
Online to Offline Conversions, Universality and Adaptive Minibatch Sizes

(Including supplementary material)

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Abstract

We present an approach towards convex optimization that relies on a novel scheme which converts adaptive online algorithms into offline methods. In the offline optimization setting, our derived methods are shown to obtain favourable adaptive guarantees which depend on the *harmonic sum* of the queried gradients. We further show that our methods implicitly adapt to the objective’s structure: in the smooth case fast convergence rates are ensured without any prior knowledge of the smoothness parameter, while still maintaining guarantees in the non-smooth setting. Our approach has a natural extension to the stochastic setting, resulting in a lazy version of SGD (stochastic GD), where minibatches are chosen *adaptively* depending on the magnitude of the gradients. Thus providing a principled approach towards choosing minibatch sizes.

1 Introduction

Over the past years *data adaptiveness* has proven to be crucial to the success of learning algorithms. The objective function underlying “big data” applications often demonstrates intricate structure: the scale and smoothness are often unknown and may change substantially in between different regions/directions, [1]. Learning methods that acclimatize to these changes may exhibit superior performance compared to non adaptive procedures.

State-of-the-art first order methods like AdaGrad, [1], and Adam, [2], adapt the learning rate on the fly according to the feedback (i.e. gradients) received during the optimization process. AdaGrad and Adam are guaranteed to work well in the *online* convex optimization setting, where loss functions may be chosen *adversarially* and change between rounds. Nevertheless, this setting is harder than the stochastic/offline settings, which may better depict practical applications. Interestingly, even in the offline convex optimization setting it could be shown that in several scenarios very simple schemes may substantially outperform the output of AdaGrad/Adam. An example of such a simple scheme is choosing the point with the smallest gradient norm among all rounds. In the first part of this work we address this issue and design adaptive methods for the offline convex optimization setting. At heart of our derivations is a novel scheme which converts adaptive online algorithms into offline methods with favourable guarantees¹. Our scheme is inspired by standard online to batch conversions, [3].

A seemingly different issue is choosing the minibatch size, b , in the stochastic setting. Stochastic optimization algorithms that can access a noisy gradient oracle may choose to invoke the oracle b times in every query point, subsequently employing an averaged gradient estimate. Theory for stochastic convex optimization suggests to use a minibatch of $b = 1$, and predicts a degradation of \sqrt{b}

¹For concreteness we concentrate in this work on converting AdaGrad, [1]. Note that our conversion scheme applies more widely to other adaptive online methods.

factor upon using larger minibatch sizes². Nevertheless in practice larger minibatch sizes are usually found to be effective. In the second part of this work we design stochastic optimization methods in which minibatch sizes are chosen *adaptively* without any theoretical degradation. These are natural extensions of the offline methods presented in the first part.

Our contributions:

Offline setting: We present two (families of) algorithms AdaNGD (Alg. 2) and SC-AdaNGD (Alg. 3) for the convex/strongly-convex settings which achieve favourable adaptive guarantees (Thms. 2.1, 2.2, 3.1, 3.2). The latter theorems also establish their universality, i.e., their ability to implicitly take advantage of the objective’s smoothness and attain rates as fast as GD would have achieved if the smoothness parameter was known. In contrast to other universal approaches such as line-search-GD, [4], and universal gradient [5], we do so *without* any line search procedure.

Concretely, without the knowledge of the smoothness parameter our algorithm ensures an $O(1/\sqrt{T})$ rate in general convex case and an $O(1/T)$ rate if the objective is also smooth (Thms. 2.1, 2.2). In the strongly-convex case our algorithm ensures an $O(1/T)$ rate in general and an $O(\exp(-\gamma T))$ rate if the objective is also smooth (Thm. 3.2), where γ is the condition number.

Stochastic setting: We present Lazy-SGD (Algorithm 4) which is an extension of our offline algorithms. Lazy-SGD employs larger minibatch sizes in points with smaller gradients, which selectively reduces the variance in the “more important” query points. Lazy-SGD guarantees are comparable with SGD in the convex/strongly-convex settings (Thms. 4.2, 4.3).

On the technical side, our online to offline conversion schemes employ three simultaneous mechanisms: an adaptive online algorithm used in conjunction with gradient normalization and with a respective importance weighting. To the best of our knowledge the combination of the above techniques is novel, and we believe it might also find use in other scenarios.

This paper is organized as follows. In Sections 2,3, we present our methods for the offline convex/strongly-convex settings. Section 4 describes our methods for the stochastic setting, and Section 5 concludes. Extensions and a preliminary experimental study appear in the Appendix.

1.1 Related Work

The authors of [1] simultaneously to [6], were the first to suggest AdaGrad—an adaptive gradient based method, and prove its efficiency in tackling online convex problems. AdaGrad was subsequently adjusted to the deep-learning setting to yield the RMSprop, [7], and Adadelta, [8], heuristics. Adam, [2], is a popular adaptive algorithm which is often the method of choice in deep-learning applications. It combines ideas from AdaGrad together with momentum machinery, [9].

An optimization procedure is called universal if it implicitly adapts to the objective’s smoothness. In [5], universal gradient methods are devised for the general convex setting. Concretely, without the knowledge of the smoothness parameter, these methods attain the standard $O(1/T)$, an accelerated $O(1/T^2)$ rates for smooth objectives, and an $O(1/\sqrt{T})$ rate in the non-smooth case. The core technique in this work is a line search procedure which estimates the smoothness parameter in every iteration. For strongly-convex and smooth objectives, line search techniques, [4], ensure linear convergence rate, *without* the knowledge of the smoothness parameter. However, line search is not “fully universal”, in the sense that it holds no guarantees in the non-smooth case. For the latter setting we present a method which is “fully universal” (Thm. 3.2), nevertheless it *requires* the strong-convexity parameter.

The usefulness of employing normalized gradients was demonstrated in several non-convex scenarios. In the context of quasi-convex optimization, [10], and [11], established convergence guarantees for the offline/stochastic settings. More recently, it was shown in [12], that normalized gradient descent is more appropriate than GD for saddle-evasion scenarios.

In the context of stochastic optimization, the effect of minibatch size was extensively investigated throughout the past years, [13, 14, 15, 16, 17, 18]. Yet, all of these studies: (i) assume a smooth expected loss, (ii) discuss fixed minibatch sizes. Conversely, our work discusses adaptive minibatch sizes, and applies to both smooth/non-smooth expected losses.

²A degradation by a \sqrt{b} factor in the general case and by a b factor in the strongly-convex case.

Algorithm 2 Adaptive Normalized Gradient Descent (AdaNGD_k)

Input: #Iterations T , $x_1 \in \mathbb{R}^d$, set \mathcal{K} , parameter k
Set: $Q_0 = 0$
for $t = 1 \dots T - 1$ **do**
 Calculate: $g_t = \nabla f(x_t)$, $\hat{g}_t = g_t / \|g_t\|^k$
 Update:

$$Q_t = Q_{t-1} + 1 / \|g_t\|^{2(k-1)}$$

 Set $\eta_t = D / \sqrt{2Q_t}$
 Update: $x_{t+1} = \Pi_{\mathcal{K}}(x_t - \eta_t \hat{g}_t)$
end for
Return: $\bar{x}_T = \sum_{t=1}^T \frac{1 / \|g_t\|^k}{\sum_{\tau=1}^T 1 / \|g_\tau\|^k} x_t$

Proof sketch. Notice that the AdaNGD_k algorithm is equivalent to applying AdaGrad to the following loss sequence: $\{\tilde{f}_t(x) := g_t^\top x / \|g_t\|^k\}_{t=1}^T$. Thus, applying Theorem 1.1, and using the definition of \bar{x}_T together with Jensen's inequality the lemma follows. \square

For $k = 0$, Algorithm 2 becomes AdaGrad (Alg. 1). Next we focus on the cases where $k = 1, 2$, showing improved adaptive rates and universality compared to GD/AdaGrad. These improved rates are attained thanks to the adaptivity of the learning rate: when query points with small gradients are encountered, AdaNGD_k (with $k \geq 1$) reduces the learning rate, thus focusing on the region around these points. The hindsight weighting further emphasizes points with smaller gradients.

2.1 AdaNGD₁

Here we show that AdaNGD₁ enjoys a rate of $O(1/\sqrt{T})$ in the non-smooth convex setting, and a fast rate of $O(1/T)$ in the smooth setting. We emphasize that the same algorithm enjoys these rates simultaneously, without any prior knowledge of the smoothness or of the gradient norms.

