
Inherent Tradeoffs in Learning Fair Representation

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Abstract

With the prevalence of machine learning in high-stakes applications, especially the ones regulated by anti-discrimination laws or societal norms, it is crucial to ensure that the predictive models do not propagate any existing bias or discrimination. Due to the ability of deep neural nets to learn rich representations, recent advances in algorithmic fairness have focused on learning fair representations with adversarial techniques to reduce bias in data while preserving utility simultaneously. In this paper, through the lens of information theory, we provide the first result that quantitatively characterizes the tradeoff between demographic parity and the joint utility across different population groups. Specifically, when the base rates differ between groups, we show that any method aiming to learn fair representation admits an information-theoretic lower bound on the joint error across these groups. To complement our negative results, we also prove that if the optimal decision functions across different groups are close, then learning fair representation leads to an alternative notion of fairness, known as the accuracy parity, which states that the error rates are close between groups. Our theoretical findings are also confirmed empirically on real-world datasets. We believe our insights contribute to better understanding of the tradeoff between utility and different notions of fairness.

1 Introduction

With the prevalence of machine learning applications in high-stakes domains, e.g., criminal judgement, medical testing, online advertising, etc., it is crucial to ensure that the automated decision making systems do not propagate existing bias or discrimination that might exist in historical data [3, 5, 32]. Among many recent proposals for achieving different notions of algorithmic fairness [13, 17, 35–37], learning fair representations has received increasing attention due to recent advances in learning rich representation with deep neural networks [6, 14, 27, 29, 34, 38]. In fact, a line of work has proposed to learn group-invariant representations with adversarial learning techniques in order to achieve statistical parity, also known as the demographic parity in the literature. This line of work dates at least back to Zemel et al. [37] where the authors proposed to learn predictive models that are independent of the group membership attribute. At a high level, the underlying idea is that if representations of instances from different groups are similar to each other, then the follow-up predictive models will certainly make decisions independent of group membership.

On the other hand, it has long been observed that there is an underlying tradeoff between utility and demographic parity:

“All methods have in common that to some extent accuracy must be traded-off for lowering the dependency.” [8]

In particular, it is easy to see that in an extreme case where the group membership coincides with the target task, a call for exact demographic parity will inevitably remove the perfect predictor [17]. Empirically, it has also been observed that a tradeoff exists between accuracy and fairness in binary

classification [40]. Clearly, methods based on learning fair representations are also bound by such inherent tradeoff between utility and fairness. But how does the fairness constraint trade for utility? Will learning fair representations help to achieve other notions of fairness besides the demographic parity? If yes, what is the fundamental limit of utility that we can hope to achieve under such constraint?

To answer the above questions, through the lens of information theory, in this paper we provide the first result that quantitatively characterizes the tradeoff between demographic parity and the joint utility across different population groups. Specifically, when the base rates differ between groups, we provide a tight information-theoretic lower bound on the joint error across these groups. Our lower bound is algorithm-independent so it holds for all methods aiming to learn fair representations. When only approximate demographic parity is achieved, we also present a family of lower bounds to quantify the tradeoff of utility introduced by such approximate constraint. As a side contribution, our proof technique is simple but general, and we expect it to have broader applications in other learning problems using adversarial techniques, e.g., unsupervised domain adaptation [15, 39] and privacy-preservation under attribute inference attacks [16].

To complement our negative results, we show that if the optimal decision functions across different groups are close, then learning fair representation helps to achieve an alternative notion of fairness, i.e., the accuracy parity, which states that the error rates are close between groups. Empirically, we conduct experiments on a real-world dataset that corroborate both our positive and negative results. We believe our theoretical insights contribute to better understanding of the tradeoff between utility and different notions of fairness, and they are also helpful in guiding the future design of representation learning algorithms to achieve algorithmic fairness.

2 Preliminary

We first introduce the notations used throughout the paper and formally describe the problem setup. We then briefly discuss some information-theoretic concepts that will be used in our analysis.

Notations We use $\mathcal{X} \subseteq \mathbb{R}^d$ and $\mathcal{Y} = \{0, 1\}$ to denote the input and output space. Accordingly, we use X and Y to denote the random variables which take values in \mathcal{X} and \mathcal{Y} , respectively. Lower case letters \mathbf{x} and y are used to denote the instantiation of X and Y . To simplify the presentation, we use $A \in \{0, 1\}$ as the sensitive attribute, e.g., race, gender, etc.¹ Let \mathcal{H} be the hypothesis class of classifiers. In other words, for $h \in \mathcal{H}$, $h = h(X) : \mathcal{X} \rightarrow \mathcal{Y}$ is the predictor that outputs a prediction. Note that even the predictor does not explicitly take the sensitive attribute A as input, this *fairness through blindness* mechanism can still be biased due to the potential correlations between X and A . In this work we study the stochastic setting where there is a joint distribution \mathcal{D} over X, Y and A from which the data are sampled. To keep the notation consistent, for $a \in \{0, 1\}$, we use \mathcal{D}_a to mean the conditional distribution of \mathcal{D} given $A = a$. For an event E , $\mathcal{D}(E)$ denotes the probability of E under \mathcal{D} .

Problem Setup Given a joint distribution \mathcal{D} , the error of a predictor h under \mathcal{D} is defined as $\text{Err}_{\mathcal{D}}(h) := \mathbb{E}_{\mathcal{D}}[|Y - h(X)|]$. Note that for binary classification problems, when $h(X) \in \{0, 1\}$, $\text{Err}_{\mathcal{D}}(h)$ reduces to the true error rate of binary classification. To make the notation more compact, we may drop the subscript \mathcal{D} when it is clear from the context. In this work we focus on group fairness where the group membership is given by the sensitive attribute A . Even in this context there are many possible definitions of *fairness* [31], and in what follows we provide a brief review of the ones that are mostly relevant to this work.

Definition 2.1 (Demographic Parity). Given a joint distribution \mathcal{D} , a classifier h satisfies *demographic parity* if $h(X)$ is independent of A .

