A Smoothed Approximate Linear Program: Technical Appendix

1 Proof of Theorem 1

Lemma 1. *For any $r \in \mathbb{R}^K$ and $\theta \geq 0$:*

(i) $\ell(r,\theta)$ *is a bounded, decreasing, piecewise linear, convex function of* θ*.*

(ii) $\ell(r,\theta) \leq (1+\alpha)\|J^* - \Phi r\|_\infty$.

(iii) $\frac{\partial}{\partial r}\ell(r,0) = -\frac{1}{\sum_{x\in\Omega(r)} \pi_{\mu^*,\nu}(x)}$, *where* $\Omega(r) = \operatorname{argmax}_{x\in\mathcal{X}} \Phi r(x) - T\Phi r(x)$.

Proof. *(i)* Given any r, clearly $\gamma = \|\Phi r - T\Phi r\|_\infty$, $s = 0$ is feasible for (5) so that $\ell(r,\theta)$ is bounded from below. To see that the LP is bounded, suppose (s,γ) is feasible. Then, for any $x \in \mathcal{X}$,

$$\gamma \geq \Phi r(x) - T\Phi r(x) - s(x) \geq \Phi r(x) - T\Phi r(x) - \theta/\pi_{\mu^*,\nu}(x).$$

Thus, the LP is bounded implying that $\ell(r,\theta)$ is bounded from above.

(ii) Let $\epsilon = \|J^* - \Phi r\|_\infty$. Then,

$$\|T\Phi r - \Phi r\|_\infty \leq \|J^* - T\Phi r\|_\infty + \|J^* - \Phi r\|_\infty \leq \alpha\|J^* - \Phi r\|_\infty + \epsilon = (1+\alpha)\epsilon.$$

Since $\gamma = \|T\Phi r - \Phi r\|_\infty$, $s = 0$ is feasible for (5), it follows that $\ell(r,\theta) \leq (1+\alpha)\|J^* - \Phi r\|_\infty$.

(iii) This claim follows immediately from standard LP sensitivity analysis; we note that $\Omega(r)$ is precisely the set of states whose constraints are binding at $\theta = 0$. ■

Lemma 2. *Let (r,s) be feasible for the LP (4). Then,*

$$\Phi r - \Delta^* s \leq J^*,$$

where

$$\Delta^* \triangleq \sum_{k=0}^{\infty} (\alpha P_{\mu^*})^k = (I - \alpha P_{\mu^*})^{-1},$$

and P_{μ^*} is the transition probability matrix corresponding to the optimal policy.

Proof. Note that

$$\Phi r \leq T_{\mu^*}\Phi r + s,$$

where T_{μ^*} is the Bellman operator corresponding to the optimal policy. Repeatedly applying T_{μ^*} and using the fact that $T_{\mu^*}^k \Phi r \to J^*$, we obtain

$$\Phi r \leq J^* + \sum_{k=0}^{\infty} (\alpha P_{\mu^*})^k s = J^* + \Delta^* s.$$

■

Theorem 1. *Let $\mathbf{1}$ be in the span of Φ and ν be a probability distribution. Let $\bar{r}$ be an optimal solution to the SALP (4). Moreover, let r^* satisfy $r^* \in \operatorname{argmin}_r \|J^* - \Phi r\|_\infty$. Then,*

$$\|J^* - \Phi\bar{r}\|_{1,\nu} \leq \|J^* - \Phi r^*\|_\infty + \frac{l(r^*,\theta) + 2\theta}{1-\alpha}.$$

Proof. First, define the weight vector $\tilde{r} \in \mathbb{R}^K$ by

$$\Phi\tilde{r} = \Phi r^* - \frac{\ell(r^*,\theta)}{1-\alpha}\mathbf{1},$$

and set $\tilde{s} = s(r^*, \theta)$, the s-component of the solution to the LP (5) with parameters r^* and θ. We will demonstrate that $(\tilde{r}, \tilde{s})$ is feasible for (3). Observe that, by the definition of the LP (5),

$$\Phi r^* \leq T\Phi r^* + \tilde{s} + \ell(r^*, \theta)\mathbf{1}.$$

Then,

$$\begin{aligned}
T\Phi\tilde{r} &= T\Phi r^* - \frac{\alpha\ell(r^*, \theta)}{1-\alpha}\mathbf{1} \\
&\geq \Phi r^* - \tilde{s} - \ell(r^*, \theta)\mathbf{1} - \frac{\alpha\ell(r^*, \theta)}{1-\alpha}\mathbf{1} \\
&= \Phi\tilde{r} + \frac{\ell(r^*, \theta)}{1-\alpha} - \tilde{s} - \ell(r^*, \theta)\mathbf{1} - \frac{\alpha\ell(r^*, \theta)}{1-\alpha}\mathbf{1} \\
&= \Phi\tilde{r} - \tilde{s}.
\end{aligned}$$