From Algorithm 2 it can be noted that for $k = 1$ the learning rate becomes independent of the gradients, i.e. $\eta_t = D/\sqrt{2t}$, the update is made according to the direction of the gradients, and the weighting is inversely proportional to the norm of the gradients. The following Theorem establishes the guarantees of AdaNGD₁,

Theorem 2.1. *Let $k = 1$, \mathcal{K} be a convex set with diameter D , and f be a convex function; Also let \bar{x}_T be the outputs of AdaNGD₁ (Alg. 2), then the following holds:*

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{\sqrt{2D^2T}}{\sum_{t=1}^T 1/\|g_t\|} \leq \frac{\sqrt{2GD}}{\sqrt{T}}.$$

Moreover, if f is also β -smooth and the global minimum $x^ = \arg \min_{x \in \mathbb{R}^n} f(x)$ belongs to \mathcal{K} , then:*

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{D\sqrt{T}}{\sum_{t=1}^T 1/\|g_t\|} \leq \frac{4\beta D^2}{T}.$$

Proof sketch. The data dependent bound is a direct corollary of Lemma 2.1. The general case bound holds by using $\|g_t\| \leq G$. The bound for the smooth case is proven by showing $\sum_{t=1}^T \|g_t\| \leq O(\sqrt{T})$. This translates to a lower bound $\sum_{t=1}^T 1/\|g_t\| \geq \Omega(T^{3/2})$, which concludes the proof. \square

The data dependent bound in Theorem 2.1 may be substantially better compared to the bound of the GD/AdaGrad. As an example, assume that half of the gradients encountered during the run of the algorithm are of $O(1)$ norms, and the other gradient norms decay proportionally to $O(1/t)$. In this case the guarantee of GD/AdaGrad is $O(1/\sqrt{T})$, whereas AdaNGD₁ guarantees a bound that behaves like $O(1/T^{3/2})$. Note that the above example presumes that all algorithms encounter the same gradient magnitudes, which might be untrue. Nevertheless in the smooth case AdaNGD₁ provably benefits due to its adaptivity.

Algorithm 3 Strongly-Convex AdaNGD (SC-AdaNGD_k)

Input: #Iterations T , $x_1 \in \mathbb{R}^d$, set \mathcal{K} , strong-convexity H , parameter k
Set: $Q_0 = 0$
for $t = 1 \dots T - 1$ **do**
 Calculate: $g_t = \nabla f(x_t)$, $\hat{g}_t = g_t / \|g_t\|^k$
 Update:

$$Q_t = Q_{t-1} + 1 / \|g_t\|^k$$

 Set $\eta_t = 1 / HQ_t$
 Update: $x_{t+1} = \Pi_{\mathcal{K}}(x_t - \eta_t \hat{g}_t)$
end for
Return: $\bar{x}_T = \sum_{t=1}^T \frac{1 / \|g_t\|^k}{\sum_{\tau=1}^T 1 / \|g_\tau\|^k} x_t$

2.2 AdaNGD₂

Here we show that AdaNGD₂ enjoys comparable guarantees to AdaNGD₁ in the general/smooth case. Similarly to AdaNGD₁ the same algorithm enjoys these rates simultaneously, without any prior knowledge of the smoothness or of the gradient norms. The following Theorem establishes the guarantees of AdaNGD₂,

Theorem 2.2. *Let $k = 2$, \mathcal{K} be a convex set with diameter D , and f be a convex function; Also let \bar{x}_T be the outputs of AdaNGD₂ (Alg. 2), then the following holds:*

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{\sqrt{2D^2}}{\sqrt{\sum_{t=1}^T 1 / \|g_t\|^2}} \leq \frac{\sqrt{2GD}}{\sqrt{T}}.$$

Moreover, if f is also β -smooth and the global minimum $x^* = \arg \min_{x \in \mathbb{R}^n} f(x)$ belongs to \mathcal{K} , then:

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{\sqrt{2D^2}}{\sqrt{\sum_{t=1}^T 1 / \|g_t\|^2}} \leq \frac{4\beta D^2}{T}.$$

It is interesting to note that AdaNGD₂ will have always performed better than AdaGrad, had both algorithms encountered the same gradient norms. This is due to the well known inequality between arithmetic and harmonic means, [19], $\frac{1}{T} \sum_{t=1}^T a_t \geq \frac{1}{\frac{1}{T} \sum_{t=1}^T 1/a_t}$, $\forall \{a_t\}_{t=1}^T \subset \mathbb{R}_+$, which directly implies, $\frac{1}{\sqrt{\sum_{t=1}^T 1 / \|g_t\|^2}} \leq \frac{1}{T} \sqrt{\sum_{t=1}^T \|g_t\|^2}$.

3 Adaptive NGD for Strongly Convex Functions

Here we discuss the offline optimization setting of strongly convex objectives. We introduce our SC-AdaNGD_k algorithm, and present convergence rates for general $k \in \mathbb{R}$. Subsequently, we elaborate on the $k = 1, 2$ cases which exhibit universality as well as adaptive guarantees that may be substantially better compared to standard methods.

Our SC-AdaNGD_k algorithm is depicted in Algorithm 3. Similarly to its non strongly-convex counterpart, SC-AdaNGD_k can be thought of as an online to offline conversion scheme which utilizes an online algorithm which we denote SC-AdaGrad (we elaborate on the latter in the appendix). The next Lemma states its guarantees,

Lemma 3.1. *Let $k \in \mathbb{R}$, and \mathcal{K} be a convex set. Let f be an H -strongly-convex function; Also let \bar{x}_T be the outputs of SC-AdaNGD_k (Alg. 3), then the following holds:*

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{1}{2H \sum_{t=1}^T \|g_t\|^{-k}} \sum_{t=1}^T \frac{\|g_t\|^{-2(k-1)}}{\sum_{\tau=1}^t \|g_\tau\|^{-k}}.$$

Proof sketch. In the appendix we present and analyze SC-AdaGrad. This is an *online* first order algorithm for strongly-convex functions in which the learning rate decays according to $\eta_t = 1 / \sum_{\tau=1}^t H_\tau$,

where H_τ is the strong-convexity parameter of the loss function at time τ . Then we show that SC-AdaNGD $_k$ is equivalent to applying SC-AdaGrad to the following loss sequence:

$$\left\{ \tilde{f}_t(x) = \frac{1}{\|g_t\|^k} g_t^\top x + \frac{H}{2\|g_t\|^k} \|x - x_t\|^2 \right\}_{t=1}^T.$$

The lemma follows by combining the regret bound of SC-AdaGrad together with the definition of \bar{x}_T and with Jensen's inequality. \square

For $k = 0$, SC-AdaNGD becomes the standard GD algorithm which uses learning rate of $\eta_t = 1/Ht$. Next we focus on the cases where $k = 1, 2$.

3.1 SC-AdaNGD $_1$

Here we show that SC-AdaNGD $_1$ enjoys a rate of $\tilde{O}(1/T)$ for strongly-convex objectives, and a faster rate of $\tilde{O}(1/T^2)$ assuming that the objective is also smooth. We emphasize that the same algorithm enjoys these rates simultaneously, without any prior knowledge of the smoothness or of the gradient norms. The following theorem establishes the guarantees of SC-AdaNGD $_1$,

Theorem 3.1. *Let $k = 1$, and \mathcal{K} be a convex set. Let f be a G -Lipschitz and H -strongly-convex function; Also let \bar{x}_T be the outputs of SC-AdaNGD $_1$ (Alg. 3), then the following holds:*

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{G \left(1 + \log \left(\sum_{t=1}^T \frac{G}{\|g_t\|} \right) \right)}{2H \sum_{t=1}^T \frac{1}{\|g_t\|}} \leq \frac{G^2(1 + \log T)}{2HT}.$$

Moreover, if f is also β -smooth and the global minimum $x^* = \arg \min_{x \in \mathbb{R}^n} f(x)$ belongs to \mathcal{K} , then,

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{(\beta/H)G^2(1 + \log T)^2}{HT^2}.$$

3.2 SC-AdaNGD $_2$

Here we show that SC-AdaNGD $_2$ enjoys the standard $\tilde{O}(1/T)$ rate for strongly-convex objectives, and a linear rate assuming that the objective is also smooth. We emphasize that the same algorithm enjoys these rates simultaneously, without any prior knowledge of the smoothness or of the gradient norms. In the case where $k = 2$ the guarantee of SC-AdaNGD is as follows,

Theorem 3.2. *Let $k = 2$, \mathcal{K} be a convex set, and f be a G -Lipschitz and H -strongly-convex function; Also let \bar{x}_T be the outputs of SC-AdaNGD $_2$ (Alg. 3), then the following holds:*

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{1 + \log(G^2 \sum_{t=1}^T \|g_t\|^{-2})}{2H \sum_{t=1}^T \|g_t\|^{-2}} \leq \frac{G^2(1 + \log T)}{2HT}.$$

Moreover, if f is also β -smooth and the global minimum $x^* = \arg \min_{x \in \mathbb{R}^n} f(x)$ belongs to \mathcal{K} , then,

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{3G^2}{2H} e^{-\frac{H}{\beta}T} \left(1 + \frac{H}{\beta}T \right).$$

Intuition: For strongly-convex objectives the appropriate GD algorithm utilizes two very extreme learning rates of $\eta_t \propto 1/t$ vs. $\eta_t = 1/\beta$ for the general/smooth settings respectively. A possible explanation to the universality of SCAdaNGD $_2$ is that it implicitly interpolate between these rates.