When h is a deterministic classifier, demographic parity reduces to the requirement that $\mathcal{D}_0(h(X) = 1) = \mathcal{D}_1(h(X) = 1)$, i.e., positive outcome is given to the two groups at the same rate. Demographic parity is also known as *statistical parity*, and it has been adopted as definition of fairness in a series of work [7, 8, 14, 18–20, 27, 29, 37]. However, as we shall quantify precisely in Section 3, demographic parity may cripple the utility that we hope to achieve, especially in the common scenario where the *base rates* differ between two groups, e.g., $\mathcal{D}_0(Y = 1) \neq \mathcal{D}_1(Y = 1)$ [17]. In light of this, an alternative definition is *accuracy parity*:

¹Our main results could also be straightforwardly extended to the setting where A is a categorical variable.

90 **Definition 2.2** (Accuracy Parity). Given a joint distribution \mathcal{D} , a classifier h satisfies *accuracy parity*
 91 if $\text{Err}_{\mathcal{D}_0}(h) = \text{Err}_{\mathcal{D}_1}(h)$.

92 In the literature, a break of accuracy parity is also known as disparate mistreatment [36]. Again, when
 93 h is a deterministic binary classifier, accuracy parity reduces to $\mathcal{D}_0(h(X) = Y) = \mathcal{D}_1(h(X) = Y)$.
 94 Different from demographic parity, the definition of accuracy parity does not eliminate the perfect
 95 predictor when $Y = A$ when the base rates differ between two groups. When costs of different error
 96 types matter, more refined definitions exist:

97 **Definition 2.3** (Positive Rate Parity). Given a joint distribution \mathcal{D} , a deterministic classifier h satisfies
 98 *positive rate parity* if $\mathcal{D}_0(h(X) = 1 \mid Y = y) = \mathcal{D}_1(h(X) = 1 \mid Y = y), \forall y \in \{0, 1\}$.

99 Positive rate parity is also known as *equalized odds* [17], which essentially requires equal true positive
 100 and false positive rates between different groups. Furthermore, Hardt et al. [17] also defined *true*
 101 *positive parity*, or *equal opportunity*, to be $\mathcal{D}_0(h(X) = 1 \mid Y = 1) = \mathcal{D}_1(h(X) = 1 \mid Y = 1)$
 102 when positive outcome is desirable. Last but not least, *predictive value parity*, also known as *test*
 103 *fairness* [9], asks for equal chance of positive outcomes across groups given predictions:

104 **Definition 2.4** (Predictive Value Parity). Given a joint distribution \mathcal{D} , a probabilistic classifier h
 105 satisfies *predictive value parity* if $\mathcal{D}_0(Y = 1 \mid h(X) = c) = \mathcal{D}_1(Y = 1 \mid h(X) = c), \forall c \in [0, 1]$.

106 When h is a deterministic binary classifier that only takes value in $\{0, 1\}$, Chouldechova [9] showed
 107 an intrinsic tradeoff between predictive value parity and positive rate parity:

108 **Theorem 2.1** (Chouldechova [9]). Assume $\mathcal{D}_0(Y = 1) \neq \mathcal{D}_1(Y = 1)$, then for any deterministic
 109 classifier $h : \mathcal{X} \rightarrow \{0, 1\}$ that is not perfect, i.e., $h(X) \neq Y$, positive rate parity and predictive value
 110 parity cannot hold simultaneously.

111 Similar tradeoff result for probabilistic classifier has also been observed by Kleinberg et al. [24],
 112 where the authors showed that for any non-perfect predictors, calibration and positive rate parity
 113 cannot be achieved simultaneously if the base rates are different across groups. Here a classifier h is
 114 said to be *calibrated* if $\mathcal{D}(Y = 1 \mid h(X) = c) = c, \forall c \in [0, 1]$, i.e., if we look at the set of data that
 115 receive a predicted probability of c by h , we would like c -fraction of them to be positive instances
 116 according to Y [33].

117 **f -divergence** Introduced by Ali and Silvey [2] and Csiszár [11, 12], f -divergence, also known as
 118 the Ali-Silvey distance, is a general class of statistical divergences to measure the difference between
 119 two probability distributions \mathcal{P} and \mathcal{Q} over the same measurable space.

120 **Definition 2.5** (f -divergence). Let \mathcal{P} and \mathcal{Q} be two probability distributions over the same space and
 121 assume \mathcal{P} is absolutely continuous w.r.t. \mathcal{Q} ($\mathcal{P} \ll \mathcal{Q}$). Then for any convex function $f : (0, \infty) \rightarrow \mathbb{R}$
 122 that is strictly convex at 1 and $f(1) = 0$, the f -divergence of \mathcal{Q} from \mathcal{P} is defined as

$$D_f(\mathcal{P} \parallel \mathcal{Q}) := \mathbb{E}_{\mathcal{Q}} \left[f \left(\frac{d\mathcal{P}}{d\mathcal{Q}} \right) \right]. \quad (1)$$

123 The function f is called the *generator function* of $D_f(\cdot \parallel \cdot)$.

124 Different choices of the generator function f recover popular statistical divergence as special cases,
 125 e.g., the KL-divergence. From Jensen's inequality it is easy to verify that $D_f(\mathcal{P} \parallel \mathcal{Q}) \geq 0$ and
 126 $D_f(\mathcal{P} \parallel \mathcal{Q}) = 0$ iff $\mathcal{P} = \mathcal{Q}$ almost surely. Note that f -divergence does not necessarily leads to
 127 a distance metric, and it is not symmetric in general, i.e., $D_f(\mathcal{P} \parallel \mathcal{Q}) \neq D_f(\mathcal{Q} \parallel \mathcal{P})$ provided
 128 that $\mathcal{P} \ll \mathcal{Q}$ and $\mathcal{Q} \ll \mathcal{P}$. We list some common choices of the generator function f and their
 129 corresponding properties in Table 1. Notably, Khosravifard et al. [22] proved that total variation is
 130 the only f -divergence that serves as a metric, i.e., satisfying the triangle inequality.

131 3 Theoretical Analysis

132 As we have briefly mentioned in Section 2, it is impossible to have imperfect predictor that is
 133 both calibrated and preserves positive rate parity when the base rates differ between two groups.
 134 Similar impossibility result also holds between positive rate parity and predictive value parity. On
 135 the other hand, while it has long been observed that demographic parity may eliminate perfect
 136 predictor [17], and previous work has empirically verified a tradeoff between the accuracy and

Table 1: List of different f -divergences and their corresponding properties. $D_{\text{KL}}(\mathcal{P} \parallel \mathcal{Q})$ denotes the KL-divergence of \mathcal{Q} from \mathcal{P} and $\mathcal{M} := (\mathcal{P} + \mathcal{Q})/2$ is the average distribution of \mathcal{P} and \mathcal{Q} . Symm. stands for Symmetric and Tri. stands for Triangle Inequality.