Now, by Lemma 2,

$$\begin{aligned}
\|J^* - \Phi\bar{r}\|_{1,\nu} &\leq \|J^* - \Phi\bar{r} + \Delta^*\bar{s}\|_{1,\nu} + \|\Delta^*\bar{s}\|_{1,\nu} \\
&= \nu^\top(J^* - \Phi\bar{r} + \Delta^*\bar{s}) + \nu^\top\Delta^*\bar{s} \\
&= \nu^\top(J^* - \Phi\bar{r}) + \frac{2\pi_{\mu^*,\nu}^\top\bar{s}}{1-\alpha} \\
&\leq \nu^\top(J^* - \Phi\bar{r}) + \frac{2\theta}{1-\alpha} \\
&\leq \nu^\top(J^* - \Phi\tilde{r}) + \frac{2\theta}{1-\alpha} \\
&\leq \|J^* - \Phi\tilde{r}\|_\infty + \frac{2\theta}{1-\alpha} \\
&\leq \|J^* - \Phi r^*\|_\infty + \|\Phi r^* - \Phi\tilde{r}\|_\infty + \frac{2\theta}{1-\alpha} \\
&\leq \|J^* - \Phi r^*\|_\infty + \frac{\ell(r^*, \theta) + 2\theta}{1-\alpha},
\end{aligned}$$

as desired. ■

2 Proof of Theorem 2

Theorem 2. *Let $\Psi \triangleq \{y \in \mathbb{R}^{|\mathcal{X}|} : y \geq \mathbf{1}\}$. For every $\psi \in \Psi$, let $\beta(\psi) = \max_\mu \left\|\frac{P_\mu\psi}{\psi}\right\|_\infty$. Then, for an optimal solution $(\bar{r}, \bar{s})$ to (6), we have:*

$$\|J^* - \Phi\bar{r}\|_{1,\nu} \leq \inf_{r,\psi\in\Psi} \|J^* - \Phi r\|_{\infty,\mathbf{1}/\psi}\left(\nu^\top\psi + \frac{2(\pi_{\mu^*,\nu}^\top\psi + 1)(\alpha\beta(\psi) + 1)}{1-\alpha}\right).$$

Proof. Let $r \in \mathbb{R}^K$ be arbitrary. Let $\epsilon_r(x) = ((\Phi r)(x) - (T\Phi r)(x))^+$ and $\psi \in \Psi$. Define s_r according to $s_r(x) = \epsilon_r(x)(1 - \frac{1}{\psi(x)})$ and notice that $0 \leq s_r(x) \leq \epsilon_r(x)$.

We next make a few observations. First, define $\tilde{r}_r$ according to $\Phi\tilde{r}_r = \Phi r - \frac{\|\epsilon_r\|_{\infty,1/\psi}}{1-\alpha}\mathbf{1}$, and observe that by construction, $(\tilde{r}_r, s_r)$ is feasible for (6). Thus,

$$\|\Phi r - \Phi\tilde{r}_r\|_\infty \leq \frac{\|\epsilon_r\|_{\infty,1/\psi}}{1-\alpha} \leq \frac{\|T\Phi r - \Phi r\|_{\infty,1/\psi}}{1-\alpha}.$$

Next, observe that

$$\begin{aligned}
\pi_{\mu^*,\nu}^\top s_r &= \sum_x \pi_{\mu^*,\nu}(x)\epsilon_r(x)(1 - 1/\psi(x)) \\
&\leq \pi_{\mu^*,\nu}^\top\epsilon_r \\
&\leq (\pi_{\mu^*,\nu}^\top\psi)\|\epsilon_r\|_{\infty,\mathbf{1}/\psi} \\
&\leq (\pi_{\mu^*,\nu}^\top\psi)\|T\Phi r - \Phi r\|_{\infty,\mathbf{1}/\psi}
\end{aligned}$$