Indeed the update rule of our algorithm can be written as follows, $x_{t+1} = x_t - \frac{1}{H \sum_{\tau=1}^t \|g_\tau\|^{-2}} g_t$. Thus, ignoring the hindsight weighting, SCAdaNGD $_2$ is equivalent to GD with an adaptive learning rate $\tilde{\eta}_t := \|g_t\|^{-2}/H \sum_{\tau=1}^t \|g_\tau\|^{-2}$. Now, when all gradient norms are of the same magnitude, then $\tilde{\eta}_t \propto 1/t$, which boils down to the standard GD for strongly-convex objectives. Conversely, assume that the gradients are exponentially decaying, i.e., that $\|g_t\| \propto q^t$ for some $q < 1$. In this case $\tilde{\eta}_t$ is approximately constant. We believe that the latter applies for strongly-convex & smooth case.

Algorithm 4 Lazy Stochastic Gradient Descent (LazySGD)

Input: #Oracle Queries T , $x_1 \in \mathbb{R}^d$, set \mathcal{K} , η_0 , p
Set: $t = 0$, $s = 0$
while $t \leq T$ **do**
 Update: $s = s + 1$
 Set $\mathcal{G} = \text{GradOracle}(x_s)$, i.e., \mathcal{G} generates i.i.d. noisy samples of $\nabla f(x_s)$
 Get: $(\tilde{g}_s, n_s) = \text{AE}(\mathcal{G}, T - t)$ % Adaptive Minibatch
 Update: $t = t + n_s$
 Calculate: $\hat{g}_s = n_s \tilde{g}_s$
 Set: $\eta_s = \eta_0 / t^p$
 Update: $x_{s+1} = \Pi_{\mathcal{K}}(x_s - \eta_s \hat{g}_s)$
end while
Return: $\bar{x}_T = \sum_{i=1}^s \frac{n_i}{T} x_i$. (Note that $\sum_{i=1}^s n_i = T$)

Algorithm 5 Adaptive Estimate (AE)

Input: random vectors generator \mathcal{G} , sample budget T_{\max} , sample factor m_0
Set: $i = 0$, $N = 0$, $\tilde{g}_0 = 0$
while $N < T_{\max}$ **do**
 Take $\tau_i = \min\{2^i, T_{\max} - N\}$ samples from \mathcal{G}
 Set $N \leftarrow N + \tau_i$
 Update: $\tilde{g}_N \leftarrow$ Average of N samples received so far from \mathcal{G}
 If $\|\tilde{g}_N\| > 3m_0/\sqrt{N}$ **then return** (\tilde{g}_N, N)
 Update $i \leftarrow i + 1$
end while
Return: (\tilde{g}_N, N)

4 Adaptive NGD for Stochastic Optimization

Here we show that using data-dependent minibatch sizes, we can adapt our (SC-)AdaNGD₂ algorithms (Algs. 2, 3 with $k = 2$) to the stochastic setting, and achieve the well know convergence rates for the convex/strongly-convex settings. Next we introduce the stochastic optimization setting, and then we present and discuss our Lazy SGD algorithm.

Setup: We consider the problem of minimizing a convex/strongly-convex function $f : \mathcal{K} \mapsto \mathbb{R}$, where $\mathcal{K} \in \mathbb{R}^d$ is a convex set. We assume that optimization lasts for T rounds; on each round $t = 1, \dots, T$, we may query a point $x_t \in \mathcal{K}$, and receive a *feedback*. After the last round, we choose $\bar{x}_T \in \mathcal{K}$, and our performance measure is the expected excess loss, defined as,

$$\mathbf{E}[f(\bar{x}_T)] - \min_{x \in \mathcal{K}} f(x).$$

Here we assume that our feedback is a first order noisy oracle $\mathcal{G} : \mathcal{K} \mapsto \mathbb{R}^d$ such that upon querying \mathcal{G} with a point $x_t \in \mathcal{K}$, we receive a bounded and unbiased gradient estimate, $\mathcal{G}(x_t)$, such $\mathbf{E}[\mathcal{G}(x_t)|x_t] = \nabla f(x_t)$; $\|\mathcal{G}(x_t)\| \leq G$. We also assume that the internal coin tosses (randomizations) of the oracle are independent. It is well known that variants of Stochastic Gradient Descent (SGD) are ensured to output an estimate \bar{x}_T such that the excess loss is bounded by $O(1/\sqrt{T})/O(1/T)$ for the setups of convex/strongly-convex stochastic optimization, [20], [21].

Notation: In this section we make a clear distinction between the number of queries to the gradient oracle, denoted henceforth by T ; and between the number of iterations in the algorithm, denoted henceforth by S . We care about the dependence of the excess loss in T .

4.1 Lazy Stochastic Gradient Descent

Data Dependent Minibatch sizes: The Lazy SGD (Alg. 4) algorithm that we present in this section, uses a minibatch size that changes in between query points. Given a query point x_s , Lazy SGD invokes the noisy gradient oracle $\tilde{O}(1/\|g_s\|^2)$ times, where $g_s := \nabla f(x_s)$ ³. Thus, in contrast to

³Note that the gradient norm, $\|g_s\|$, is unknown to the algorithm. Nevertheless it is estimated on the fly.

SGD which utilizes a fixed number of oracle calls per query point, our algorithm tends to stall in points with smaller gradients, hence the name Lazy SGD.

Here we give some intuition regarding our adaptive minibatch size rule: Consider the stochastic optimization setting. However, imagine that instead of the noisy gradient oracle \mathcal{G} , we may access an improved (imaginary) oracle which provides us with unbiased estimates, $\tilde{g}(x)$, that are accurate up to some *multiplicative factor*, e.g., $\mathbf{E}[\tilde{g}(x)|x] = \nabla f(x)$, and $\frac{1}{2}\|\nabla f(x)\| \leq \|\tilde{g}(x)\| \leq 2\|\nabla f(x)\|$. Then intuitively we could have used these estimates instead of the exact normalized gradients inside our (SC-)AdaNGD₂ algorithms (Algs. 2, 3 with $k = 2$), and still get similar (in expectation) data dependent bounds. Quite nicely, we may use our original noisy oracle \mathcal{G} to generate estimates from this imaginary oracle. This can be done by invoking \mathcal{G} for $\tilde{O}(1/\|g_s\|^2)$ times at each query point. Using this minibatch rule, the total number of calls to \mathcal{G} (along all iterations) is equal to $T = \sum_{s=1}^S 1/\|g_s\|^2$. Plugging this into the data dependent bounds of (SC-)AdaNGD₂ (Thms. 2.2, 3.2), we get the well known $\tilde{O}(1/\sqrt{T})/\tilde{O}(1/T)$ rates for the stochastic convex settings.

The imaginary oracle: The construction of the imaginary oracle from the original oracle appears in Algorithm 5 (AE procedure). It receives as an input, \mathcal{G} , a generator of independent random vectors with an (unknown) expected value $g \in \mathbb{R}^d$. The algorithm outputs two variables: N which is an estimate of $1/\|g\|^2$, and \tilde{g}_N an average of N random vectors from \mathcal{G} . Thus, it is natural to think of $N\tilde{g}_N$ as an estimate for $g/\|g\|^2$. Moreover, it can be shown that $E[N(\tilde{g}_N - g)] = 0$. Thus in a sense we receive an unbiased estimate. The guarantees of Algorithm 5 appear below,

Lemma 4.1 (Informal). *Let $T_{\max} \geq 1, \delta \in (0, 1)$. Suppose an oracle $\mathcal{G} : \mathcal{K} \mapsto \mathbb{R}^d$ that generates G -bounded i.i.d. random vectors with an (unknown) expected value $g \in \mathbb{R}^d$. Then w.p. $\geq 1 - \delta$, invoking AE (Algorithm 5), with $m_0 = \Theta(G \log(1/\delta))$, it is ensured that:*

$$N = \Theta(\min\{m_0/\|g\|^2, T_{\max}\}), \text{ and } E[N(\tilde{g}_N - g)] = 0.$$

Lazy SGD: Now, plugging the output of the AE algorithm into our offline algorithms (SC-)AdaNGD₂, we get their stochastic variants which appears in Algorithm 4 (Lazy SGD). This algorithm is equivalent to the offline version of (SC-)AdaNGD₂, with the difference that we use n_s instead of $1/\|\nabla f(x_s)\|^2$ and $n_s\tilde{g}_s$ instead of $\nabla f(x_s)/\|\nabla f(x_s)\|^2$.

