
An Approximate Inference Approach to Temporal Optimization in Optimal Control

Supplementary Material

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1 Detailed E-&M-Step

1.1 E-Step

In the E-step we need to calculate

$$q(\mathbf{x}_{0:K}) = \mathcal{P}(\mathbf{x}_{0:K} | r_{0:K} = 1, \theta_{0:K-1}^i). \quad (1)$$

We make the approximation

$$q(\mathbf{x}_{0:K}) \approx \prod_k q_k(\mathbf{x}_k), \quad (2)$$

with

$$q_k(\mathbf{x}_k) = \alpha_k(\mathbf{x}_k) \rho_k(\mathbf{x}_k) \beta_k(\mathbf{x}_k), \quad (3)$$

where $\alpha_k = \mathcal{N}(\gamma_k, \mathbf{\Gamma}_k)$, $\beta_k = \mathcal{N}(\lambda_k, \mathbf{\Lambda}_k)$ and $\rho_k = \mathcal{N}[\mathbf{v}_k, \mathbf{V}_k]$ are the approximate messages send to \mathbf{x}_k from \mathbf{x}_{k-1} , \mathbf{x}_{k+1} and r_k respectively, with $\mathcal{N}()$ and $\mathcal{N}[]$ being Gaussians in standard and canonical form respectively. To refine the messages one iterates over the time steps. At each k , based on the current posterior approximation $q(\mathbf{x}_k)$, a point of approximation $\hat{\mathbf{x}}$ is picked and local approximations

$$\begin{aligned} \mathcal{F}(\mathbf{x}) &\approx \mathbf{a}_k + \mathbf{A}_k \mathbf{x} \\ \mathcal{J}(\mathbf{x}) &\approx \frac{1}{2} \mathbf{x}_k^T \mathbf{J}_k \mathbf{x}_k - \mathbf{j}_k^T \mathbf{x}_k \end{aligned} \quad (4)$$

and if necessary

$$\mathcal{C}_k(\mathbf{x}) \approx \frac{1}{2} \mathbf{x}^T \mathbf{V}_k \mathbf{x} - \mathbf{v}_k^T \mathbf{x} \quad (5)$$

are computed. Writing $\tilde{\mathbf{a}}_k = \mathbf{a}_k \theta_k$, $\tilde{\mathbf{A}}_k = \mathbf{I} + \mathbf{A}_k \theta_k$ and $\tilde{\mathbf{v}}_{\theta_k} = \mathbf{v}_k + \mathbf{j}_k \theta_k$, $\tilde{\mathbf{V}}_k = \mathbf{V}_k + \mathbf{J}_k \theta_k$ it can be shown that the updates for the parameters of α_k and β_k are given by

$$\begin{aligned} \gamma_k &= \tilde{\mathbf{a}}_{k-1} + \tilde{\mathbf{A}}_{k-1} (\mathbf{\Gamma}_{k-1}^{-1} + \tilde{\mathbf{V}}_{k-1})^{-1} (\mathbf{\Gamma}_{k-1}^{-1} \gamma_{k-1} + \tilde{\mathbf{v}}_{k-1}) \\ \mathbf{\Gamma}_k &= \tilde{\mathbf{W}}_k + \tilde{\mathbf{A}}_{k-1} (\mathbf{\Gamma}_{k-1}^{-1} + \tilde{\mathbf{V}}_{k-1})^{-1} \tilde{\mathbf{A}}_{k-1}^T \end{aligned} \quad (6)$$

and

$$\begin{aligned} \lambda_k &= -\tilde{\mathbf{A}}_k^{-1} \tilde{\mathbf{a}}_k + \tilde{\mathbf{A}}_k^{-1} (\mathbf{\Lambda}_{k+1}^{-1} + \tilde{\mathbf{V}}_{k+1})^{-1} (\mathbf{\Lambda}_{k+1}^{-1} \lambda_{k+1} + \tilde{\mathbf{v}}_{k+1}) \\ \mathbf{\Lambda}_k &= \tilde{\mathbf{A}}_k^{-1} \left[\tilde{\mathbf{W}} + (\mathbf{\Lambda}_{k+1}^{-1} + \tilde{\mathbf{V}}_{k+1})^{-1} \right] (\tilde{\mathbf{A}}_k^{-1})^T \end{aligned} \quad (7)$$