Finally, observe that,

$$\pi_{\mu^*,\nu}^\top (J^* - \Phi r) \le (\pi_{\mu^*,\nu}^\top \psi) \|J^* - \Phi r\|_{\infty, \mathbf{1}/\psi}.$$

Now, we have from the last set of inequalities in the proof of Theorem 1 and the above observations:

$$\begin{aligned} \|J^* - \Phi \bar{r}\|_{1,\nu} &\le \nu^\top (J^* - \Phi \bar{r}) + \frac{2\pi_{\mu^*,\nu}^\top \bar{s}}{1-\alpha} \\ &\le \nu^\top (J^* - \Phi \tilde{r}_r) + \frac{2\pi_{\mu^*,\nu}^\top s_r}{1-\alpha} \\ &\le \nu^\top (J^* - \Phi r) + \nu^\top (\Phi r - \Phi \tilde{r}_r) + \frac{2\pi_{\mu^*,\nu}^\top s_r}{1-\alpha} \\ &\le \nu^\top (J^* - \Phi r) + \|\Phi r - \Phi \tilde{r}_r\|_\infty + \frac{2\pi_{\mu^*,\nu}^\top s_r}{1-\alpha} \\ &\le (\nu^\top \psi) \|J^* - \Phi r\|_{\infty, \mathbf{1}/\psi} + (2\pi_{\mu^*,\nu}^\top \psi + 1) \frac{\|T\Phi r - \Phi r\|_{\infty, 1/\psi}}{1-\alpha} \end{aligned} \tag{9}$$

Since our choice of r and ψ were arbitrary, we have:

$$\begin{aligned} \|J^* - \Phi \bar{r}\|_{1,\nu} \le \inf_{r,\psi \in \Psi} &(\nu^\top \psi) \|J^* - \Phi r\|_{\infty, \mathbf{1}/\psi} + \\ &\frac{\|T\Phi r - \Phi r\|_{\infty, 1/\psi}}{1-\alpha} + \frac{2(\pi_{\mu^*,\nu}^\top \psi) \|T\Phi r - \Phi r\|_{\infty, 1/\psi}}{1-\alpha}. \end{aligned} \tag{10}$$

We now relate the Bellman error on the right hand side in the above bound to the optimal approximation error. Before doing so we recall:

$$|TJ - T\bar{J}| \le \alpha \max_u P_u |J - \bar{J}|,$$

and derive the following intermediate result,

$$\begin{aligned} \max_\mu \|P_\mu |\Phi r - J^*|\|_{\infty, \mathbf{1}/\psi} &= \max_{\mu, x \in \mathcal{X}} \left(\frac{\sum_{y \in \mathcal{X}} P_\mu(x,y) |(\Phi r)(y) - J^*(y)|}{\psi(x)} \right) \\ &= \max_{\mu, x \in \mathcal{X}} \left(\frac{\sum_{y \in \mathcal{X}} P_\mu(x,y) \psi(y) \frac{|(\Phi r)(y) - J^*(y)|}{\psi(y)}}{\psi(x)} \right) \\ &\le \max_{\mu, x \in \mathcal{X}} \left(\frac{\sum_{y \in \mathcal{X}} P_\mu(x,y) \psi(y)}{\psi(x)} \right) \|\Phi r - J^*\|_{\infty, \mathbf{1}/\psi} \\ &= \max_\mu \left\| \frac{P_\mu \psi}{\psi} \right\|_\infty \|\Phi r - J^*\|_{\infty, \mathbf{1}/\psi} \\ &= \beta(\psi) \|\Phi r - J^*\|_{\infty, \mathbf{1}/\psi}. \end{aligned}$$

Now,

$$\begin{aligned} \|T\Phi r - \Phi r\|_{\infty, \mathbf{1}/\psi} &\le \left(\|T\Phi r - J^*\|_{\infty, \mathbf{1}/\psi} + \|J^* - \Phi r\|_{\infty, \mathbf{1}/\psi} \right) \\ &\le \left(\alpha \max_\mu \|P_\mu |\Phi r - J^*|\|_{\infty, \mathbf{1}/\psi} + \|J^* - \Phi r\|_{\infty, \mathbf{1}/\psi} \right) \\ &\le \left(\alpha \beta(\psi) \|\Phi r - J^*\|_{\infty, \mathbf{1}/\psi} + \|J^* - \Phi r\|_{\infty, \mathbf{1}/\psi} \right) \\ &= \|\Phi r - J^*\|_{\infty, \mathbf{1}/\psi} (\alpha \beta(\psi) + 1). \end{aligned} \tag{11}$$

Using (10) and (11), we get the result. ■

3 Sample Complexity Results: Proof of Theorem 3

Our proof will rely on the following lemma, which provides a Chernoff bound for the *uniform* convergence of a certain class of functions and the proof of this lemma is based on bounding the pseudo-dimension of this class of functions.