Name	$D_f(\mathcal{P} \parallel \mathcal{Q})$	Generator $f(t)$	Symm.	Tri.
Kullback-Leibler	$D_{\text{KL}}(\mathcal{P} \parallel \mathcal{Q})$	$t \log t$	✗	✗
Reverse-KL	$D_{\text{KL}}(\mathcal{Q} \parallel \mathcal{P})$	$-\log t$	✗	✗
Jensen-Shannon	$D_{\text{JS}}(\mathcal{P}, \mathcal{Q}) := \frac{1}{2}(D_{\text{KL}}(\mathcal{P} \parallel \mathcal{M}) + D_{\text{KL}}(\mathcal{Q} \parallel \mathcal{M}))$	$t \log t - (t+1) \log(\frac{t+1}{2})$	✓	✗
Squared Hellinger	$H^2(\mathcal{P}, \mathcal{Q}) := \frac{1}{2} \int (\sqrt{d\mathcal{P}} - \sqrt{d\mathcal{Q}})^2$	$(1 - \sqrt{t})^2/2$	✓	✗
Total Variation	$d_{\text{TV}}(\mathcal{P}, \mathcal{Q}) := \sup_E \mathcal{P}(E) - \mathcal{Q}(E) $	$ t - 1 /2$	✓	✓

demographic parity [8, 19, 40] on various datasets, so far a quantitative characterization on the exact tradeoff between accuracy and various notions of parity is still missing. In what follows we shall prove a family of information theoretic lower bounds on the accuracy that hold for *all* the methods. Due to space limit, we defer most of the proofs to appendix, while only leaving one to showcase the high-level idea of our proof technique.

3.1 Tradeoff between Accuracy and Demographic Parity

Essentially, every prediction function induces a Markov chain: $X \xrightarrow{g} Z \xrightarrow{h} \hat{Y}$, where g is the feature transformation, h is the classifier on feature space, Z is the feature and \hat{Y} is the predicted target variable by $h \circ g$. Note that simple models, e.g., linear classifiers, are also included by specifying g to be the identity map. With this notation, we first state the following theorem that quantifies an information-theoretic lower bound on the joint error across different groups:

Theorem 3.1. Let $\hat{Y} = h(g(X))$ be the predictor. If \hat{Y} satisfies demographic parity, then $\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$.

Remark First of all, $d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$ essentially measures the discrepancy of base rates across groups, and $d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$ achieves its maximum value of 1 iff $Y = A$ almost surely, i.e., Y indicates group membership. Second, Theorem 3.1 applies to all possible feature transformation g and predictor h . In particular, if we choose g to be the identity map, then Theorem 3.1 says that when the base rates differ, *no algorithm* can achieve a small joint error on both groups, and it also recovers the previous observation that demographic parity can eliminate the perfect predictor [17]. Third, the lower bound in Theorem 3.1 is insensitive to the marginal distribution of A , i.e., it treats the errors from both groups equally. As a comparison, let $\alpha := \mathcal{D}(A = 1)$, then $\text{Err}_{\mathcal{D}}(h \circ g) = (1 - \alpha)\text{Err}_{\mathcal{D}_0}(h \circ g) + \alpha\text{Err}_{\mathcal{D}_1}(h \circ g)$. In this case $\text{Err}_{\mathcal{D}}(h \circ g)$ could still be small even if the minority group suffers a large error.

Before we give the proof, we first present a useful lemma that lower bounds the prediction error by the total variation distance.

Lemma 3.1. Let $\hat{Y} = h(g(X))$ be the predictor, then for $a \in \{0, 1\}$, $d_{\text{TV}}(\mathcal{D}_a(Y), \mathcal{D}_a(\hat{Y})) \leq \text{Err}_{\mathcal{D}_a}(h \circ g)$.

Proof of Theorem 3.1. First of all, we show that if $\hat{Y} = h(g(X))$ satisfies demographic parity, then:

$$\begin{aligned} d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) &= \max \{ |\mathcal{D}_0(\hat{Y} = 0) - \mathcal{D}_1(\hat{Y} = 0)|, |\mathcal{D}_0(\hat{Y} = 1) - \mathcal{D}_1(\hat{Y} = 1)| \} \\ &= |\mathcal{D}_0(\hat{Y} = 1) - \mathcal{D}_1(\hat{Y} = 1)| \\ &= |\mathcal{D}(\hat{Y} = 1 \mid A = 0) - \mathcal{D}(\hat{Y} = 1 \mid A = 1)| = 0, \end{aligned}$$

where the last equality follows from the definition of demographic parity. Now from Table 1, $d_{\text{TV}}(\cdot, \cdot)$ is symmetric and satisfies the triangle inequality, we have:

$$\begin{aligned} d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) &\leq d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_0(\hat{Y})) + d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) + d_{\text{TV}}(\mathcal{D}_1(\hat{Y}), \mathcal{D}_1(Y)) \\ &= d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_0(\hat{Y})) + d_{\text{TV}}(\mathcal{D}_1(\hat{Y}), \mathcal{D}_1(Y)). \end{aligned} \quad (2)$$

The last step is to bound $d_{\text{TV}}(\mathcal{D}_a(Y), \mathcal{D}_a(\hat{Y}))$ in terms of $\text{Err}_{\mathcal{D}_a}(h \circ g)$ for $a \in \{0, 1\}$ using Lemma 3.1:

$$d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_0(\hat{Y})) \leq \text{Err}_{\mathcal{D}_0}(h \circ g), \quad d_{\text{TV}}(\mathcal{D}_1(Y), \mathcal{D}_1(\hat{Y})) \leq \text{Err}_{\mathcal{D}_1}(h \circ g).$$

Combining the above two inequalities and (2) completes the proof. ■

170 It is not hard to show that our lower bound in Theorem 3.1 is tight. To see this, consider the case
 171 $A = Y$, where the lower bound achieves its maximum value $d_{TV}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) = 1$. Now consider
 172 a constant predictor $\hat{Y} \equiv 1$ or $\hat{Y} \equiv 0$, which clearly satisfies demographic parity by definition. But in
 173 this case either $\text{Err}_{\mathcal{D}_0}(h \circ g) = 1$, $\text{Err}_{\mathcal{D}_1}(h \circ g) = 0$ or $\text{Err}_{\mathcal{D}_0}(h \circ g) = 0$, $\text{Err}_{\mathcal{D}_1}(h \circ g) = 1$, hence
 174 $\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \equiv 1$, achieving the lower bound.