As all messages are Gaussian, by (2) the posterior approximation is a factorial Gaussian. To calculate $\langle \mathbf{x}_k \mathbf{x}'_{k+1} \rangle$ we require the covariance of the two slice posterior. We have

$$\begin{aligned}
q(\mathbf{x}_k, \mathbf{x}_{k+1}) &= \alpha_k(\mathbf{x}_k) \rho_k(\mathbf{x}_k) \mathcal{P}(\mathbf{x}_{k+1} | \mathbf{x}_k) \rho_k(\mathbf{x}_{k+1}) \beta_k(\mathbf{x}_{k+1}) \\
&= \mathcal{N}(\mathbf{x}_k | \gamma_k, \mathbf{\Gamma}_k) \mathcal{N}[\mathbf{x}_k | \tilde{\mathbf{v}}_k, \tilde{\mathbf{V}}_k] \mathcal{N}(\mathbf{x}_{k+1} | \tilde{a}_k + \tilde{\mathbf{A}}_k \mathbf{x}_k, \tilde{\mathbf{W}}_k) \mathcal{N}[\mathbf{x}_{k+1} | \tilde{\mathbf{v}}_{k+1}, \tilde{\mathbf{V}}_{k+1}] \mathcal{N}(\mathbf{x}_{k+1} | \lambda_{k+1}, \mathbf{\Lambda}_{k+1}) \\
&\propto \mathcal{N}[\mathbf{x}_k | \tilde{\gamma}, \tilde{\mathbf{\Gamma}}] \mathcal{N}(\mathbf{x}_{k+1} | \tilde{a}_k + \tilde{\mathbf{A}}_k \mathbf{x}_k, \tilde{\mathbf{W}}_k) \mathcal{N}[\mathbf{x}_{k+1} | \tilde{\lambda}, \tilde{\mathbf{\Lambda}}] \\
&\propto \mathcal{N} \left[\begin{pmatrix} \mathbf{x}_k \\ \mathbf{x}_{k+1} \end{pmatrix} \middle| \begin{pmatrix} \tilde{\gamma} + \tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{a}_k \\ \tilde{\lambda} + \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k^T \tilde{a}_k \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{\Gamma}} + \tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k & -\tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \\ -\tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k & \tilde{\mathbf{W}}_k^{-1} + \tilde{\mathbf{\Lambda}} \end{pmatrix} \right], \tag{8}
\end{aligned}$$

with

$$\tilde{\gamma} = \mathbf{\Gamma}_k^{-1} \gamma_k + \tilde{\mathbf{v}}_k \tag{9}$$

$$\tilde{\lambda} = \mathbf{\Lambda}_{k+1}^{-1} \lambda_{k+1} + \tilde{\mathbf{v}}_{k+1} \tag{10}$$

Hence using the Schur complement we have

$$\text{Cov}(\mathbf{x}_k, \mathbf{x}_{k+1}) = (\tilde{\mathbf{\Gamma}} + \tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k)^{-1} \tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \text{Var}(\mathbf{x}_{k+1}) \tag{11}$$

Finally

$$\langle \mathbf{x}_k \mathbf{x}_k^T \rangle = \text{Var}(\mathbf{x}_k) + \langle \mathbf{x}_k \rangle \langle \mathbf{x}_k^T \rangle \tag{12}$$

$$\langle \mathbf{x}_k \mathbf{x}_{k+1}^T \rangle = \text{Cov}(\mathbf{x}_k, \mathbf{x}_{k+1}) + \langle \mathbf{x}_k \rangle \langle \mathbf{x}_{k+1}^T \rangle. \tag{13}$$