Lemma 3. *Given a constant $B > 0$, define the function $\zeta\colon \mathbb{R} \to [0, B]$ by*

$$\zeta(t) \triangleq \max\left(\min(t, B), 0\right).$$

Consider a pair of random variables $(Y, Z) \in \mathbb{R}^K \times \mathbb{R}$. For each $i = 1, \ldots, n$, let the pair $\left(Y^{(i)}, Z^{(i)}\right)$ be an i.i.d. sample drawn according to the distribution of (Y, Z). Then, for all $\epsilon \in (0, B]$,

$$\mathsf{P}\left(\sup_{r \in \mathbb{R}^K} \left| \frac{1}{n} \sum_{i=1}^{n} \zeta\left(r^\top Y^{(i)} + Z^{(i)}\right) - \mathsf{E}\left[\zeta\left(r^\top Y + Z\right)\right] \right| > \epsilon\right) \leq 8\left(\frac{32eB}{\epsilon} \log \frac{32eB}{\epsilon}\right)^{K+2} \exp\left(-\frac{\epsilon^2 n}{64B^2}\right).$$

Moreover, given $\delta \in (0, 1)$, if

$$n \geq \frac{64B^2}{\epsilon^2}\left(2(K+2)\log\frac{16eB}{\epsilon} + \log\frac{8}{\delta}\right),$$

then this probability is at most δ.

Before presenting the proof, we present a few definitions and intermediate results. Consider a family $\mathcal{F}$ of functions from a set $\mathcal{S}$ to $\{0, 1\}$. Define the *Vapnik-Chervonenkis (VC) dimension* $\dim_{\mathrm{VC}}(\mathcal{F})$ to be the cardinality d of the largest set $\{x_1, x_2, \ldots, x_d\} \subset \mathcal{S}$ satisfying:

$$\forall e \in \{0, 1\}^d,\ \exists f \in \mathcal{F} \text{ such that } \forall i,\ f(x_i) = 1 \text{ iff } e_i = 1.$$

Now, let $\mathcal{F}$ be some set of *real*-valued functions mapping $\mathcal{S}$ to $[0, B]$. The *pseudo-dimension* $\dim_P(\mathcal{F})$ is the following generalization of VC dimension: for each function $f \in \mathcal{F}$ and scalar $c \in \mathbb{R}$, define a function $g\colon \mathcal{S} \times \mathbb{R} \to \{0, 1\}$ according to:

$$g(x, c) \triangleq \mathbb{I}_{\{f(x) - c \geq 0\}}.$$

Let $\mathcal{G}$ denote the set of all such functions. Then, we define $\dim_P(\mathcal{F}) \triangleq \dim_{\mathrm{VC}}(\mathcal{G})$.

In order to prove Lemma 3, define the $\mathcal{F}$ to be the set of functions $f\colon \mathbb{R}^K \times \mathbb{R} \to [0, B]$, where, for all $x \in \mathbb{R}^K$ and $y \in \mathbb{R}$,

$$f(y, z) \triangleq \zeta\left(r^\top y + z\right).$$

Here, $\zeta(t) \triangleq \max\left(\min(t, B), 0\right)$, and $r \in \mathbb{R}^K$ is a vector that parameterizes f. We will show that $\dim_P(\mathcal{F}) \leq K + 2$.