Let T be a bound on the *total number of queries* to the the first order oracle \mathcal{G} , and δ be the confidence parameter used to set m_0 in the AE procedure. Next we present the guarantees of LazySGD,

Lemma 4.2. *Let $\delta = O(T^{-3/2})$; let \mathcal{K} be a convex set with diameter D , and f be a convex function; and assume $\|\mathcal{G}(x)\| \leq G$ w.p.1. Then using LazySGD with $\eta_0 = D/\sqrt{2}G, p = 1/2$, ensures:*

$$\mathbf{E}[f(\bar{x}_T)] - \min_{x \in \mathcal{K}} f(x) \leq O\left(\frac{GD \log(T)}{\sqrt{T}}\right).$$

Lemma 4.3. *Let $\delta = O(T^{-2})$, let \mathcal{K} be a convex set, and f be an H -strongly-convex convex function; and assume $\|\mathcal{G}(x)\| \leq G$ w.p.1. Then using LazySGD with $\eta_0 = 1/H, p = 1$, ensures:*

$$\mathbf{E}[f(\bar{x}_T)] - \min_{x \in \mathcal{K}} f(x) \leq O\left(\frac{G^2 \log^2(T)}{HT}\right).$$

Note that LazySGD uses minibatch sizes that are adapted to the magnitude of the gradients, and still maintains the optimal $\tilde{O}(1/\sqrt{T})/\tilde{O}(1/T)$ rates. In contrast using a fixed minibatch size b for SGD might degrade the convergence rates, yielding $O(\sqrt{b}/\sqrt{T})/O(b/T)$ guarantees. This property of LazySGD may be beneficial when considering distributed computations (see [13]).

5 Discussion

We have presented a new approach based on a conversion scheme, which exhibits universality and new adaptive bounds in the offline convex optimization setting, and provides a principled approach towards minibatch size selection in the stochastic setting. Among the many questions that remain open is whether we can devise “accelerated” universal methods. Furthermore, our universality results only apply when the global minimum is inside the constraints. Thus, it is natural to seek for methods that ensure universality when this assumption is violated. Moreover, our algorithms depend on a parameter $k \in \mathbb{R}$, but only the cases where $k \in \{0, 1, 2\}$ are well understood. Investigating a wider spectrum of k values is intriguing. Lastly, it is interesting to modify and test our methods in non-convex scenarios, especially in the context of deep-learning applications.

Acknowledgments

I would like to thank Elad Hazan and Shai Shalev-Shwartz for fruitful discussions during the early stages of this work.

This work was supported by the ETH Zürich Postdoctoral Fellowship and Marie Curie Actions for People COFUND program.

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A Extensions

Acceleration: The catalyst approach, [22], enables to take any first order method that ensures linear convergence rates in the strongly-convex and smooth case and transform it into an accelerated method obtaining $O(\exp(-\sqrt{\gamma}T))$ rate in the strongly-convex and smooth case, and $O(1/T^2)$ rate in the smooth case. In particular, this acceleration applies to our SC-AdaNGD₂ Algorithm. Unfortunately, the catalyst approach requires the smoothness parameter, and the resulting accelerated SC-AdaNGD₂ is no longer universal.

Other adaptive online schemes: The adaptive methods that we have presented so far lean on AdaGrad (Alg. 1). Nevertheless, we may base our methods on other online algorithms with adaptive regret guarantees, and obtain convergence rates of the form,

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{\mathcal{R}^{\mathcal{A}}(g_1/\|g_1\|^k, \dots, g_T/\|g_T\|^k)}{\sum_{t=1}^T \|g_t\|^k},$$

where $\mathcal{R}^{\mathcal{A}}(\theta_1, \dots, \theta_T)$ is the regret bound of algorithm \mathcal{A} with respect to the linear loss sequence $\{\theta_t^\top x\}_{t=1}^T$. For example we can use the very popular version of AdaGrad, which employs a separate learning rate to different directions. Also noteworthy is the Multiplicative Weights (MW) online algorithm, which over the simplex ensures a regret bound of the form (see [23], [24]),

$$\mathcal{R}^{MW} \leq \sqrt{\sum_{t=1}^T \|g_t\|_\infty^2 \log(d)}.$$

Using AdaNGD_k with the appropriate modifications: AdaGrad \leftrightarrow MW, and $\ell_2 \leftrightarrow \ell_\infty$, yields similar adaptive guarantees as in Theorems 2.1, 2.2, with the difference that, $D \leftrightarrow \log d$, and $\ell_2 \leftrightarrow \ell_\infty$.

B Experiments

As a preliminary experimental investigation we compare our SC-AdaNGD_k to GD accelerated-GD, and line-search for two strongly-convex objectives⁴. Concretely, we compare the above methods for the following quadratic (smooth) minimization problem,

$$\min_{x \in \mathbb{R}^d} R(x) := \frac{1}{2} \sum_{i=1}^d i \cdot x_i^2.$$

,and also for the following non-smooth problem,

$$\min_{\|x\| \leq 1} F(x) := \frac{1}{2} \sum_{i=1}^d i \cdot x_i^2 + \|x\|_1.$$

where x_i is the i 'th component of x , and $\|x\|_1$ is the ℓ_1 norm. Note that both R and F are 1-strongly-convex, however R is d -smooth while F is non-smooth. Also, for both R and F the unique global minimum is in $x = 0$. We initialize all of the methods at the same random point, and take $d = 100$.

The results are depicted in Fig. 1. In Fig. 1(a) we present our results for the smooth quadratic objective R . We compare three SC-AdaNGD_k variants $k \in \{1, 1.1, 2\}$, to GD which uses a constant learning rate $\eta_t = 1/\beta$ (recall $\beta = d = 100$), and to Nesterov's accelerated method. While this is not surprising that the latter demonstrates the best performance, it is surprising that all SC-AdaNGD_k variants are performing better than GD/lines-search, and the $k = 1.1$ variant substantially outperforms GD. Also, in contrast to GD, SC-AdaNGD_k are not descent methods, in the sense that the losses are not necessarily monotonically decreasing from one iteration to another.

Fig. 1(b) shows the results for the non-smooth objective F , where we compare two SC-AdaNGD_k variants $k \in \{1, 2\}$, with two variants of GD, **(i)** const learning rate $\eta_t = 1/\beta$, and **(ii)** decaying

⁴Line-search may invoke the gradient oracle several times in each iteration. To make a fair comparison, we present performance vs. #calls to the gradient oracle

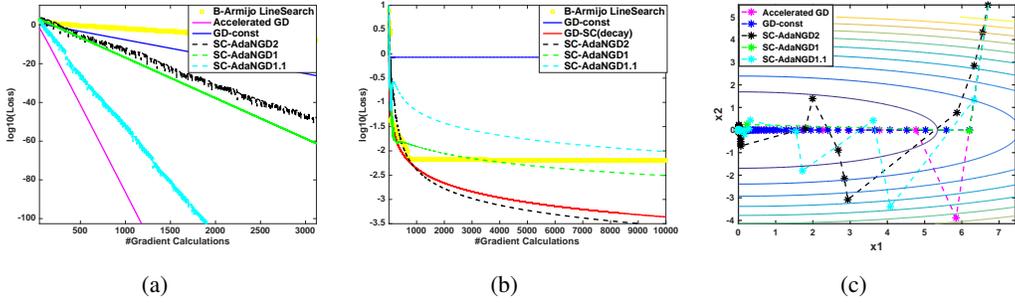


Figure 1: SC-AdaNGD_k compared to GD, accelerated-GD and line-search. Left: strongly-convex and smooth objective, $R(\cdot)$. Middle: strongly-convex and non-smooth objective, $F(\cdot)$. Right: iterates of these methods for a 2D quadratic objective, $Z(\cdot)$.

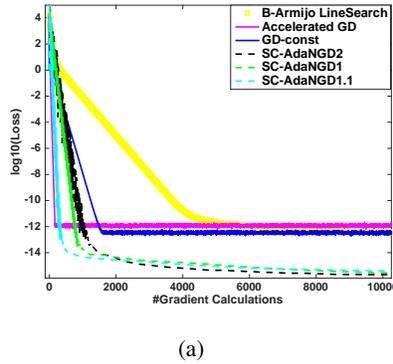


Figure 2: Robustness experiments comparing SC-AdaNGD_k with GD and accelerated-GD for the strongly-convex and smooth objective, $R(\cdot)$. Gradient oracle is perturbed with $\propto 10^{-6}$ noise magnitude.

learning rate $\eta_t = 1/Ht$. We have also compared to accelerated-GD and found its performance to be similar to GD-const (and therefore omitted). As can be seen, GD with a constant learning rate is doing very poorly, SC-AdaNGD₂ demonstrates the best performance, and GD-SC (decay) lags behind only by little. Note that for GD-SC (decay) we present results for a moving average over the GD iterates (which improve its performance).

The universality of SC-AdaNGD_k for $k \in \{1, 2\}$ is clearly evident from Figures 1(a), 1(b). In order to learn more about the character of SC-AdaNGD, we have applied the above methods to a simple 2D quadratic objective,

$$Z(x) = x_1^2 + 10x_2^2.$$

The progress (iterates) of these methods is presented in Fig. 1(c). It can be seen that GD and accelerated-GD converge quickly to the x_1 axis and progress along it towards $(0, 0)$. Conversely, SC-AdaNGD methods progress diagonally, however take larger steps in the x_1 directions compared to GD and accelerated-GD.