175 To conclude this subsection, we point out that the choice of total variation in the lower bound is not
 176 unique. As we will see shortly, similar lower bounds could be attained using specific choices of the
 177 general f -divergence with some desired properties.

178 3.2 Tradeoff in Adversarial Representation Learning

179 Theorem 3.1 is an impossibility result on achieving accuracy and exact demographic parity jointly.
 180 But what if we only aim to achieve approximate demographic parity? What is the tradeoff between
 181 demographic parity and accuracy in this scenario? In fact, from the perspective of representation
 182 learning, recent work [6, 14, 29, 38] have proposed to learn intermediate feature Z through deep
 183 neural networks, aiming to maintain task-relevant information while at the same time removing
 184 sensitive information related to A . To study the tradeoff in this setting, we introduce a relaxed version
 185 of total variation, known as the \mathcal{H} -divergence [4]:

186 **Definition 3.1** (\mathcal{H} -divergence). Let \mathcal{H} be a hypothesis class on feature space \mathcal{Z} , and $\mathcal{A}_{\mathcal{H}}$ be the
 187 collection of subsets of \mathcal{Z} that are the support of some hypothesis in \mathcal{H} , i.e., $\mathcal{A}_{\mathcal{H}} := \{h^{-1}(1) \mid$
 188 $h \in \mathcal{H}\}$. The distance between two distributions \mathcal{D} and \mathcal{D}' over \mathcal{Z} based on \mathcal{H} is: $d_{\mathcal{H}}(\mathcal{D}, \mathcal{D}') :=$
 189 $\sup_{A \in \mathcal{A}_{\mathcal{H}}} |\mathcal{D}(A) - \mathcal{D}'(A)|$.

190 \mathcal{H} -divergence is particularly favorable in the analysis of adversarial representation learning with
 191 binary classification problems, and it had also been generalized to the *discrepancy distance* [10, 30]
 192 for general loss functions. When \mathcal{H} has a finite VC-dimension, \mathcal{H} -divergence can be estimated using
 193 finite unlabeled samples from \mathcal{D} and \mathcal{D}' [23]. From an algorithmic viewpoint, \mathcal{H} -divergence admits a
 194 natural interpretation that $1 - d_{\mathcal{H}}(\mathcal{D}, \mathcal{D}')$ corresponds to the minimum sum of Type-I and Type-II
 195 error in distinguishing \mathcal{D} and \mathcal{D}' . To see this, realize

$$\begin{aligned} 1 - d_{\mathcal{H}}(\mathcal{D}, \mathcal{D}') &= 1 - \sup_{A \in \mathcal{A}_{\mathcal{H}}} |\mathcal{D}(A) - \mathcal{D}'(A)| \\ &= \inf_{A \in \mathcal{A}_{\mathcal{H}}} 1 - \mathcal{D}(A) + \mathcal{D}'(A) = \inf_{h \in \mathcal{H}} \mathcal{D}(h(Z) = 0) + \mathcal{D}'(h(Z) = 1). \end{aligned} \quad (3)$$

196 The second equality holds because for $h \in \mathcal{H}$, we also have $1 - h \in \mathcal{H}$. In (3) the hypothesis h acts
 197 as a discriminator trying to distinguish between \mathcal{D} and \mathcal{D}' . The above probabilistic interpretation
 198 exactly serves as the theoretical justification of recent work on using adversarial training to learn
 199 group-invariant representation Z through transformation g such that $d_{\mathcal{H}}(\mathcal{D}_0^g, \mathcal{D}_1^g)$ is small, where \mathcal{D}_a^g
 200 is the induced distribution of \mathcal{D}_a under g . The following proposition exactly characterizes an intrinsic
 201 tradeoff of these methods:

202 **Proposition 3.1.** Let $\hat{Y} = h(g(X))$ be the predictor and $\lambda_{\mathcal{H}} := 1 - d_{\mathcal{H}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y}))$. Then
 203 $\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq d_{\mathcal{H}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) + \lambda_{\mathcal{H}} - 1$.

204 **Remark** As we show above, $\lambda_{\mathcal{H}}$ is the minimum sum of Type-I and Type-II errors in discriminating
 205 $\mathcal{D}_0(\hat{Y})$ and $\mathcal{D}_1(\hat{Y})$ using discriminators from \mathcal{H} . Hence if the optimal discriminator from \mathcal{H} fails
 206 to distinguish between $\mathcal{D}_0(\hat{Y})$ and $\mathcal{D}_1(\hat{Y})$, i.e., larger $\lambda_{\mathcal{H}}$, the lower bound on the joint error across
 207 different groups will also get larger.

208 In fact, a close scrutiny of the proof above shows that the lower bound in Proposition 3.1 holds even
 209 if different transformation functions are used on the corresponding groups:

210 **Corollary 3.1.** Let $\hat{Y} = h(g_a(X))$ be the predictors for group $A = a, a \in \{0, 1\}$ and $\lambda_{\mathcal{H}}$ as defined
 211 in Proposition 3.1. Then $\text{Err}_{\mathcal{D}_0}(h \circ g_0) + \text{Err}_{\mathcal{D}_1}(h \circ g_1) \geq d_{\mathcal{H}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) + \lambda_{\mathcal{H}} - 1$.

212 One interesting fact implied by Proposition 3.1 is that the lower bound of the joint error across groups
 213 scales linearly with $\lambda_{\mathcal{H}}$, the optimal sum of Type-I and Type-II errors in distinguishing between
 214 $\mathcal{D}_0(\hat{Y})$ and $\mathcal{D}_1(\hat{Y})$. In the work of Zhang et al. [38], the authors proposed a model (Fig. 1 in [38]) that
 215 precisely tries to maximize $\lambda_{\mathcal{H}}$ by learning the model parameters of g through adversarial techniques.
 216 In this case our lower bound directly quantifies the loss of utility due to the increase of $\lambda_{\mathcal{H}}$.