We will use the following standard result from convex geometry:

Lemma 4 (Radon's Lemma)**.** *A set $A \subset \mathbb{R}^m$ of $m + 2$ points can be partitioned into two disjoint sets A_1 and A_2, such that the convex hulls of A_1 and A_2 intersect.*

Lemma 5. $\dim_P(\mathcal{F}) \leq K + 2$

Proof. Assume, for the sake of contradiction, that $\dim_P(\mathcal{F}) > K+2$. It must be that there exists a 'shattered' set

$$\left\{\left(y^{(1)}, z^{(1)}, c^{(1)}\right), \left(y^{(2)}, z^{(2)}, c^{(2)}\right), \ldots, \left(y^{(K+3)}, z^{(K+3)}, c^{(K+3)}\right)\right\} \subset \mathbb{R}^K \times \mathbb{R} \times \mathbb{R},$$

such that, for all $e \in \{0, 1\}^{K+3}$, there exists a vector $r_e \in \mathbb{R}^K$ with

$$\zeta\left(r_e^\top y^{(i)} + z^{(i)}\right) \geq c^{(i)} \text{ iff } e_i = 1, \quad \forall\ 1 \leq i \leq K+3.$$

Observe that we must have $c^{(i)} \in (0, B]$ for all i, since if $c^{(i)} \leq 0$ or $c^{(i)} > B$, then no such shattered set can be demonstrated. But if $c^{(i)} \in (0, B]$, for all $r \in \mathbb{R}^K$,

$$\zeta\left(r^\top y^{(i)} + z^{(i)}\right) \geq c^{(i)} \implies r_e^\top y^{(i)} \geq c^{(i)} - z^{(i)},$$

and

$$\zeta\left(r^\top y^{(i)} + z^{(i)}\right) < c^{(i)} \implies r_e^\top y^{(i)} < c^{(i)} - z^{(i)}.$$

For each $1 \leq i \leq K+3$, define $x^{(i)} \in \mathbb{R}^{K+1}$ component-wise according to

$$x_j^{(i)} \triangleq \begin{cases} y_j^{(i)} & \text{if } j < K+1, \\ c^{(i)} - z^{(i)} & \text{if } j = K+1. \end{cases}$$

Let $A = \{x^{(1)}, x^{(2)}, \ldots, x^{(K+3)}\} \subset \mathbb{R}^{K+1}$, and let A_1 and A_2 be subsets of A satisfying the conditions of Radon's lemma. Define a vector $\tilde{e} \in \{0,1\}^{K+3}$ component-wise according to

$$\tilde{e}_i \triangleq \mathbb{I}_{\left\{x^{(i)} \in A_1\right\}}.$$

Define the vector $\tilde{r} \triangleq r_{\tilde{e}}$. Then, we have

$$\sum_{j=1}^{K} \tilde{r}_j x_j \geq x_{K+1}, \quad \forall\, x \in A_1,$$

$$\sum_{j=1}^{K} \tilde{r}_j x_j < x_{K+1}, \quad \forall\, x \in A_2.$$

Now, let $\bar{x} \in \mathbb{R}^{K+1}$ be a point contained in both the convex hull of A_1 and the convex hull of A_2. Such a point must exist by Radon's lemma. By virtue of being contained in the convex hull of A_1, we must have

$$\sum_{j=1}^{K} \tilde{r}_j \bar{x}_j \geq \bar{x}_{K+1}.$$

Yet, by virtue of being contained in the convex hull of A_2, we must have

$$\sum_{j=1}^{K} \tilde{r}_j \bar{x}_j < \bar{x}_{K+1},$$

which is impossible. ∎

With the above pseudo-dimension estimate, Lemma 3 follows immediately from Corollary 2 of Haussler [11, Section 4]. Armed with this Lemma, we are ready to prove Theorem 3.

Proof of Theorem 3. Define the vectors

$$\hat{s}_{\mu^*} \triangleq \left(\Phi\hat{r}_{\text{SALP}} - T_{\mu^*}\Phi\hat{r}_{\text{SALP}}\right)^+, \quad \text{and} \quad \hat{s} \triangleq \left(\Phi\hat{r}_{\text{SALP}} - T\Phi\hat{r}_{\text{SALP}}\right)^+.$$

One has, via Lemma 2, that

$$\Phi\hat{r}_{\text{SALP}} - J^* \leq \Delta^* \hat{s}_{\mu^*}$$

Thus, as in the last set of inequalities in the proof of Theorem 1, we have

$$\|J^* - \Phi\hat{r}_{\text{SALP}}\|_{1,\nu} \leq \nu^\top (J^* - \Phi\hat{r}_{\text{SALP}}) + \frac{2\pi_{\mu^*,\nu}^\top \hat{s}_{\mu^*}}{1-\alpha}. \tag{12}$$

