

A Proofs

Proof of Lemma 1. We wish to prove $\mathbb{E} [q(X_i^{\text{Unc}})Y_i^{\text{Unc}}|X_i^{\text{Unc}}, E^{\text{Unc}}] = \tau(X_i^{\text{Unc}})$. We have that:

$$\mathbb{E} [q(X_i^{\text{Unc}})Y_i^{\text{Unc}}|X_i^{\text{Unc}}, E^{\text{Unc}}] = \mathbb{E} [q(X_i^{\text{Unc}})Y_i^{\text{Unc}}|X_i^{\text{Unc}}, E^{\text{Unc}}, T = 1] e^{\text{Unc}}(X_i^{\text{Unc}}) \quad (3)$$

$$+ \mathbb{E} [q(X_i^{\text{Unc}})Y_i^{\text{Unc}}|X_i^{\text{Unc}}, E^{\text{Unc}}, T = 0] (1 - e^{\text{Unc}}(X_i^{\text{Unc}})) \quad (4)$$

$$= \mathbb{E} [Y_i^{\text{Unc}}|X_i^{\text{Unc}}, E^{\text{Unc}}, T = 1] - \mathbb{E} [Y_i^{\text{Unc}}|X_i^{\text{Unc}}, E^{\text{Unc}}, T = 0] \quad (5)$$

$$\stackrel{(a)}{=} \mathbb{E} [Y(1)|X_i^{\text{Unc}}] - \mathbb{E} [Y(0)|X_i^{\text{Unc}}] = \tau(X_i^{\text{Unc}}),$$

where equality (a) is by Assumption 1 and the definition of Y . □

Proof of Thm. 1. Let $\mathbf{X}^{\text{Unc}} = (X_1^{\text{Unc}}, \dots, X_n^{\text{Unc}})$ be the design matrix in the experimental data and let $u_i^{\text{Unc}} = \left(\frac{T_i^{\text{Unc}}}{e^{\text{Unc}}(X_i^{\text{Unc}})} - \frac{1 - T_i^{\text{Unc}}}{1 - e^{\text{Unc}}(X_i^{\text{Unc}})} \right) Y_i^{\text{Unc}} - \hat{\omega}(X_i^{\text{Unc}})$ be the regression outcome and $\mathbf{u} = (u_1^{\text{Unc}}, \dots, u_n^{\text{Unc}})$ so that $\hat{\theta} = (\mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top})^{-1}(\mathbf{X}^{\text{Unc}})^{\top} \mathbf{u}^{\text{Unc}}$. Let $a_i^{\text{Unc}} = \omega(X_i^{\text{Unc}}) - \hat{\omega}(X_i^{\text{Unc}})$ and $b_i^{\text{Unc}} = \left(\frac{T_i^{\text{Unc}}}{e^{\text{Unc}}(X_i^{\text{Unc}})} - \frac{1 - T_i^{\text{Unc}}}{1 - e^{\text{Unc}}(X_i^{\text{Unc}})} \right) Y_i^{\text{Unc}} - \tau(X_i^{\text{Unc}})$. Note that $u_i^{\text{Unc}} = \tau(X_i^{\text{Unc}}) - \omega(X_i^{\text{Unc}}) + b_i^{\text{Unc}} - a_i^{\text{Unc}}$, which by condition 3 we can write as $u_i^{\text{Unc}} = \theta^{\top} X_i^{\text{Unc}} + b_i^{\text{Unc}} - a_i^{\text{Unc}}$. Hence, we have

$$\begin{aligned} \hat{\theta} - \theta &= (\mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top})^{-1}(\mathbf{X}^{\text{Unc}})^{\top} \mathbf{u} - \theta \\ &= \left(\left(\frac{1}{n^{\text{Unc}}} \mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top} \right)^{-1} \left(\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{X}^{\text{Unc}} - I \right) \right) \\ &\quad - \left(\frac{1}{n^{\text{Unc}}} \mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top} \right)^{-1} \left(\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{a}^{\text{Unc}} \right) \\ &\quad + \left(\frac{1}{n^{\text{Unc}}} \mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top} \right)^{-1} \left(\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{b}^{\text{Unc}} \right) \end{aligned}$$

Let $M = \mathbb{E}[X X^{\top} | E^{\text{Unc}}]$. By condition 5, we have that $\left\| \frac{1}{n^{\text{Unc}}} \mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top} - M \right\|_F^2 = O_p(1/n)$.

