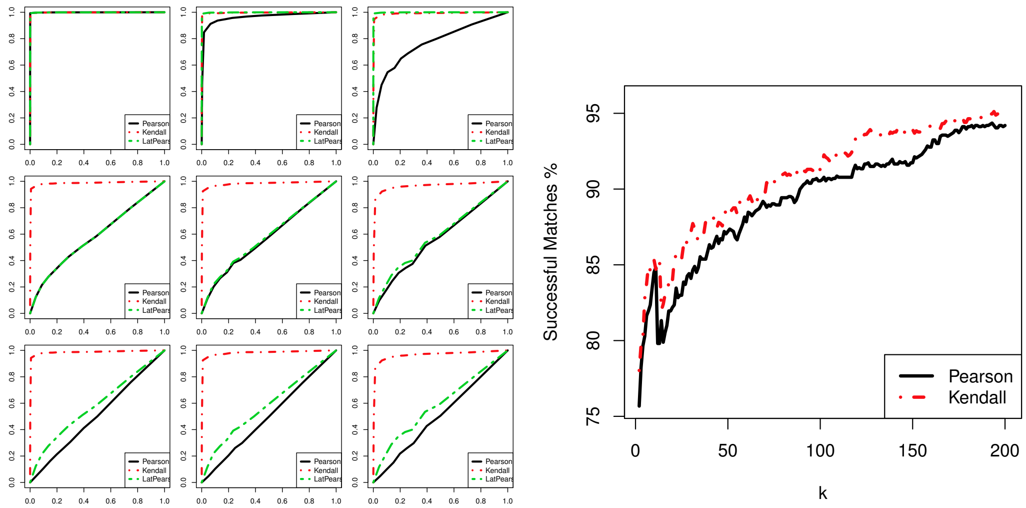
[Scheme 2] $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$ with $\xi =^d \sqrt{m}\xi_1^*/\xi_2^*$ and $f_1(x) = \ldots = f_d(x) = x$. Here $\xi_1^* \sim \chi_d$, $\xi_2^* \sim \chi_m$, ξ_1^* is independent of ξ_2^* and $m \in \mathbb{N}$. This is equivalent to say that $X \sim Mt_d(m, \mathbf{0}, \Sigma^0)$, i.e. X following a multivariate-t distribution with degree of freedom m, mean $\mathbf{0}$ and covariance matrix Σ^0 (Example 2.5 of [5]). Here we consider $m = 3$.

[Scheme 3] $X \sim TE_d(\Sigma^0, \xi; f_1, \ldots, f_d)$ with $\xi =^d \sqrt{m}\xi_1^*/\xi_2^*$. Here $\xi_1^* \sim \chi_d$, $\xi_2^* \sim \chi_m$, ξ_1^* is independent of ξ_2^* and $m = 3$. Moreover, $\{f_1, \ldots, f_d\} = \{h_1, h_2, h_3, h_4, h_5, h_1, h_2, h_3, h_4, h_5, \ldots\}$, where

$$h_1^{-1}(x) := x, \quad h_2^{-1}(x) := \frac{\text{sign}(x)|x|^{1/2}}{\sqrt{\int |t|\phi(t)dt}}, h_3^{-1}(x) := \frac{\Phi(x) - \int \Phi(t)\phi(t)dt}{\sqrt{\int \left(\Phi(y) - \int \Phi(t)\phi(t)dt\right)^2 \phi(y)dy}},$$
$$h_4^{-1}(x) := \frac{x^3}{\sqrt{\int t^6\phi(t)dt}}, h_5^{-1}(x) := \frac{\exp(x) - \int \exp(t)\phi(t)dt}{\sqrt{\int \left(\exp(y) - \int \exp(t)\phi(t)dt\right)^2 \phi(y)dy}}.$$

This is equivalent to say that X is transelliptically distributed with the latent elliptical distribution $Z \sim Mt_d(3, \mathbf{0}, \Sigma^0)$.