3.3 A Family of Information-Theoretic Lower Bounds

In the last subsection we show that any adversarial discriminator that tries to distinguish between $\mathcal{D}_0(\hat{Y})$ and $\mathcal{D}_1(\hat{Y})$ by taking the predicted target variable \hat{Y} as input admits an inherent lower bound in terms of joint target error. This is the algorithm proposed by Zhang et al. [38] for mitigating biases. As a comparison, most other variants [1, 6, 14, 27] build an adversarial discriminator that takes as input the feature vector $Z = g(X)$ instead. In this subsection we generalize our previous analysis with f -divergence to prove a family of lower bounds on the joint target prediction error for the latter variants. Based on our theoretical analysis, we conclude that matching the distributions from different groups within the feature space does not remove the tradeoff. In fact, a family of lower bounds also exist for these approaches.

We require one last piece of ingredient before we state and prove the main results in this section. The following lemma is proved by Liese and Vajda [25] as a generalization of the data processing inequality for f -divergence:

Lemma 3.2 (Liese and Vajda [25]). Let $\mu(\mathcal{Z})$ be the space of all probability distributions over \mathcal{Z} . Then for any f -divergence $D_f(\cdot \parallel \cdot)$, any stochastic kernel $\kappa : \mathcal{X} \rightarrow \mu(\mathcal{Z})$, and any distributions \mathcal{P} and \mathcal{Q} over \mathcal{X} , $D_f(\kappa\mathcal{P} \parallel \kappa\mathcal{Q}) \leq D_f(\mathcal{P} \parallel \mathcal{Q})$.

Roughly speaking, Lemma 3.2 says that data processing cannot increase discriminating information. Define $d_{\text{JS}}(\mathcal{P}, \mathcal{Q}) := \sqrt{D_{\text{JS}}(\mathcal{P}, \mathcal{Q})}$ and $H(\mathcal{P}, \mathcal{Q}) := \sqrt{H^2(\mathcal{P}, \mathcal{Q})}$. Both $d_{\text{JS}}(\cdot, \cdot)$ and $H(\cdot, \cdot)$ form a bounded distance metric over the space of probability distributions. Realize that $d_{\text{TV}}(\cdot, \cdot)$, $H^2(\cdot, \cdot)$ and $D_{\text{JS}}(\cdot, \cdot)$ are all f -divergence. The following corollary holds:

Corollary 3.2. Let $h : \mathcal{Z} \rightarrow \mathcal{Y}$ to any (randomized) hypothesis, and \mathcal{D}_a^g be the induced distribution of \mathcal{D}_a by g , $\forall a \in \{0, 1\}$. Let $\hat{Y} = h(g(X))$ be the predictor, then 1). $d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) \leq d_{\text{TV}}(\mathcal{D}_0^g, \mathcal{D}_1^g)$. 2). $H(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) \leq H(\mathcal{D}_0^g, \mathcal{D}_1^g)$. 3). $d_{\text{JS}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) \leq d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g)$.

Now we are ready to present the following main theorem of this subsection:

Theorem 3.2. Let $\hat{Y} = h(g(X))$ be the predictor. Assume $d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g) \leq d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$ and $H(\mathcal{D}_0^g, \mathcal{D}_1^g) \leq H(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$, then the following three inequalities hold:

1. Total variation lower bound:

$$\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) - d_{\text{TV}}(\mathcal{D}_0^g, \mathcal{D}_1^g).$$

2. Jensen-Shannon lower bound:

$$\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq (d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) - d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g))^2 / 2.$$

3. Hellinger lower bound:

$$\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq (H(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) - H(\mathcal{D}_0^g, \mathcal{D}_1^g))^2 / 2.$$

Remark All the three lower bounds in Theorem 3.2 imply a tradeoff between demographic parity and the joint error across groups through learning group-invariant feature representations. When $\mathcal{D}_0^g = \mathcal{D}_1^g$, which also implies $\mathcal{D}_0(\hat{Y}) = \mathcal{D}_1(\hat{Y})$, all three lower bounds get larger, in this case we have $\max\{d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)), \frac{1}{2}d_{\text{JS}}^2(\mathcal{D}_0(Y), \mathcal{D}_1(Y)), \frac{1}{2}H^2(\mathcal{D}_0(Y), \mathcal{D}_1(Y))\} = d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$, and this reduces to the tight lower bound in Theorem 3.1.

3.4 Group-Invariant Representation Leads to Accuracy Parity

In previous subsections we prove a family of information-theoretic lower bounds that demonstrate an inherent tradeoff between demographic parity and joint error across groups. Specifically, we show that group-invariant representation will also inevitably compromise utility. A natural question to ask then, is, what kind of parity can group-invariant representation bring us? To complement our negative results, in this subsection we show that learning group-invariant representation helps to reduce discrepancy of errors (utilities) across groups.

First of all, since we work under the stochastic setting where \mathcal{D}_a is a joint distribution over X and Y conditioned on $A = a$, then any function mapping $h : \mathcal{X} \rightarrow \mathcal{Y}$ will inevitably incur an error due to the noise existed in the distribution \mathcal{D}_a . Formally, for $a \in \{0, 1\}$, define the optimal function

261 $h_a^* : \mathcal{X} \rightarrow \mathcal{Y}$ under the absolute error to be $h_a^*(X) := m_{\mathcal{D}_a}(Y | X)$, where $m_{\mathcal{D}_a}(Y | X)$ denotes
 262 the median of Y given X under distribution \mathcal{D}_a . Now define the noise of distribution \mathcal{D}_a to be
 263 $n_{\mathcal{D}_a} := \mathbb{E}_{\mathcal{D}_a}[|Y - h_a^*(X)|]$. With these notations, we are now ready to present the following theorem:

264 **Theorem 3.3.** For any hypothesis $\mathcal{H} \ni h : \mathcal{X} \rightarrow \mathcal{Y}$, the following inequality holds:

$$|\text{Err}_{\mathcal{D}_0}(h) - \text{Err}_{\mathcal{D}_1}(h)| \leq n_{\mathcal{D}_0} + n_{\mathcal{D}_1} + d_{\text{TV}}(\mathcal{D}_0(X), \mathcal{D}_1(X)) + \min\{\mathbb{E}_{\mathcal{D}_0}[|h_0^* - h_1^*|], \mathbb{E}_{\mathcal{D}_1}[|h_0^* - h_1^*|]\}.$$

