Supplementary material
Switched Latent Force Models
for Movement Segmentation

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1 Introduction

The motor primitive idea is similar to the latent force model one. We want to use a set of templates for basic motions in order to generate more complex ones. The analogy we can think of is the generation of speech, in which phonemes are used to generate words and sentences.

Motor primitive model

Motor primitives employ the concept of autonomous dynamical system in which the independent variable is first parameterized by a first order homogenous dynamical system. The output of this system is used as the independent variable of the inducing force of a second order differential equation [2]. The first system is known as the *canonical system* and its form depends on the type of movement that is to be represented: point attractive and limit cycle behaviors are the two most basic behaviors of nonlinear dynamical systems. In motor control these correspond to discrete and rythmic movements.

Latent force model

The latent force model was first introduced in [1]. A set of coupled second order ordinary differential equations was employed for human-balancing movement representation. Here we only review the basic form for the covariance function in the Gaussian process formulation of the Latent force model. More details and applications can be found in [1].

A set of D outputs $\{f_d(t)\}_{d=1}^D$ (where each of them describes the relative position of a particle wrt to a set of reference points in a spring-damper-mass system) is represented by a Gaussian process with covariance function,

$$k_{f_d f_{d'}}(t,t') = \sum_{q=1}^{Q} \frac{S_{qd}S_{qd'}\sqrt{\pi\ell_q^2}}{8A_d A_{d'}\omega_d\omega_{d'}} k_{f_d f_{d'}}^{(q)}(t,t'),$$

with A_d the mass of system d, ω_d the angular frequency, S_{qd} the relative strength of latent force q over output d, ℓ_q the length-scale of the RBF covariance for the Gaussian process that describes the latent force q and $k_{f_d f_{d'}}^{(q)}(t,t')$, the cross-covariance between the d-th and d'-th outputs under the effect of the q-th latent force, and is given by

$$k_{f_d f_{d'}}^{(q)}(t,t') = h^q(\widetilde{\gamma}_{d'},\gamma_d,t,t') + h^q(\gamma_d,\widetilde{\gamma}_{d'},t',t) + h^q(\gamma_{d'},\widetilde{\gamma}_d,t,t') + h^q(\widetilde{\gamma}_d,\gamma_{d'},t',t)$$
$$- h^q(\widetilde{\gamma}_{d'},\widetilde{\gamma}_d,t,t') - h^q(\widetilde{\gamma}_d,\widetilde{\gamma}_{d'},t',t) - h^q(\gamma_{d'},\gamma_d,t,t') - h^q(\gamma_d,\gamma_{d'},t',t),$$

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where $\gamma_d = \alpha_d + j\omega_d$, $\widetilde{\gamma}_d = \alpha_d - j\omega_d$, and

$$h^q(\gamma_{d'}, \gamma_d, t, t') = \frac{1}{\gamma_d + \gamma_{d'}} \left[\Upsilon^q(\gamma_{d'}, t', t) - \exp(-\gamma_d t)\Upsilon^q(\gamma_{d'}, t', 0) \right].$$

$$\Upsilon^q(\gamma_{d'}, t, t') = \underbrace{2\exp\left(\frac{\ell_q^2 \gamma_{d'}^2}{4}\right) \exp(-\gamma_{d'}(t - t'))}_{\psi^q(\gamma_{d'}, t, t')} - \underbrace{\exp\left(-\frac{(t - t')^2}{\ell_q^2}\right) \mathrm{w}(jz(t, t'))}_{\upsilon^q(\gamma_{d'}, t, t')}$$

$$- \underbrace{\exp\left(-\frac{(t')^2}{\ell_q^2}\right) \exp(-\gamma_{d'}t)\mathrm{w}(-jz(0, t'))}_{\varphi^q(\gamma_{d'}, t, t')}$$