Robustness: We have also examined the robustness of SC-AdaNGD compared to GD, accelerated-GD and line-search. We applied these methods to the quadratic objective R , however instead of the exact gradients we provided them with a slightly noisy and (unbiased) gradient feedback. The results when using noise perturbation magnitude of 10^{-6} appear in Fig. 2. This behaviour persisted when we employed other noise magnitudes.

Stochastic setting: We made a few experiments in the stochastic setting. While examining LazySGD, we have found out that using the n_s output of the AE procedure (Alg. 5) is a too crude estimate for $1/\|g_s\|^2$ (due to the doubling procedure), which lead to unsatisfactory performance. Instead, we found that using $1/\|\tilde{g}_s\|^2$ is a much better approximation, that works very well in practice.

An initial experimental study on several simple stochastic problems shows that LazySGD (with the above modification) compares with minibatch SGD, for various values of minibatch sizes. A more elaborate examination of LazySGD is left for future work.

C Proofs for Section 2 (AdaNGD)

C.1 Proof of Theorem 1.1 (AdaGrad)

Proof. Let $x \in \mathcal{K}$ and Consider the update rule $x_{t+1} = \Pi_{\mathcal{K}}(x_t - \eta_t g_t)$. We can write:

$$\|x_{t+1} - x\|^2 \leq \|x_t - x\|^2 - 2\eta_t g_t^\top (x_t - x) + \eta_t^2 \|g_t\|^2$$

Re-arranging the above we get:

$$g_t^\top (x_t - x) \leq \frac{1}{2\eta_t} (\|x_t - x\|^2 - \|x_{t+1} - x\|^2) + \frac{\eta_t}{2} \|g_t\|^2.$$

Combined with the convexity of f_t and summing over all rounds we conclude that $\forall x \in \mathcal{K}$,

$$\begin{aligned} \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x) &\leq \sum_{t=1}^T \frac{\|x_t - x\|^2}{2} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \sum_{t=1}^T \frac{\eta_t}{2} \|g_t\|^2 \\ &\leq \frac{D^2}{2} \sum_{t=1}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{D}{2\sqrt{2}} \sum_{t=1}^T \frac{\|g_t\|^2}{\sqrt{\sum_{\tau=1}^t \|g_\tau\|^2}} \\ &\leq \frac{D}{2} \sqrt{2 \sum_{t=1}^T \|g_t\|^2} + \frac{D}{\sqrt{2}} \sqrt{\sum_{t=1}^T \|g_t\|^2} \\ &= \sqrt{2D^2 \sum_{t=1}^T \|g_t\|^2} \end{aligned}$$

here in the first inequality we denote $\eta_0 = \infty$, the second inequality uses $\text{diam}\mathcal{K} = D$ and $\eta_t \leq \eta_{t-1}$, the third inequality uses the following lemma from [6]:

Lemma C.1. *For any non-negative numbers a_1, \dots, a_n the following holds:*

$$\sum_{i=1}^n \frac{a_i}{\sqrt{\sum_{j=1}^i a_j}} \leq 2 \sqrt{\sum_{i=1}^n a_i}$$

□

C.2 Proof of Lemma 2.1

Proof. Notice that AdaNGD $_k$ described in Algorithm 2, is equivalent to applying AdaGrad (Algorithm 1) to the following sequence of linear loss functions:

$$\left\{ \tilde{f}_t(x) := \frac{1}{\|g_t\|^k} g_t^\top x \right\}_{t=1}^T.$$

The regret bound of AdaGrad appearing in Theorem 1.1 implies the following for any $x \in \mathcal{K}$:

$$\sum_{t=1}^T \frac{1}{\|g_t\|^k} g_t^\top (x_t - x) \leq \sqrt{2D^2 \sum_{t=1}^T 1/\|g_t\|^{2(k-1)}}. \quad (1)$$

Using the above bound together with Jensen's inequality, enables to bound the excess loss of AdaNGD_k:

$$\begin{aligned}
f(\bar{x}_T) - f(x^*) &\leq \sum_{t=1}^T \frac{\|g_t\|^{-k}}{\sum_{\tau=1}^T \|g_\tau\|^{-k}} (f(x_t) - f(x^*)) \\
&\leq \sum_{t=1}^T \frac{\|g_t\|^{-k}}{\sum_{\tau=1}^T \|g_\tau\|^{-k}} g_t^\top (x_t - x^*) \\
&= \frac{1}{\sum_{\tau=1}^T \|g_\tau\|^{-k}} \sum_{t=1}^T \frac{1}{\|g_t\|^k} g_t^\top (x_t - x^*) \\
&\leq \frac{\sqrt{2D^2 \sum_{t=1}^T 1/\|g_t\|^{2(k-1)}}}{\sum_{\tau=1}^T 1/\|g_\tau\|^k},
\end{aligned}$$

where the second line uses the gradient inequality. \square

C.3 Proof of Theorem 2.1

Proof. The data dependent bound,

$$f(\bar{x}_T) - f(x^*) \leq \frac{\sqrt{2D^2 T}}{\sum_{t=1}^T 1/\|g_t\|}, \quad (2)$$

is a direct corollary of Lemma 2.1 with $k = 1$. Note that the above bound holds for both smooth/non-smooth cases. The general case bound holds directly by using $\|g_t\| \leq G$.

Next we focus on the second part of the theorem regarding the smooth case. We will first require the following lemma regarding smooth objectives,

Lemma C.2. *Let $F : \mathbb{R}^d \mapsto \mathbb{R}$ be a β -smooth function, and let $x^* = \arg \min_{x \in \mathbb{R}^d} F(x)$, then,*

$$\|\nabla F(x)\|^2 \leq 2\beta (F(x) - F(x^*)), \quad \forall x \in \mathbb{R}^d.$$

The above lemma enables to upper bound sum of gradient norms in the query points of AdaNGD₁,

$$\begin{aligned}
\sum_{t=1}^T \|g_t\| &= \sum_{t=1}^T \frac{\|g_t\|^2}{\|g_t\|} \\
&\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|} (f(x_t) - f(x^*)) \\
&\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|} g_t^\top (x_t - x^*) \\
&= 2\beta \sum_{t=1}^T \hat{g}_t^\top (x_t - x^*) \\
&\leq 2\sqrt{2}\beta D\sqrt{T},
\end{aligned} \quad (3)$$

where the last line follows by the regret guarantee of AdaGrad for the following sequence (see Equation (1)),

$$\left\{ \tilde{f}_t(x) := \frac{1}{\|g_t\|} g_t^\top x \right\}_{t=1}^T.$$

The second line is a consequence of Lemma C.2 regarding smooth objectives. Now utilizing the convexity of the function $H(z) = 1/z$ for $z > 0$, and applying Equation (3), we may bound the sum

of inverse gradients:

$$\begin{aligned} \sum_{\tau=1}^T \frac{1}{\|g_\tau\|} &= T \frac{1}{T} \sum_{\tau=1}^T \frac{1}{\|g_\tau\|} \geq T \frac{1}{\frac{1}{T} \sum_{\tau=1}^T \|g_\tau\|} \\ &\geq T \frac{1}{2\sqrt{2}\beta D/\sqrt{T}}. \end{aligned}$$

Rearranging the latter equation, and using Equation (2) concludes the proof,

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{D\sqrt{2T}}{\sum_{\tau=1}^T 1/\|g_\tau\|} \leq \frac{4\beta D^2}{T}.$$

□

C.4 Proof of Theorem 2.2

Proof. The data dependent bound,

$$f(\bar{x}_T) - f(x^*) \leq \frac{\sqrt{2D^2}}{\sqrt{\sum_{t=1}^T 1/\|g_t\|^2}}, \quad (4)$$

is a direct corollary of Lemma 2.1 with $k = 2$. Note that the above bound holds for both smooth/non-smooth cases. The general case bound holds directly by using $\|g_t\| \leq G$.