265 **Remark** Theorem 3.3 upper bounds the discrepancy of accuracy across groups by three terms: the
 266 noise, the distance of representation across groups and the discrepancy of optimal decision functions.
 267 In an ideal setting where both distributions are noiseless, i.e., same people in the same group are
 268 always treated equally, the upper bound simplifies to the latter two terms. If we further require that
 269 the optimal decision functions h_0^* and h_1^* are close to each other, i.e., optimal decisions are insensitive
 270 to the group membership, then Theorem 3.3 implies that a sufficient condition to guarantee accuracy
 271 parity is to find group-invariant representation that minimizes $d_{\text{TV}}(\mathcal{D}_0(X), \mathcal{D}_1(X))$.
 272
 273

274 4 Experiments

275 Our theoretical results on the lower bound between demographic parity and the sum of joint error
 276 across groups imply that over-training the feature transformation function to achieve group-invariant
 277 representation will lead to large joint errors. On the other hand, our upper bound also implies that
 278 group-invariant representation help to achieve accuracy parity. To verify these theoretical implications,
 279 in this section we conduct experiments on a real-world benchmark dataset, the UCI Adult dataset², to
 280 present empirical results with various metrics.

281 **Dataset** The Adult dataset contains 30,162/15,060 training/test instances for income prediction.
 282 Each instance in the dataset describes an adult from the 1994 US Census. Attributes include gender,
 283 education level, age, etc. In this experiment we use gender (binary) as the sensitive attribute, and we
 284 preprocess the dataset to convert categorical variables into one-hot representations. The processed
 285 data contains 114 attributes. The target variable (income) is also binary: 1 if $\geq 50\text{K}/\text{year}$ otherwise 0.
 286 For the sensitive attribute A , $A = 0$ means Male otherwise Female. In this dataset, the base rates
 287 across groups are different: $\Pr(Y = 1 | A = 0) = 0.310$ while $\Pr(Y = 1 | A = 1) = 0.113$. Also,
 288 the group ratios are different: $\Pr(A = 0) = 0.673$.

289 **Network Architecture** To validate the effect of learning group-invariant representation with adver-
 290 sarial debiasing techniques [6, 29, 38], we perform a controlled experiment by fixing the baseline
 291 network architecture to be a three hidden-layer feed-forward network with ReLU activations. The
 292 number of units in each hidden layer are 500, 200, and 100, respectively. The output layer corresponds
 293 to a logistic regression model. This baseline without debiasing is denoted as NoDebias. For debiasing
 294 with adversarial learning techniques, the adversarial discriminator network takes the feature from
 295 the last hidden layer as input, and connects it to a hidden-layer with 50 units, followed by a binary
 296 classifier whose goal is to predict the sensitive attribute A . This model is denoted as AdvDebias.
 297 Compared with NoDebias, the only difference of AdvDebias in terms of objective function is that
 298 besides the cross-entropy loss for target prediction, the AdvDebias also contains a classification loss
 299 from the adversarial discriminator to predict the sensitive attribute A . In the experiment, all the
 300 other factors are fixed to be the same for these two methods, including learning rate, optimization
 301 algorithm, training epoch, and also batch size. To see how the adversarial loss affects the joint error,
 302 the demographic parity as well as the accuracy parity, we vary the coefficient ρ for the adversarial
 303 loss between 0.1, 1.0 and 5.0.

304 The experimental results are listed in Table 2. Note that in the table $|\text{Err}_{\mathcal{D}_0} - \text{Err}_{\mathcal{D}_1}|$ could be un-
 305 derstood as measuring an approximate version of accuracy parity, and similarly $d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y}))$
 306 measures the closeness of the prediction function to demographic parity. From the table, it is then
 307 clear that with increasing ρ , both the overall error $\text{Err}_{\mathcal{D}}$ (sensitive to the marginal distribution of A)
 308 and the joint error $\text{Err}_{\mathcal{D}_0} + \text{Err}_{\mathcal{D}_1}$ (insensitive to the imbalance of A) are increasing. As expected,
 309 $d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y}))$ is drastically decreasing with the increasing of ρ . Furthermore, $|\text{Err}_{\mathcal{D}_0} - \text{Err}_{\mathcal{D}_1}|$
 310 is also gradually decreasing, but much slowly than $d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y}))$. This is due to the existing
 311 noise in the data as well as the shift between the optimal decision functions across groups, as indicated
 312 by our upper bound in Theorem 3.3. To conclude, all the empirical results are consistent with our
 313 theoretical findings.

²<https://archive.ics.uci.edu/ml/datasets/adult>

Table 2: Adversarial debiasing on demographic parity, joint error across groups, and accuracy parity.

	$\text{Err}_{\mathcal{D}}$	$\text{Err}_{\mathcal{D}_0} + \text{Err}_{\mathcal{D}_1}$	$ \text{Err}_{\mathcal{D}_0} - \text{Err}_{\mathcal{D}_1} $	$d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y}))$
NoDebias	0.157	0.275	0.115	0.189
AdvDebias, $\rho = 0.1$	0.159	0.278	0.116	0.190
AdvDebias, $\rho = 1.0$	0.162	0.286	0.106	0.113
AdvDebias, $\rho = 5.0$	0.166	0.295	0.106	0.032

5 Related Work

Fairness Frameworks There is a broad literature on fairness, notably in social choice theory, ethics, economics and machine learning. Two central notions of fairness have been extensively studied, i.e., group fairness and individual fairness. In a seminal work, Dwork et al. [13] define individual fairness as a measure of smoothness of the classification function. Under the assumption that number of individuals is finite, the authors proposed a linear programming framework to maximize the utility under their fairness constraint. However, their framework requires apriori a distance function that computes the similarity between individuals, and their optimization formulation does not produce an inductive rule to generalize to unseen data. Based on the definition of positive rate parity, Hardt et al. [17] proposed a post-processing method to achieve fairness by taking as input the prediction and the sensitive attribute. In a concurrent work, Kleinberg et al. [24] offer a calibration technique to achieve fairness as well. However, both of the aforementioned two approaches require sensitive attribute during the inference phase, which is not available in many real-world scenarios.