$$= \psi^q(\gamma_{d'}, t, t') - \upsilon^q(\gamma_{d'}, t, t') - \varphi^q(\gamma_{d'}, t, t'),$$

and $z(t, t') = (t - t')/\ell_q - (\ell_q\gamma_{d'})/2$. Note that $z(t, t') \in \mathbb{C}$, and $\mathrm{w}(jz)$ in the above equation, for $z \in \mathbb{C}$, denotes Faddeeva's function $\mathrm{w}(jz) = \exp(z^2)\mathrm{erfc}(z)$, where $\mathrm{erfc}(z)$ is the complex version of the complementary error function, $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-v^2)\mathrm{d}v$. Faddeeva's function is usually considered the complex equivalent of the error function, since $|\mathrm{w}(jz)|$ is bounded whenever the imaginary part of jz is greater or equal than zero, and is the key to achieving a good numerical stability when computing $\Upsilon^q(\gamma_{d'}, t, t')$ and its gradients.

2 Switching forces

Figure 1 shows a cartoon representation of output $z_d(t)$ switching its behavior between points t_0, t_1, t_2 and t_3. For each interval (t_{i-1}, t_i), only the latent force $u_{i-1}(t)$ is active.
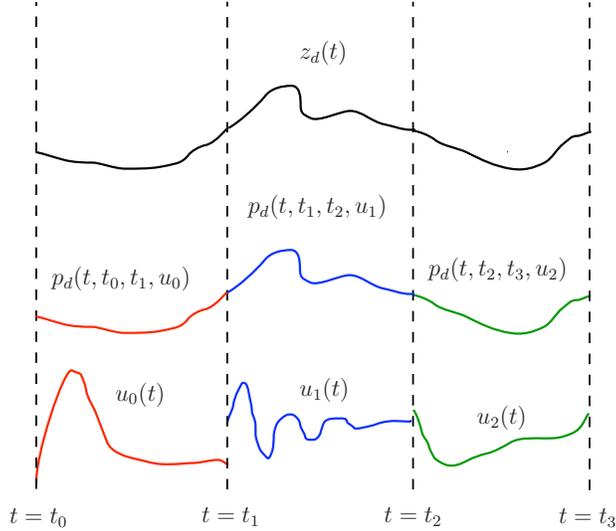
Figure 1: A pictorial representation of the switching scenario for $z_d(t)$

2.1 Definition of the model

Taking into account the initial conditions, the solution to the second order model is given as

$$y_d(t) = y_d(0)e^{-\alpha_d t}\left[\cos(\omega_d t) + \frac{\alpha_d}{\omega_d}\sin(\omega_d t)\right] + \dot{y}_d(0)\left[\frac{e^{-\alpha_d t}}{\omega_d}\sin(\omega_d t)\right]$$

$$+ \frac{S_d}{A_d\omega_d}\int_0^t e^{-\alpha_d(t-\tau)}\sin[(t-\tau)\omega_d]u(\tau)\mathrm{d}\tau,$$

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where $y_d(0)$ and $\dot{y}_d(0)$ are the initial conditions. This is the basic equation we need to use to express the covariance function for the switching model. The uncertainty in this model is due to the latent force $u(t)$ and the initial conditions $y_d(0)$ and $\dot{y}_d(0)$. For simplicity, we write the above equation as

$$y_d(t) = c_d(t)y_d(0) + e_d(t)\dot{y}_d(0) + f_d(t), \tag{1}$$

with

$$c_d(t) = e^{-\alpha_d t}\left[\cos(\omega_d t) + \frac{\alpha_d}{\omega_d}\sin(\omega_d t)\right]$$

$$e_d(t) = \left[\frac{e^{-\alpha_d t}}{\omega_d}\sin(\omega_d t)\right]$$

$$f_d(t) = \frac{S_d}{A_d\omega_d}\int_0^t e^{-\alpha_d(t-\tau)}\sin[(t-\tau)\omega_d]u(\tau)\mathrm{d}\tau = \int_0^t G_d(t-\tau)u(\tau)\mathrm{d}\tau.$$