We will now focus on the second part of the theorem regarding the smooth case. Let us lower bound $\sum_{t=1}^T 1/\|g_t\|^2$ for AdaNGD₂:

$$\begin{aligned} T &= \sum_{t=1}^T \frac{\|g_t\|^2}{\|g_t\|^2} \\ &\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|^2} (f(x_t) - f(x^*)) \\ &\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|^2} g_t^\top (x_t - x^*) \\ &= 2\beta \sum_{t=1}^T (\tilde{f}_t(x_t) - \tilde{f}_t(x^*)) \\ &\leq 2\sqrt{2}\beta D \sqrt{\sum_{t=1}^T \frac{1}{\|g_t\|^2}}, \end{aligned} \quad (5)$$

where the last line follows by the regret guarantee of AdaGrad for the following sequence (see Equation (1)),

$$\left\{ \tilde{f}_t(x) = \frac{1}{\|g_t\|^2} g_t^\top x \right\}_{t=1}^T.$$

The second line is a consequence of Lemma C.2. Combining Equation (5) together with Equation (4) concludes the proof. □

C.5 Proof of Lemma C.2

Proof. The β smoothness of F means the following to hold $\forall x, u \in \mathbb{R}^d$,

$$F(x+u) \leq F(x) + \nabla F(x)^\top u + \frac{\beta}{2} \|u\|^2.$$

Taking $u = -\frac{1}{\beta} \nabla F(x)$ we get,

$$F(x+u) \leq F(x) - \frac{1}{\beta} \|\nabla F(x)\|^2 + \frac{1}{2\beta} \|\nabla F(x)\|^2.$$

Combining the latter bound with the definition of \bar{x}_T , and applying Jensen's inequality we conclude:

$$\begin{aligned}
f(\bar{x}_T) - f(x^*) &\leq \sum_{t=1}^T \frac{\|g_t\|^{-k}}{\sum_{\tau=1}^T \|g_\tau\|^{-k}} (f(x_t) - f(x^*)) \\
&\leq \frac{1}{\sum_{t=1}^T \|g_t\|^{-k}} \sum_{t=1}^T \|g_t\|^{-k} \left(g_t^\top (x_t - x^*) - \frac{H}{2} \|x_t - x^*\|^2 \right) \\
&= \frac{1}{\sum_{t=1}^T \|g_t\|^{-k}} \sum_{t=1}^T \left(\tilde{f}_t(x_t) - \tilde{f}_t(x^*) \right) \\
&\leq \frac{1}{2H \sum_{t=1}^T \|g_t\|^{-k}} \sum_{t=1}^T \frac{\|g_t\|^{-2(k-1)}}{\sum_{\tau=1}^t \|g_\tau\|^{-k}},
\end{aligned}$$

where we used the H -strong-convexity of f in the second line. □

D.2 Proof of Theorem 3.1

Proof. We will require the following lemma, its proof is provided in Section D.5.

Lemma D.2. For any non-negative real numbers $a_1, \dots, a_n \geq 1$,

$$\sum_{i=1}^n \frac{a_i}{\sum_{j=1}^i a_j} \leq 1 + \log \left(\sum_{i=1}^n a_i \right).$$

Combining the above lemma together with Lemma 3.1 and using $k = 1$, we obtain,

$$\begin{aligned}
f(\bar{x}_T) - f(x^*) &\leq \frac{1}{2H \sum_{t=1}^T \|g_t\|^{-1}} \sum_{t=1}^T \frac{1}{\sum_{\tau=1}^t \|g_\tau\|^{-1}} \\
&\leq \frac{1}{2H \sum_{t=1}^T \|g_t\|^{-1}} \sum_{t=1}^T \frac{G \|g_t\|^{-1}}{\sum_{\tau=1}^t \|g_\tau\|^{-1}} \\
&\leq \frac{G}{2H \sum_{t=1}^T \|g_t\|^{-1}} \sum_{t=1}^T \frac{G \|g_t\|^{-1}}{\sum_{\tau=1}^t G \|g_\tau\|^{-1}} \\
&\leq \frac{G}{2H \sum_{t=1}^T \|g_t\|^{-1}} \left(1 + \log \left(\sum_{t=1}^T \frac{G}{\|g_t\|} \right) \right)
\end{aligned}$$

where the second line uses $\|g_t\| \leq G$, and the last line uses Lemma D.2. Note that the above bound holds for both smooth/non-smooth cases.

We now turn to prove the second part of the theorem regarding the smooth case. First let us bound the sum of gradient norms in the query points of SC-AdaNGD₁:

$$\begin{aligned}
\sum_{t=1}^T \|g_t\| &= \sum_{t=1}^T \frac{\|g_t\|^2}{\|g_t\|} \\
&\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|} (f(x_t) - f(x^*)) \\
&\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|} \left(g_t^\top (x_t - x^*) - \frac{H}{2} \|x_t - x^*\|^2 \right) \\
&= 2\beta \sum_{t=1}^T (\tilde{f}_t(x_t) - \tilde{f}_t(x^*)) \\
&\leq \frac{\beta}{H} \sum_{t=1}^T \frac{1}{\sum_{\tau=1}^t \|g_\tau\|^{-1}} \\
&\leq \frac{\beta}{H} G \left(1 + \log \left(\sum_{t=1}^T \frac{G}{\|g_t\|} \right) \right),
\end{aligned}$$

where the second line uses Lemma C.2, the third line uses the strong-convexity of f , the fourth line uses the regret bound of the SC-AdaGrad algorithm over the following sequence (see Equation (??)),

$$\left\{ \tilde{f}_t(x) = \frac{1}{\|g_t\|} g_t^\top x + \frac{H}{2\|g_t\|} \|x - x_t\|^2 \right\}_{t=1}^T,$$

and the last line uses Lemma D.2. Combining the convexity of the function $H(z) = 1/z$ for $z > 0$, together with the above inequality, we may bound the sum of inverse gradient norms,

$$\begin{aligned}
\sum_{\tau=1}^T \frac{1}{\|g_\tau\|} &= T \frac{1}{T} \sum_{\tau=1}^T \frac{1}{\|g_\tau\|} \geq T \frac{1}{\frac{1}{T} \sum_{\tau=1}^T \|g_\tau\|} \\
&\geq T^2 \frac{1}{(\beta/H)G \left(1 + \log \left(\sum_{t=1}^T \frac{G}{\|g_t\|} \right) \right)}.
\end{aligned}$$

Rearranging the latter equation, and using the data dependent bound for SC-AdaNGD₁ concludes the proof,

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{(\beta/H)G^2 (1 + \log T)^2}{HT^2}.$$

□

D.3 Proof of Theorem 3.2

Proof. The data dependent bound,

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{1 + \log(G^2 \sum_{t=1}^T \|g_t\|^{-2})}{2H \sum_{t=1}^T \|g_t\|^{-2}} \quad (6)$$

is a direct corollary of Lemma 3.1 with $k = 2$, combined with Lemma D.2. Note that the above bound holds for both smooth/non-smooth cases.

We now turn to prove the second part of the theorem regarding the smooth case. Let us lower bound $\sum_{t=1}^T 1/\|g_t\|^2$, for SC-AdaNGD₂:

$$\begin{aligned}
T &= \sum_{t=1}^T \frac{\|g_t\|^2}{\|g_t\|^2} \\
&\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|^2} (f(x_t) - f(x^*)) \\
&\leq \sum_{t=1}^T \frac{2\beta}{\|g_t\|^2} \left(g_t^\top (x_t - x^*) - \frac{H}{2} \|x_t - x^*\|^2 \right) \\
&= 2\beta \sum_{t=1}^T \left(\tilde{f}_t(x_t) - \tilde{f}_t(x^*) \right) \\
&\leq \frac{\beta}{H} \sum_{t=1}^T \frac{\|g_t\|^{-2}}{\sum_{\tau=1}^t \|g_\tau\|^{-2}} \\
&\leq \frac{\beta}{H} \left(1 + \log(G^2 \sum_{t=1}^T \|g_t\|^{-2}) \right), \tag{7}
\end{aligned}$$

where the second line uses Lemma C.2, the third line uses the strong-convexity of f , the fifth line uses the regret bound of the SC-AdaGrad algorithm for the following sequence (see Equation (??)),

$$\left\{ \tilde{f}_t(x) = \frac{1}{\|g_t\|^2} g_t^\top x + \frac{H}{2\|g_t\|^2} \|x - x_t\|^2 \right\}_{t=1}^T,$$

and the last line uses Lemma D.2. Now Equation (7) implies,

$$G^2 \sum_{t=1}^T \|g_t\|^{-2} \geq \frac{1}{3} e^{\frac{H}{\beta} T}. \tag{8}$$

Now let $z \in \mathbb{R}$ and note that the function $A(z) := \frac{1+\log(z)}{z}$ is monotonically decreasing for $z \geq 1$. Let $z = G^2 \sum_t \|g_t\|^{-2}$ and assume $\frac{1}{3} e^{\frac{H}{\beta} T} \geq 1$; combining this with Equation (6),(8), concludes the proof. Note that the case where $\frac{1}{3} e^{\frac{H}{\beta} T} \leq 1$ is not too interesting. □

D.4 Proof of Lemma D.1

Proof. Let $x \in \mathcal{K}$ and Consider the update rule $x_{t+1} = \Pi_{\mathcal{K}}(x_t - \eta_t g_t)$. We can write:

$$\|x_{t+1} - x\|^2 \leq \|x_t - x\|^2 - 2\eta_t g_t^\top (x_t - x) + \eta_t^2 \|g_t\|^2.$$

Re-arranging the above we get:

$$g_t^\top (x_t - x) \leq \frac{1}{2\eta_t} (\|x_t - x\|^2 - \|x_{t+1} - x\|^2) + \frac{\eta_t}{2} \|g_t\|^2.$$

Combining the above with the H_t -strong-convexity of f_t and summing over all rounds we conclude that,

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x) \leq \sum_{t=1}^T \frac{\|x_t - x\|^2}{2} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} - H_t \right) + \sum_{t=1}^T \frac{\eta_t}{2} \|g_t\|^2,$$

where we denote $\eta_0 = \infty$. Recalling $\eta_t = (\sum_{\tau=1}^t H_\tau)^{-1}$, the lemma follows. □

D.5 Proof of Lemma D.2

Proof. We will prove the statement by induction over n . The base case $n = 1$ naturally holds. For the induction step, let us assume that the guarantee holds for $n - 1$, which implies that for any $a_1, \dots, a_n \geq 1$,

$$\sum_{i=1}^n \frac{a_i}{\sum_{j=1}^i a_j} \leq 1 + \log\left(\sum_{i=1}^{n-1} a_i\right) + \frac{a_n}{\sum_{i=1}^n a_i}.$$

The above suggests that establishing following inequality concludes the proof,

$$1 + \log\left(\sum_{i=1}^{n-1} a_i\right) + \frac{a_n}{\sum_{i=1}^n a_i} \leq 1 + \log\left(\sum_{i=1}^n a_i\right). \quad (9)$$

Using the notation $x = a_n / \sum_{i=1}^{n-1} a_i$, Equation (9) is equivalent to the following,

$$\log(x + 1) - \frac{x}{1 + x} \geq 0.$$

However, it is immediate to validate that the function $M(x) = \log(x + 1) - \frac{x}{1+x}$, is non-negative for any $x \geq 0$, which establishes the lemma. \square

E Proofs for Section 4.1 (Lazy SGD)

E.1 Proof of Lemma 4.1

We first provide the exact statement rather than the informal one appearing in Lemma 4.1.