1.2 M-Step

In the M-Step we find $\text{argmax}_{\theta_{0:K-1}} \mathcal{Q}(\theta_{0:K-1} | \theta_{0:K-1}^{i+1})$ or at least $\nabla_{\theta_{0:K-1}} \mathcal{Q}(\theta_{0:K-1} | \theta_{0:K-1}^{i+1})$. We have

$$\begin{aligned}
\mathcal{Q}(\theta_{0:K-1} | \theta_{0:K-1}^{i+1}) &= \langle \log \mathcal{P}(\mathbf{x}_{0:K}, r_{0:K} | \theta) | r_{0:K} = 1 \rangle \\
&= \left\langle \log \mathcal{P}(\mathbf{x}_0) + \log \mathcal{P}(r_{k+1} = 1 | \mathbf{x}_{k+1}) + \sum_{k=0}^{K-1} [\log \mathcal{P}(\mathbf{x}_{k+1} | \mathbf{x}_k, \theta_k) + \log \mathcal{P}(r_k = 1 | \mathbf{x}_k, \theta_k)] \right\rangle \\
&\quad + \log \mathcal{P}(r_K | x_K) \\
&= \sum_{k=0}^{K-1} \langle \log \mathcal{P}(\mathbf{x}_{k+1} | \mathbf{x}_k, \theta_k) \rangle - \sum_{k=1}^{K-1} [T(\theta_k) + \theta_k \langle \mathcal{J}(\mathbf{x}_k) \rangle] + \text{constant}. \tag{14}
\end{aligned}$$

As $q^i(\mathbf{x}_{0:K+1}) = \mathcal{N}(\mathbf{x}_{0:K+1} | \mathbf{m}, \mathbf{S})$ the marginals $q^i(\mathbf{x}_{k,k+1})$ can be easily formed (cf. E-Step) and we have, under the approximations in (4) (cf. main text) for the individual terms of the first sum

$$\begin{aligned}
\langle \log \mathcal{P}(\mathbf{x}_{k+1} | \mathbf{x}_k, \theta_k) \rangle_{q^i(\mathbf{x}_{k,k+1})} &= \left\langle -\frac{D_x}{2} \log |\tilde{\mathbf{W}}_k| - \frac{1}{2} (\mathbf{x}_{k+1} - \tilde{a}_k - \tilde{\mathbf{A}}_k \mathbf{x}_k)^T \tilde{\mathbf{W}}_k^{-1} (\mathbf{x}_{k+1} - \tilde{a}_k - \tilde{\mathbf{A}}_k \mathbf{x}_k) \right\rangle_{q^i(\mathbf{x}_{k,k+1})} \\
&= -\frac{D_x}{2} \log |\tilde{\mathbf{W}}_k| - \frac{1}{2} \left\langle (\mathbf{x}_{k+1} - \tilde{a}_k - \tilde{\mathbf{A}}_k \mathbf{x}_k)^T \tilde{\mathbf{W}}_k^{-1} (\mathbf{x}_{k+1} - \tilde{a}_k - \tilde{\mathbf{A}}_k \mathbf{x}_k) \right\rangle_{q^i(\mathbf{x}_{k,k+1})} \tag{15}
\end{aligned}$$

where $D_x = \dim\{\mathbf{x}\}$. Here the required expectation over the quadratic form can be calculated as

$$\begin{aligned}
& \left\langle (\mathbf{x}_{k+1} - \tilde{\mathbf{a}}_k - \tilde{\mathbf{A}}_k \mathbf{x}_k)^T \tilde{\mathbf{W}}_k^{-1} (\mathbf{x}_{k+1} - \tilde{\mathbf{a}}_k - \tilde{\mathbf{A}}_k \mathbf{x}_k) \right\rangle \\
&= \left\langle \mathbf{x}_{k+1}^T \tilde{\mathbf{W}}_k^{-1} \mathbf{x}_{k+1} \right\rangle - 2 \left\langle \mathbf{x}_{k+1}^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k \mathbf{x}_k \right\rangle + \left\langle \mathbf{x}_k^T \tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k \mathbf{x}_k \right\rangle \\
&\quad - 2 \left\langle \mathbf{x}_{k+1}^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k \right\rangle + \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k + 2 \left\langle \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k \mathbf{x}_k \right\rangle \\
&= \left\langle \text{Tr}(\tilde{\mathbf{W}}_k^{-1} \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T) \right\rangle - 2 \left\langle \text{Tr}((\tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k)^T \mathbf{x}_{k+1} \mathbf{x}_k^T) \right\rangle + \left\langle \text{Tr}((\tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k)^T \mathbf{x}_k \mathbf{x}_k^T) \right\rangle \quad (16) \\
&\quad - 2 \left\langle \mathbf{x}_{k+1}^T \right\rangle \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k + \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k + 2 \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k \langle \mathbf{x}_k \rangle \\
&= \text{Tr}(\tilde{\mathbf{W}}_k^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T \rangle) - 2 \text{Tr}(\tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_k^T \rangle) + \text{Tr}(\tilde{\mathbf{A}}_k \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k^T \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) \\
&\quad - 2 \left\langle \mathbf{x}_{k+1}^T \right\rangle \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k + \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k + 2 \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k \langle \mathbf{x}_k \rangle
\end{aligned}$$