Now, let $\hat{\pi}_{\mu^*,\nu}$ be the empirical measure induced by the collection of sampled states $\hat{\mathcal{X}}$. Given a state $x \in \mathcal{X}$, define a vector $Y(x) \in \mathbb{R}^K$ and a scalar $Z(x) \in \mathbb{R}$ according to

$$Y(x) \triangleq \Phi(x)^\top - \alpha P_{\mu^*}\Phi(x)^\top, \quad Z(x) \triangleq -g(x, \mu^*(x)),$$

so that, for any vector of weights $r \in \mathcal{N}$,

$$(\Phi r(x) - T_{\mu^*}\Phi r(x))^+ = \zeta\left(r^\top Y(x) + Z(x)\right).$$

Then,

$$\left|\hat{\pi}_{\mu^*,\nu}^\top \hat{s}_{\mu^*} - \pi_{\mu^*,\nu}^\top \hat{s}_{\mu^*}\right| \leq \sup_{r\in\mathcal{N}} \left| \frac{1}{S}\sum_{x\in\hat{\mathcal{X}}} \zeta\left(r^\top Y(x) + Z(x)\right) - \sum_{x\in\mathcal{X}} \pi_{\mu^*,\nu}(x)\zeta\left(r^\top Y(x)+Z(x)\right)\right|.$$

Applying Lemma 3, we have that

$$\mathsf{P}\left(\left|\hat{\pi}_{\mu^*,\nu}^\top \hat{s}_{\mu^*} - \pi_{\mu^*,\nu}^\top \hat{s}_{\mu^*}\right| > \epsilon\right) \leq \delta. \tag{13}$$

Next, suppose $(r_{\mathrm{SALP}}, \bar{s})$ is an optimal solution to the SALP (6). Then, with probability at least $1-\delta$,

$$\begin{aligned}
\nu^\top(J^* - \Phi\hat{r}_{\mathrm{SALP}}) + \frac{2\pi_{\mu^*,\nu}^\top \hat{s}_{\mu^*}}{1-\alpha} &\leq \nu^\top(J^* - \Phi\hat{r}_{\mathrm{SALP}}) + \frac{2\hat{\pi}_{\mu^*,\nu}^\top \hat{s}_{\mu^*}}{1-\alpha} + \frac{2\epsilon}{1-\alpha} \\
&\leq \nu^\top(J^* - \Phi\hat{r}_{\mathrm{SALP}}) + \frac{2\hat{\pi}_{\mu^*,\nu}^\top \hat{s}}{1-\alpha} + \frac{2\epsilon}{1-\alpha} \\
&\leq \nu^\top(J^* - \Phi r_{\mathrm{SALP}}) + \frac{2\hat{\pi}_{\mu^*,\nu}^\top \bar{s}}{1-\alpha} + \frac{2\epsilon}{1-\alpha},
\end{aligned} \tag{14}$$

where the first inequality follows from (13), and the final inequality follows from the optimality of $(\hat{r}_{\mathrm{SALP}}, \hat{s})$ for the sampled SALP (7).

Notice that, without loss of generality, we can assume that $\bar{s}(x) = (\Phi r_{\mathrm{SALP}}(x) - T\Phi r_{\mathrm{SALP}}(x))^+$, for each $x \in \mathcal{X}$. Thus, $0 \leq \bar{s}(x) \leq B$. Applying Hoeffding's inequality,

$$\mathsf{P}\left(\left|\hat{\pi}_{\mu^*,\nu}^\top \bar{s} - \pi_{\mu^*,\nu}^\top \bar{s}\right| \geq \epsilon\right) \leq 2\exp\left(-\frac{2S\epsilon^2}{B^2}\right) < 2^{-383}\delta^{128},$$

where final inequality follows from our choice of S. Combining this with (12) and (14), with probability at least $1-\delta-2^{-383}\delta^{128}$, we have

$$\begin{aligned}
\|J^* - \Phi\hat{r}_{\mathrm{SALP}}\|_{1,\nu} &\leq \nu^\top(J^* - \Phi r_{\mathrm{SALP}}) + \frac{2\hat{\pi}_{\mu^*,\nu}^\top \bar{s}}{1-\alpha} + \frac{2\epsilon}{1-\alpha} \\
&\leq \nu^\top(J^* - \Phi r_{\mathrm{SALP}}) + \frac{2\pi_{\mu^*,\nu}^\top \bar{s}}{1-\alpha} + \frac{4\epsilon}{1-\alpha}.
\end{aligned}$$

The result then follows from (9)–(11) in the proof of Theorem 2. ■