Lemma E.1. *Let $T_{\max} \geq 1$. Suppose an oracle $\mathcal{G} : \mathcal{K} \mapsto \mathbb{R}^d$ that generates i.i.d. random vectors with an (unknown) expected value $g \in \mathbb{R}^d$. Assume that w.p. 1 the Euclidean norm of the sampled vectors is bounded by G . Then w.p. $\geq 1 - \delta$, invoking AE (Algorithm 5), with $m_0 = 6G \left(1 + \sqrt{\log(\delta^{-1}(1 + \log_2 T_{\max}))}\right)$, it is ensured that:*

$$\min\{m_0^2/\|g\|^2, T_{\max}\} \leq N \leq \min\{32m_0^2/\|g\|^2, T_{\max}\}. \quad (1)$$

Moreover, w.p. $\geq 1 - \delta$, the following holds for the output of the algorithm:

$$\sqrt{N}\|\tilde{g}_N\| \leq 8m_0. \quad (2)$$

and also,

$$E[N(\tilde{g}_N - g)] = 0. \quad (3)$$

We will require the following Hoeffding type inequality regarding vector valued random variables, by [25] (see also [26])

Theorem E.1. *Suppose that $X_1, X_2, \dots, X_n \in \mathbb{R}^d$ are i.i.d. random vectors, and that $\forall i \in [n]; \|X_i\| \leq M$ almost surely. Then w.p. $\geq 1 - \delta$*

$$\left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbf{E}[X_1] \right\| \leq \frac{6M}{\sqrt{n}} \left(1 + \sqrt{\log \delta^{-1}}\right).$$

We are now ready to prove Lemma E.1.

Proof of Lemma E.1. Define $V = \left\{ \{2^i - 1\}_{i=1}^{\log_2 T_{\max}}, T_{\max} \right\}$, and note that N is a discrete random variable taking one of the $1 + \log_2 T_{\max}$ possible values among V . By Theorem E.1 combined with the union bound, it follows that w.p. $\geq 1 - \delta$, for every $n \in V$ we have $\|\tilde{g}_n - g\| \leq \frac{m_0}{\sqrt{n}}$. This means the following to hold:

$$\|\tilde{g}_n\| \leq \|g\| + \|\tilde{g}_n - g\| \leq \frac{2m_0}{\sqrt{n}}, \quad \forall n \in V \text{ such that } \|g\| \leq m_0/\sqrt{n} \quad (10)$$

Furthermore,

$$\|\tilde{g}_n\| \geq \|g\| - \|\tilde{g}_n - g\| \geq \frac{3m_0}{\sqrt{n}}, \quad \forall n \in V \text{ such that } \|g\| \geq 4m_0/\sqrt{n} \quad (11)$$

The above together with the stopping criteria of Algorithm 5 directly implies the first part of the lemma.

For the second part of the lemma, recall that N is the total number of samples, and let N_{prev} be the number of samples up to the iteration before stopping. Then necessarily, $N_{\text{prev}} \geq (N - 1)/2$. Since the loop did not stop at the iteration before setting N , it follows that $\sqrt{N_{\text{prev}}}\|\tilde{g}_{N_{\text{prev}}}\| \leq 3m_0$ (i.e. the stopping criteria of the loop at the round prior to setting N fails). Recalling that w.p. $\geq 1 - \delta$, for every $n \in V$ we have $\|\tilde{g}_n - g\| \leq \frac{m_0}{\sqrt{n}}$, and combining this with the above implies:

$$\begin{aligned} \sqrt{N}\|\tilde{g}_N\| &\leq \sqrt{N}(\|\tilde{g}_N - g\| + \|g - \tilde{g}_{N_{\text{prev}}}\|) + \sqrt{N}\|\tilde{g}_{N_{\text{prev}}}\| \\ &\leq \sqrt{N}\left(\frac{m_0}{\sqrt{N}} + \frac{m_0}{\sqrt{N_{\text{prev}}}}\right) + \sqrt{\frac{N}{N_{\text{prev}}}}\sqrt{N_{\text{prev}}}\|\tilde{g}_{N_{\text{prev}}}\| \\ &\leq m_0 + \sqrt{3}m_0 + \sqrt{3} \cdot 3m_0 \\ &\leq 8m_0 \end{aligned}$$

Where we have used $N \leq 3\frac{N-1}{2} \leq 3N_{\text{prev}}$; which holds since $N_{\text{prev}} \geq (N - 1)/2$ and also $N \geq 3$. The latter is ensured since for any $n \leq 3$ then $\|\tilde{g}_n\| \leq G < 3m_0/\sqrt{n}$.

For the third part of the lemma, it is easy to notice that for any fixed n then $n(\tilde{g}_n - g)$ is a sum of n i.i.d. random variables, and that $\mathbf{E}[n(\tilde{g}_n - g)] = 0$. Since N is a bounded stopping time, Doob's optional stopping theorem [27] implies that $E[N(\tilde{g}_N - g)] = 0$. \square

E.2 Proof of Lemma 4.2

Proof. Let S be the total number of times that LazySGD invokes the AE procedure. We will first upper bound the expectation of following sum (weighted regret):

$$\begin{aligned} \sum_{s=1}^S n_s (f(x_s) - f(x^*)) &\leq \sum_{s=1}^S n_s g_s^\top (x_s - x^*) \\ &\leq \underbrace{\sum_{s=1}^S n_s \tilde{g}_s^\top (x_s - x^*)}_{(a)} + \underbrace{\sum_{s=1}^S n_s (g_s - \tilde{g}_s)^\top (x_s - x^*)}_{(b)} \quad (12) \end{aligned}$$

where we have used the gradient inequality. The proof goes on by bounding the expectation of terms (a), (b) appearing above.

Bounding term (a): Assume that LazySGD uses the AE procedure with some $\delta > 0$. Since LazySGD is equivalent to AdaNGD₂ with $\|g_s\|^2 \leftarrow n_s$ and $g_s \leftarrow n_s g_s$, then a similar analysis to AdaNGD₂ may show that this sum is bounded by $O(\sqrt{T})$. For completeness we provide the full analysis here. Consider the update rule of LazySGD: $x_{s+1} = \Pi_{\mathcal{K}}(x_s - \eta_s n_s \tilde{g}_s)$. We can write:

$$\|x_{s+1} - x^*\|^2 \leq \|x_s - x^*\|^2 - 2\eta_s n_s \tilde{g}_s^\top (x_s - x^*) + \eta_s^2 n_s^2 \|\tilde{g}_s\|^2$$

Re-arranging the above we get:

$$n_s \tilde{g}_s^\top (x_s - x^*) \leq \frac{1}{2\eta_s} (\|x_s - x^*\|^2 - \|x_{s+1} - x^*\|^2) + \frac{\eta_s}{2} n_s^2 \|\tilde{g}_s\|^2$$

Summing over all rounds we conclude that w.p. $\geq 1 - \delta T$:

$$\begin{aligned}
(\mathbf{a}) &= \sum_{s=1}^S n_s \tilde{g}_s^\top (x_s - x^*) \\
&\leq \sum_{s=1}^S \frac{\|x_s - x^*\|^2}{2} \left(\frac{1}{\eta_s} - \frac{1}{\eta_{s-1}} \right) + \sum_{s=1}^S \frac{\eta_s}{2} n_s^2 \|\tilde{g}_s\|^2 \\
&\leq \frac{D^2}{2} \sum_{s=1}^S \left(\frac{1}{\eta_s} - \frac{1}{\eta_{s-1}} \right) + 64m_0^2 \sum_{s=1}^S \eta_s n_s \\
&\leq \frac{DG}{2} \sqrt{2T} + \frac{64m_0^2 D}{G} \sum_{s=1}^S \frac{n_s}{\sqrt{\sum_{i=1}^s n_i}} \\
&= \frac{DG}{2} \sqrt{2T} + \frac{128m_0^2 D}{G} \sqrt{\sum_{s=1}^S n_s} \\
&\leq O(GD\sqrt{T} \log(1/\delta)).
\end{aligned}$$

here in the first inequality we denote $\eta_0 = \infty$, the second inequality uses $n_s \|\tilde{g}_s\|^2 \leq 64m_0^2$, which follows by Theorem E.1, and it also uses $\eta_s \leq \eta_{s-1}$; the fourth inequality uses Lemma C.1. We also make use of $\sum_{s=1}^S n_s = T$, and $1/\eta_s = \sqrt{\sum_{i=1}^s n_i}$.