Hence up to an additive constant we have

$$\begin{aligned}
\mathcal{Q}(\theta_{0:K} | \theta_{0:K}^i) &= - \sum_{k=0}^{K-1} \frac{D_x}{2} \log |\tilde{\mathbf{W}}_k| \\
&\quad - \frac{1}{2} \sum_{k=0}^{K-1} \left[\text{Tr}(\tilde{\mathbf{W}}_k^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T \rangle) - 2 \text{Tr}(\tilde{\mathbf{A}}_k^T \tilde{\mathbf{W}}_k^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_k^T \rangle) + \text{Tr}(\tilde{\mathbf{A}}_k \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k^T \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) \right. \\
&\quad \left. - 2 \left\langle \mathbf{x}_{k+1}^T \right\rangle \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k + \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{a}}_k + 2 \tilde{\mathbf{a}}_k^T \tilde{\mathbf{W}}_k^{-1} \tilde{\mathbf{A}}_k \langle \mathbf{x}_k \rangle \right] \\
&\quad - \sum_{k=0}^{K-1} \mathcal{T}(\theta_k) + \theta_k \left[\text{Tr}(\frac{1}{2} \mathbf{J}_k \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) - \mathbf{j}_k \langle \mathbf{x}_k \rangle \right] \quad (17)
\end{aligned}$$

where we used (4) to also approximate $\langle \mathcal{J}(\mathbf{x}_k) \rangle$. Substituting $\tilde{\mathbf{a}}_k = \theta_k \mathbf{a}_k$, $\tilde{\mathbf{A}}_k = (\mathbf{I} - \theta_k \mathbf{A})$ and $\tilde{\mathbf{W}} = \theta_k \mathbf{W}$ we obtain