Regularization Techniques The line of work on fairness-aware learning through regularization dates at least back to Kamishima et al. [21], where the authors argue that simple deletion of sensitive features in data is insufficient for eliminating biases in automated decision making, due to the possible correlations among attributes and sensitive information [28]. In light of this, the authors proposed a *prejudice remover* regularizer that essentially penalizes the mutual information between the predicted goal and the sensitive information. In a more recent approach, Zafar et al. [35] leveraged a measure of decision boundary fairness and incorporated it via constraints into the objective function of logistic regression as well as support vector machines. As discussed in Section 2, both approaches essentially reduce to achieving demographic parity through regularization.

Representation Learning In a pioneer work, Zemel et al. [37] proposed to preserve both group and individual fairness through the lens of representation learning, where the main idea is to find a good representation of the data with two competing goals: to encode the data for utility maximization while at the same time to obfuscate any information about membership in the protected group. Due to the power of learning rich representations offered by deep neural nets, recent advances in building fair automated decision making systems focus on using adversarial techniques to learn fair representation that also preserves enough information for the prediction vendor to achieve his utility [1, 6, 14, 27, 34, 38]. Madras et al. [29] further extended this approach by incorporating reconstruction loss given by an autoencoder into the objective function to preserve demographic parity, equalized odds, and equal opportunity.

6 Conclusion

In this paper we theoretically and empirically study the important problem of quantifying the tradeoff between utility and fairness in learning group-invariant representations. Specifically, we prove a novel lower bound to characterize the tradeoff between demographic parity and the joint utility across different population groups when the base rates differ between groups. In particular, our results imply that any method aiming to learn fair representation admits an information-theoretic lower bound on the joint error, and the better the representation, the larger the joint error. Complementary to our negative results, we also show that learning fair representation leads to accuracy parity if the optimal decision functions across different groups are close. These theoretical findings are also confirmed empirically on real-world datasets. We believe our results take an important step towards better understanding the tradeoff between utility and different notions of fairness. Inspired by our lower bound, one interesting direction for future work is to design instance-weighting algorithm to balance the base rates during representation learning.

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A Proofs

Lemma 3.1. Let $\hat{Y} = h(g(X))$ be the predictor, then for $a \in \{0, 1\}$, $d_{\text{TV}}(\mathcal{D}_a(Y), \mathcal{D}_a(\hat{Y})) \leq \text{Err}_{\mathcal{D}_a}(h \circ g)$.

Proof. For $a \in \{0, 1\}$, we have:

$$\begin{aligned} d_{\text{TV}}(\mathcal{D}_a(Y), \mathcal{D}_a(\hat{Y})) &= |\mathcal{D}_a(Y = 1) - \mathcal{D}_a(h(g(X)) = 1)| = |\mathbb{E}_{\mathcal{D}_a}[Y] - \mathbb{E}_{\mathcal{D}_a}[h(g(X))]| \\ &\leq \mathbb{E}_{\mathcal{D}_a}[|Y - h(g(X))|] = \text{Err}_{\mathcal{D}_a}(h \circ g). \end{aligned} \quad (4)$$

Proposition 3.1. Let $\hat{Y} = h(g(X))$ be the predictor and $\lambda_{\mathcal{H}} := 1 - d_{\mathcal{H}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y}))$. Then $\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq d_{\mathcal{H}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) + \lambda_{\mathcal{H}} - 1$.

Proof. First, it is easy to show that $\forall \mathcal{H}$ and any distributions $\mathcal{D}, \mathcal{D}'$ and \mathcal{D}'' , the following triangle inequality holds:

$$\begin{aligned} d_{\mathcal{H}}(\mathcal{D}, \mathcal{D}') &= \sup_{A \in \mathcal{A}_{\mathcal{H}}} |\mathcal{D}(A) - \mathcal{D}'(A)| \leq \sup_{A \in \mathcal{A}_{\mathcal{H}}} |\mathcal{D}(A) - \mathcal{D}''(A)| + |\mathcal{D}''(A) - \mathcal{D}'(A)| \\ &\leq \sup_{A \in \mathcal{A}_{\mathcal{H}}} |\mathcal{D}(A) - \mathcal{D}''(A)| + \sup_{A \in \mathcal{A}_{\mathcal{H}}} |\mathcal{D}''(A) - \mathcal{D}'(A)| = d_{\mathcal{H}}(\mathcal{D}, \mathcal{D}'') + d_{\mathcal{H}}(\mathcal{D}'', \mathcal{D}'). \end{aligned}$$

Again, we apply a chain of triangle inequalities as we did in the proof of Theorem 3.1:

$$\begin{aligned} d_{\mathcal{H}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) &\leq d_{\mathcal{H}}(\mathcal{D}_0(Y), \mathcal{D}_0(\hat{Y})) + d_{\mathcal{H}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) + d_{\mathcal{H}}(\mathcal{D}_1(\hat{Y}), \mathcal{D}_1(Y)) \\ &\leq d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_0(\hat{Y})) + d_{\text{TV}}(\mathcal{D}_1(\hat{Y}), \mathcal{D}_1(Y)) + d_{\mathcal{H}}(\mathcal{D}_1(\hat{Y}), \mathcal{D}_1(Y)) \\ &\leq \text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) + 1 - \lambda_{\mathcal{H}}. \end{aligned}$$

The second inequality follows from the definition of \mathcal{H} -divergence and the last one is due to (4). Arranging the terms on both sides completes the proof. \blacksquare

Theorem 3.2. Let $\hat{Y} = h(g(X))$ be the predictor. Assume $d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g) \leq d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$ and $H(\mathcal{D}_0^g, \mathcal{D}_1^g) \leq H(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$, then the following three inequalities hold:

1. Total variation lower bound:

$$\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq d_{\text{TV}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) - d_{\text{TV}}(\mathcal{D}_0^g, \mathcal{D}_1^g).$$

2. Jensen-Shannon lower bound:

$$\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq (d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) - d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g))^2 / 2.$$

3. Hellinger lower bound:

$$\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g) \geq (H(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) - H(\mathcal{D}_0^g, \mathcal{D}_1^g))^2 / 2.$$

Proof. We prove the three inequalities respectively. The total variation lower bound follows the same idea as the proof of Theorem 3.1 and the inequality $d_{\text{TV}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) \leq d_{\text{TV}}(\mathcal{D}_0^g, \mathcal{D}_1^g)$ from Corollary 3.2. To prove the Jensen-Shannon lower bound, realize that $d_{\text{JS}}(\cdot, \cdot)$ is a distance metric over probability distributions. Combining with the inequality $d_{\text{JS}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) \leq d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g)$ from Corollary 3.2, we have:

$$d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) \leq d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_0(\hat{Y})) + d_{\text{JS}}(\mathcal{D}_0(\hat{Y}), \mathcal{D}_1(\hat{Y})) + d_{\text{JS}}(\mathcal{D}_1(\hat{Y}), \mathcal{D}_1(Y)).$$