Since (\mathbf{a}) is bounded by $2GDT$, then taking $\delta = 1/T^{3/2}$ ensures that,

$$\mathbf{E}[(\mathbf{a})] \leq O(GD\sqrt{T} \log(T)). \quad (13)$$

Bounding term (b): Here we show that $\mathbf{E}[(\mathbf{b})] = 0$. Without loss of generality we will make the following two assumptions which do not affect the output of LazySGD:

- We assume that LazySGD invokes the AE procedure exactly T times. Note that in practice the algorithm invokes the AE procedure S times, where $S \leq T$ is a random variable, after which $T - t = 0$. Nevertheless calling AE for any $s \in \{S+1, \dots, T\}$ yields $\tilde{g}_s = 0, n_s = 0$, which does not affect the output of LazySGD.
- We assume that at each time $s \in [T]$ that LazySGD calls the AE procedure, it samples exactly T times from $\text{GradOracle}(x_s)$. We denote these samples by $\{\tilde{g}_s^{(i)}\}_{i=1}^T$. Nevertheless the output of the procedure only uses the first n_s samples, where n_s is set according to the AE procedure. Thus the remaining $T - n_s$ samples do not affect the output of AE and LazySGD. Note that $\forall s \in [T], n_s \leq T - t \leq T$,

Thus, for any $s \in [T]$ let $\{\tilde{g}_s^{(i)}\}_{i=1}^T$ be the samples drawn from the noisy first order oracle $\text{GradOracle}(x_s)$ during the s 'th call to AE at this iteration. This implies that $n_s \tilde{g}_s = \sum_{i=1}^{n_s} \tilde{g}_s^{(i)}$. Term (b) can be therefore written as follows:

$$(\mathbf{b}) = \sum_{s=1}^T n_s (g_s - \tilde{g}_s)^\top (x_s - x^*) = \sum_{s=1}^T \sum_{i=1}^{n_s} (g_s - \tilde{g}_s^{(i)})^\top (x_s - x^*)$$

Given $s \in [T]$ define the following filtration:

$$\begin{aligned}
\mathcal{F}_0^{(s)} &= \sigma\text{-field } \{x_s, t\} \\
\mathcal{F}_j^{(s)} &= \sigma\text{-field } \{x_s, t, g_s^{(1)}, \dots, g_s^{(j)}\}, \quad \forall j \in [T]
\end{aligned}$$

Also define the following sequence $\{B_j^{(s)}\}_{j=0}^T$:

$$B_0^{(s)} = 0, \quad B_j^{(s)} = \sum_{i=1}^j (g_s - \tilde{g}_s^{(i)})^\top (x_s - x^*), \quad \forall j \in [T]$$

Since $\mathbf{E}[\tilde{g}_s^{(i)} | x_s] = g_s$, $\forall i, s \in [T]$, then it immediately follows that $\{B_j^{(s)}\}_{t=0}^T$ is a martingale with respect to the above filtration. Also it is immediate to see that n_s is a bounded stopping time with respect to the above filtration. Thus, Doob's optional stopping theorem (see [27]) implies that

$$\mathbf{E}[B_{n_s}^{(s)} | \mathcal{F}_0] = \mathbf{E} \left[\sum_{i=1}^{n_s} (g_s - \tilde{g}_s^{(i)})^\top (x_s - x^*) | \mathcal{F}_0 \right] = 0.$$

which directly implies,

$$\mathbf{E}[\mathbf{(b)}] = \mathbf{E} \left[\sum_{s=1}^T B_{n_s}^{(s)} \right] = 0.$$

Using Jensen's inequality and combining the above with Equations (12), (13), establishes the lemma:

$$\begin{aligned} \mathbf{E}[f(\bar{x}_T)] - f(x^*) &\leq \mathbf{E} \left[\sum_{s=1}^S \frac{n_s}{T} (f(x_s) - f(x^*)) \right] \\ &\leq \frac{1}{T} O(GD\sqrt{T} \log(T)) \\ &\leq O(GD \log(T) / \sqrt{T}). \end{aligned}$$

□

E.3 Proof of Lemma 4.3

Proof. Let S be the total number of times that LazySGD invokes the AE procedure. We will first upper bound the expectation of the following sum (weighted regret):

$$\begin{aligned} &\sum_{s=1}^S n_s (f(x_s) - f(x^*)) \\ &\leq \sum_{s=1}^S n_s (g_s^\top (x_s - x^*) - \frac{H}{2} \|x_s - x^*\|^2) \\ &\leq \underbrace{\sum_{s=1}^S n_s (\tilde{g}_s^\top (x_s - x^*) - \frac{H}{2} \|x_s - x^*\|^2)}_{(a)} + \underbrace{\sum_{s=1}^S n_s (g_s - \tilde{g}_s)^\top (x_s - x^*)}_{(b)} \end{aligned} \quad (14)$$

where we have used the H -strong-convexity of $f(\cdot)$. The proof goes on by bounding the expectation of terms (a), (b) appearing above.

Bounding term (a): Assume that LazySGD uses the AE procedure with some $\delta > 0$. Since LazySGD is equivalent to SC-AdaNGD₂ with $\|g_s\|^2 \leftarrow n_s$ and $g_s \leftarrow n_s g_s$, then a similar analysis to SC-AdaNGD₂ may show that this sum is bounded by $O(\log T)$. For completeness we provide the full analysis here. Consider the update rule of LazySGD: $x_{s+1} = \Pi_{\mathcal{K}}(x_s - \eta_s n_s \tilde{g}_s)$. We can write:

$$\|x_{s+1} - x^*\|^2 \leq \|x_s - x^*\|^2 - 2\eta_s n_s \tilde{g}_s^\top (x_s - x^*) + \eta_s^2 n_s^2 \|\tilde{g}_s\|^2$$

Re-arranging the above we get:

$$n_s \tilde{g}_s^\top (x_s - x^*) \leq \frac{1}{2\eta_s} (\|x_s - x^*\|^2 - \|x_{s+1} - x^*\|^2) + \frac{\eta_s}{2} n_s^2 \|\tilde{g}_s\|^2$$

Summing over all rounds we conclude that w.p. $\geq 1 - \delta T$:

$$\begin{aligned}
(\mathbf{a}) &= \sum_{s=1}^S n_s \tilde{g}_s^\top (x_s - x^*) - n_s \frac{H}{2} \|x_s - x^*\|^2 \\
&\leq \sum_{s=1}^S \frac{\|x_s - x^*\|^2}{2} \left(\frac{1}{\eta_s} - \frac{1}{\eta_{s-1}} - n_s H \right) + \sum_{s=1}^S \frac{\eta_s}{2} n_s^2 \|\tilde{g}_s\|^2 \\
&\leq 0 + 32m_0^2 \sum_{s=1}^S \eta_s n_s \\
&\leq \frac{32m_0^2}{H} \sum_{s=1}^S \frac{n_s}{\sum_{k=1}^s n_s} \\
&= \frac{32m_0^2}{H} \left(1 + \log \left(\sum_{s=1}^S n_s \right) \right) \\
&\leq \tilde{O} \left(\frac{G^2 \log T}{H} \log(1/\delta) \right). \tag{15}
\end{aligned}$$

here in the first inequality we denote $\eta_0 = \infty$, the second inequality uses $1/\eta_s = H \sum_{i=1}^s n_i$, and also $n_s \|\tilde{g}_s\|^2 \leq 64m_0^2$, which follows by Theorem E.1; the fourth inequality uses Lemma D.2. We also make use of $\sum_{s=1}^S n_s = T$.

Since (\mathbf{a}) is bounded by $2GDT$, then taking $\delta = O(1/T^2)$ ensures that,

$$\mathbf{E}[(\mathbf{a})] \leq O(G^2 \log^2(T)/H). \tag{16}$$

Bounding term (b): Similarly the proof of Lemma 4.2 (see Section E.2) we can show that,

$$\mathbf{E}[(\mathbf{b})] = 0.$$

Using Jensen's inequality and combining the above with Equations (14), (16), establishes the lemma:

$$\begin{aligned}
\mathbf{E}[f(\bar{x}_T)] - f(x^*) &\leq \mathbf{E} \left[\sum_{s=1}^S \frac{n_s}{T} (f(x_s) - f(x^*)) \right] \\
&\leq O \left(\frac{G^2 \log^2(T)}{HT} \right).
\end{aligned}$$

□