To evaluate the robustness of different methods, let $r \in [0, 1)$ represent the proportion of samples being contaminated. For each dimension, we randomly select $\lfloor nr \rfloor$ entries and replace them with either 5 or -5 with equal probability. The final data matrix we obtained is $\boldsymbol{X} \in \mathbb{R}^{n \times d}$. Here we pick $r = 0, 0.02$ or 0.05. Under the Scheme 1 to Scheme 3 with different levels of contamination ($r = 0, 0.02$ or 0.05), we repeatedly generate the data matrix $\boldsymbol{X}$ for 1,000 times and compute the averaged False Positive Rates and False Negative Rates using a path of tuning parameters k from 5 to 90. The feature selection performances of different methods are then evaluated by plotting $(\text{FPR(k)}, 1 - \text{FNR(k)})$. The corresponding ROC curves are presented in Figure 1 (A). More results are shown in the long version of this paper [8]. It can be observed that Kendall is generally better and more resistance to the outliers compared with Pearson.



(A) (B)

Figure 1: (A) ROC curves under Scheme 1, Scheme 2 and Scheme 3 (top, middle, bottom) and data contamination at different levels ($r = 0, 0.02, 0.05$ from left to right). $x-$axis is FPR and $y-$axis is TPR. Here $n = 100$ and $d = 100$. (B) Successful matches of the market trend proportions only using the stocks in A_k and B_k. The $x-$axis represents the tuning parameter k scaling from 1 to 200; the $y-$axis represents the % of successful matches. The curve denoted by 'Kendall' represents the points of (k, ρ_{A_k}) and the curves denoted by 'Pearson' represents the points of (k, ρ_{B_k}).

5.2 Equities Data

In this section we apply the TCA on the stock price data from Yahoo! Finance (`finance.yahoo.com`). We collected the daily closing prices for J=452 stocks that were consistently in the S&P 500 index between January 1, 2003 through January 1, 2008. This gave us altogether T=1,257 data points, each data point corresponds to the vector of closing prices on a trading day. Let $St = [St_{t,j}]$ denote by the closing price of stock j on day t.

We wish to evaluate the ability of using the only k stocks to represent the trend of the whole stock market. To this end, we run Kendall and Pearson on St and obtain the leading eigenvectors $\widetilde{\theta}_{Kendall}$ and $\widetilde{\theta}_{Pearson}$ using the tuning parameter $k \in \mathbb{N}$. Let $A_k := \text{supp}(\widetilde{\theta}_{Kendall})$ and $B_k := \text{supp}(\widetilde{\theta}_{Pearson})$. And then we let T_t^W, $T_t^{A_k}$ and $T_t^{B_k}$ denote by the trend of the whole stocks, A_k stocks and B_k stocks in tth day compared with $t-1$th date, i.e:

$$T_t^W := I(\sum_j St_{t,j} - \sum_j St_{t-1,j} >), T_t^{A_k} := I(\sum_{j \in A_k} St_{t,j} - \sum_{j \in A_k} St_{t-1,j} > 0)$$

and

$$T_t^{B_k} := I(\sum_{j \in B_k} St_{t,j} - \sum_{j \in B_k} St_{t-1,j} > 0),$$

here I is the indicator function. In this way, we can calculate the proportion of successful matches of the market trend using the stocks in A_k and B_k as: $\rho_{A_k} := \frac{1}{T}\sum_t I(T_t^W = T_t^{A_k})$ and $\rho_{B_k} := \frac{1}{T}\sum_t I(T_t^W = T_t^{B_k})$. We visualize the result by plotting (k, ρ_{A_k}) and (k, ρ_{B_k}) on a 2D figure. The result is presented in Figure 1 (B).

It can be observed from Figure 1 (B) that Kendall summarizes the trend of the whole stock market constantly better than Pearson. Moreover, the averaged difference between the two methods are $\frac{1}{200}\sum_k(\rho_{A_k} - \rho_{B_k}) = 1.4025$ with the standard deviation 0.6743. Therefore, the difference is significant.

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