Now by Lin's lemma [26, Theorem 3], for any two distributions \mathcal{P} and \mathcal{Q} , we have $d_{\text{JS}}^2(\mathcal{P}, \mathcal{Q}) \leq d_{\text{TV}}(\mathcal{P}, \mathcal{Q})$. Combine Lin's lemma with Lemma 3.1, we get the following lower bound:

$$\sqrt{\text{Err}_{\mathcal{D}_0}(h \circ g)} + \sqrt{\text{Err}_{\mathcal{D}_1}(h \circ g)} \geq d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y)) - d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g).$$

Apply a simple AM-GM inequality, we can further bound the L.H.S. by

$$\sqrt{2(\text{Err}_{\mathcal{D}_0}(h \circ g) + \text{Err}_{\mathcal{D}_1}(h \circ g))} \geq \sqrt{\text{Err}_{\mathcal{D}_0}(h \circ g)} + \sqrt{\text{Err}_{\mathcal{D}_1}(h \circ g)}.$$

Under the assumption that $d_{\text{JS}}(\mathcal{D}_0^g, \mathcal{D}_1^g) \leq d_{\text{JS}}(\mathcal{D}_0(Y), \mathcal{D}_1(Y))$, taking the square at both sides then completes the proof for the second inequality. The proof for Hellinger's lower bound follows exactly as the one for Jensen-Shannon's lower bound, except that we need to use $H^2(\mathcal{P}, \mathcal{Q}) \leq d_{\text{TV}}(\mathcal{P}, \mathcal{Q}) \leq \sqrt{2}H(\mathcal{P}, \mathcal{Q})$, $\forall \mathcal{P}, \mathcal{Q}$, instead of Lin's lemma. \blacksquare

488 **Theorem 3.3.** For any hypothesis $\mathcal{H} \ni h : \mathcal{X} \rightarrow \mathcal{Y}$, the following inequality holds:

$$|\text{Err}_{\mathcal{D}_0}(h) - \text{Err}_{\mathcal{D}_1}(h)| \leq n_{\mathcal{D}_0} + n_{\mathcal{D}_1} + d_{\text{TV}}(\mathcal{D}_0(X), \mathcal{D}_1(X)) + \min\{\mathbb{E}_{\mathcal{D}_0}[|h_0^* - h_1^*|], \mathbb{E}_{\mathcal{D}_1}[|h_0^* - h_1^*|]\}.$$

489 *Proof.* First, we show that for $a \in \{0, 1\}$, $\text{Err}_{\mathcal{D}_a}(h)$ cannot be too large if h is close to h_a^* :

$$\begin{aligned} |\text{Err}_{\mathcal{D}_a}(h) - n_{\mathcal{D}_a}| &= |\text{Err}_{\mathcal{D}_a}(h) - \text{Err}_{\mathcal{D}_a}(h_a^*)| = |\mathbb{E}_{\mathcal{D}_a}[|Y - h(X)|] - \mathbb{E}_{\mathcal{D}_a}[|Y - h_a^*(X)|]| \\ &\leq \mathbb{E}_{\mathcal{D}_a}[|h(X) - h_a^*(X)|]. \end{aligned}$$

491 Next, we bound $|\text{Err}_{\mathcal{D}_0}(h) - \text{Err}_{\mathcal{D}_1}(h)|$ by:

$$|\text{Err}_{\mathcal{D}_0}(h) - \text{Err}_{\mathcal{D}_1}(h)| \leq n_{\mathcal{D}_0} + n_{\mathcal{D}_1} + |\mathbb{E}_{\mathcal{D}_0}[|h(X) - h_0^*(X)|] - \mathbb{E}_{\mathcal{D}_1}[|h(X) - h_1^*(X)|]|.$$

492 To simplify the notation, define $\varepsilon_a(h, h') := \mathbb{E}_{\mathcal{D}_a}[|h(X) - h'(X)|]$ so that

$$|\mathbb{E}_{\mathcal{D}_0}[|h(X) - h_0^*(X)|] - \mathbb{E}_{\mathcal{D}_1}[|h(X) - h_1^*(X)|]| = |\varepsilon_0(h, h_0^*) - \varepsilon_1(h, h_1^*)|.$$

493 To bound $|\varepsilon_0(h, h_0^*) - \varepsilon_1(h, h_1^*)|$, realize that $|h(X) - h_a^*(X)| \in \{0, 1\}$. On one hand, we have:

$$\begin{aligned} |\varepsilon_0(h, h_0^*) - \varepsilon_1(h, h_1^*)| &= |\varepsilon_0(h, h_0^*) - \varepsilon_0(h, h_1^*) + \varepsilon_0(h, h_1^*) - \varepsilon_1(h, h_1^*)| \\ &\leq |\varepsilon_0(h, h_0^*) - \varepsilon_0(h, h_1^*)| + |\varepsilon_0(h, h_1^*) - \varepsilon_1(h, h_1^*)| \\ &\leq \varepsilon_0(h_0^*, h_1^*) + d_{\text{TV}}(\mathcal{D}_0(X), \mathcal{D}_1(X)), \end{aligned}$$

494 where the last inequality is due to $|\varepsilon_0(h, h_1^*) - \varepsilon_1(h, h_1^*)| = |\mathcal{D}_0(|h - h_1^*| = 1) - \mathcal{D}_1(|h - h_1^*| = 1)| \leq$
495 $\sup_E |\mathcal{D}_0(E) - \mathcal{D}_1(E)| = d_{\text{TV}}(\mathcal{D}_0, \mathcal{D}_1)$. Similarly, by subtracting and adding back $\varepsilon_1(h, h_0^*)$ instead,
496 we can also show that $|\varepsilon_0(h, h_0^*) - \varepsilon_1(h, h_1^*)| \leq \varepsilon_1(h_0^*, h_1^*) + d_{\text{TV}}(\mathcal{D}_0(X), \mathcal{D}_1(X))$. Combining all
497 the inequalities above finishes the proof. ■