By condition 4, $\left\| \left(\frac{1}{n^{\text{Unc}}} \mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top} \right)^{-1} \left(\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{X} \right) - I \right\|_F^2 = O_p(1/n)$.

Next, consider the second term:

$$\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{a}^{\text{Unc}} = \frac{1}{n^{\text{Unc}}} \sum_{i=1}^n (\omega(X_i^{\text{Unc}}) - \hat{\omega}(X_i^{\text{Unc}})) X_i^{\text{Unc}}$$

By Cauchy-Schwartz and condition 5,

$$\mathbb{E}[\|(\omega(X_i^{\text{Unc}}) - \hat{\omega}(X_i^{\text{Unc}})) X_i^{\text{Unc}}\|_2^2] \leq \mathbb{E}[(\omega(X_i^{\text{Unc}}) - \hat{\omega}(X_i^{\text{Unc}}))^4] \mathbb{E}[\|X_i^{\text{Unc}}\|_2^4] < \infty.$$

And again by Cauchy-Schwartz,

$$\|\mathbb{E}[(\omega(X_i^{\text{Unc}}) - \hat{\omega}(X_i^{\text{Unc}})) X_i^{\text{Unc}}]\|_2^2 \leq \mathbb{E}[(\omega(X) - \hat{\omega}(X))^2 | E^{\text{Unc}}] \mathbb{E}[\|X\|_2^2 | E^{\text{Unc}}].$$

Conditions 1 and 2 give that $\mathbb{E}[(\hat{\omega}(X) - \omega(X))^2 | E^{\text{Unc}}] = O(r(n))$. Hence, by above finiteness of second moment,

$$\left\| \left(\frac{1}{n^{\text{Unc}}} \mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top} \right)^{-1} \left(\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{a}^{\text{Unc}} \right) \right\|_2^2 = O_p(r(n) + 1/n).$$

Finally, consider the third term:

$$\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{b}^{\text{Unc}} = \frac{1}{n^{\text{Unc}}} \sum_{i=1}^n b_i^{\text{Unc}} X_i^{\text{Unc}}$$

First note that $\mathbb{E}[b_i^{\text{Unc}} | X_i^{\text{Unc}}] = 0$ by the outcome weighting formula and hence $\mathbb{E}[b_i^{\text{Unc}} X_i^{\text{Unc}}] = 0$.

By Cauchy-Schwartz and conditions 5 and 6, we have

$$\mathbb{E}[\|b_i^{\text{Unc}} X_i^{\text{Unc}}\|_2^2] \leq \mathbb{E}[(b_i^{\text{Unc}})^4] \mathbb{E}[\|X_i^{\text{Unc}}\|_2^4] < \infty$$

and hence

$$\left\| \left(\frac{1}{n^{\text{Unc}}} \mathbf{X}^{\text{Unc}}(\mathbf{X}^{\text{Unc}})^{\top} \right)^{-1} \left(\frac{1}{n^{\text{Unc}}} (\mathbf{X}^{\text{Unc}})^{\top} \mathbf{b}^{\text{Unc}} \right) \right\|_2^2 = O_p(1/n).$$

We conclude that $\|\hat{\theta} - \theta_0\|_2^2 = O_p(r(n) + 1/n)$. Since condition 1 implies that $((\hat{\omega}(X) - \omega(X))^2 | E^{\text{Conf}}) = O_p(r(n))$ by Markov's theorem, we also conclude that $((\hat{\tau}(X) - \tau(X))^2 | E^{\text{Conf}}) = O_p(r(n) + 1/n)$. □