$$\begin{aligned}
\mathcal{Q}(\theta_{0:K-1} | \theta_{0:K-1}^i) &= - \sum_{k=0}^{K-1} \frac{D_x}{2} \log |\theta_k \mathbf{W}| - \frac{1}{2} \sum_{k=0}^{K-1} [\text{Tr}(\theta_k^{-1} \mathbf{W}^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T \rangle)] \\
&\quad + \sum_{k=0}^{K-1} [\text{Tr}((\mathbf{I} + \theta_k \mathbf{A})^T \theta_k^{-1} \mathbf{W}^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_k^T \rangle)] - \frac{1}{2} \sum_{k=0}^{K-1} [\text{Tr}((\mathbf{I} + \theta_k \mathbf{A}) \theta_k^{-1} \mathbf{W}^{-1} (\mathbf{I} + \theta_k \mathbf{A})^T \langle \mathbf{x}_k \mathbf{x}_k^T \rangle)] \\
&\quad - \frac{1}{2} \sum_{k=0}^{K-1} [\theta_k \mathbf{a}_k^T \mathbf{W}^{-1} \mathbf{a}_k + 2 \theta_k \mathbf{a}_k^T \mathbf{W}^{-1} (\mathbf{I} + \theta_k \mathbf{A}) \langle \mathbf{x}_k \rangle] \\
&\quad - \sum_{k=0}^{K-1} \mathcal{T}(\theta_k) + \theta_k \left[\frac{1}{2} \text{Tr}(\mathbf{J}_k \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) - \mathbf{j}_k \langle \mathbf{x}_k \rangle \right] + \text{constant} \quad (18)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\mathcal{Q}(\theta_{0:K-1}|\theta_{0:K-1}^i) &= - \sum_{k=0}^{K-1} \frac{D_x}{2} \log \theta_k^{D_x} - \frac{1}{2} \sum_{k=0}^{K-1} \theta_k^{-1} \text{Tr}(\mathbf{W}^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T \rangle) \\
&+ \sum_{k=0}^{K-1} \theta_k^{-1} \text{Tr}(\mathbf{W}^{-1} \langle \mathbf{x}_{k+1} \mathbf{x}_k^T \rangle) - \frac{1}{2} \sum_{k=0}^{K-1} \theta_k^{-1} \text{Tr}(\mathbf{W}^{-1} \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) \\
&- \frac{1}{2} \sum_{k=0}^{K-1} \theta_k \text{Tr}(\mathbf{A} \mathbf{W}^{-1} \mathbf{A}^T \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) - \frac{1}{2} \sum_{k=0}^{K-1} \theta_k \mathbf{a}_k^T \mathbf{W}^{-1} \mathbf{a}_k + 2\theta_k \mathbf{a}_k^T \mathbf{W}^{-1} \mathbf{A}_k \langle \mathbf{x}_k \rangle \\
&- \sum_{k=0}^{K-1} \mathcal{T}(\theta_k) + \theta_k \left[\frac{1}{2} \text{Tr}(\mathbf{J}_k \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) - \mathbf{j}_k \langle \mathbf{x}_k \rangle \right] + \text{constant} \tag{19} \\
&= - \sum_{k=0}^{K-1} \frac{D_x^2}{2} \log \theta_k - \frac{1}{2} \sum_{k=0}^{K-1} \theta_k^{-1} \text{Tr}(\mathbf{W}^{-1} (\langle \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T \rangle - 2 \langle \mathbf{x}_{k+1} \mathbf{x}_k^T \rangle + \langle \mathbf{x}_k \mathbf{x}_k^T \rangle)) \\
&- \frac{1}{2} \sum_{k=0}^{K-1} \theta_k \text{Tr}(\mathbf{A} \mathbf{W}^{-1} \mathbf{A}^T \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) - \sum_{k=0}^{K-1} \frac{1}{2} \theta_k \mathbf{a}_k^T \mathbf{W}^{-1} \mathbf{a}_k + \theta_k \mathbf{a}_k^T \mathbf{W}^{-1} \mathbf{A}_k \langle \mathbf{x}_k \rangle \\
&- \sum_{k=0}^{K-1} \mathcal{T}(\theta_k) + \theta_k \left[\frac{1}{2} \text{Tr}(\mathbf{J}_k \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) - \mathbf{j}_k \langle \mathbf{x}_k \rangle \right] + \text{constant}
\end{aligned}$$

Now taking the partial derivatives w.r.t. the individual θ 's we have (cf. main text)

$$\begin{aligned}
\frac{\partial \mathcal{Q}}{\partial \theta_k} &= \frac{1}{2} \theta_k^{-2} \text{Tr}(\mathbf{W}^{-1} (\langle \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T \rangle - 2 \langle \mathbf{x}_{k+1} \mathbf{x}_k^T \rangle + \langle \mathbf{x}_k \mathbf{x}_k^T \rangle)) - \frac{D_x^2}{2} \theta_k^{-1} \\
&- \frac{1}{2} \left[\text{Tr}(\mathbf{A} \mathbf{W}^{-1} \mathbf{A}^T \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) + 2 \frac{d\mathcal{T}}{d\theta} \Big|_{\theta_k} + \mathbf{a}_k^T \mathbf{W}^{-1} \mathbf{a}_k + 2\mathbf{a}_k^T \mathbf{W}^{-1} \mathbf{A}_k \langle \mathbf{x}_k \rangle \right. \\
&\left. + \text{Tr}(\mathbf{J}_k \langle \mathbf{x}_k \mathbf{x}_k^T \rangle) - 2\mathbf{j}_k \langle \mathbf{x}_k \rangle \right]. \tag{20}
\end{aligned}$$