We'll need also the velocity $v_d(t)$, which is is given as

$$v_d(t) = \frac{\mathrm{d}y_d(t)}{\mathrm{d}t} = g_d(t)y_d(0) + h_d(t)\dot{y}_d(0) + m_d(t), \tag{2}$$

with

$$g_d(t) = \frac{\mathrm{d}c_d(t)}{\mathrm{d}t} = -e^{-\alpha_d t}\sin(\omega_d t)\left(\frac{\alpha_d^2}{\omega_d} + \omega_d\right)$$

$$h_d(t) = \frac{\mathrm{d}e_d(t)}{\mathrm{d}t} = -e^{-\alpha_d t}\left[\frac{\alpha_d}{\omega_d}\sin(\omega_d t) - \cos(\omega_d t)\right]$$

$$m_d(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t G_d(t-\tau)u(\tau)\mathrm{d}\tau\right).$$

Furthermore, we also need the acceleration, given as

$$a_d(t) = \frac{\mathrm{d}v_d(t)}{\mathrm{d}t} = r_d(t)y_d(0) + b_d(t)\dot{y}_d(0) + w_d(t), \tag{3}$$

with

$$r_d(t) = \frac{\mathrm{d}h_d(t)}{\mathrm{d}t} = e^{-\alpha_d t}\left(\frac{\alpha_d^2}{\omega_d} + \omega_d\right)\left[\alpha_d\sin(\omega_d t) - \omega_d\cos(\omega_d t)\right]$$

$$b_d(t) = \frac{\mathrm{d}g_d(t)}{\mathrm{d}t} = e^{-\alpha_d t}\left[\left(\frac{\alpha_d^2}{\omega_d} - \omega_d\right)\sin(\omega_d t) - 2\alpha_d\cos(\omega_d t)\right]$$

$$w_d(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\int_0^t G_d(t-\tau)u(\tau)\mathrm{d}\tau\right).$$

The input space is divided in non-overlapping intervals $[t_{q-1}, t_q]_{q=1}^Q$ and for each one of these intervals, only one force $u_{q-1}(t)$ out of Q forces is active, this is, there are $\{u_{q-1}\}_{q=1}^Q$ forces. The force $u_{q-1}(t)$ is activated after time t_{q-1} and desactivated after time t_q. We can use the basic model in the equation before to describe the contribution to the output due to the sequential activation of these forces. An output $z_d(t)$ at a particular time instant t, in the interval (t_{q-1}, t_q), is expressed as

$$z_d(t, t_{q-1}, t_q) = p_d(t, t_{q-1}, t_q, u_{q-1}), \quad \text{for} \quad 1 \le d \le D,$$

where $p_d(t, t_{q-1}, t_q, u_{q-1})$ uses the model for $y_d(t)$ in equation (1) as

$$p_d(t, t_{q-1}, t_q, u_{q-1}) = y_d(t)\big|_{t_{q-1}} = c_d(t - t_{q-1})y_d(t_{q-1}) + e_d(t - t_{q-1})\dot{y}_d(t_{q-1})$$
$$+ f_d(t, t_{q-1}, t_q, u_{q-1}).$$

Notice that there are as many intervals $\{(t_{q-1}, t_q)\}_{q=1}^Q$ as latent forces $\{u_q(t)\}_{q=1}^Q$. For simplicity, we write $z_d(t, t_{q-1}, t_q)$ as $z_d(t)$. In the above equation, $y_d(t)\big|_{t_{q-1}}$ expresses that $y_d(t)$ has to be evaluated with the initial condition specified at t_{q-1} and

$$f_d(t, t_{q-1}, t_q, u_{q-1}) = \int_0^{t-t_{q-1}} G_d(t - t_{q-1} - \tau)u_{q-1}(\tau)\mathrm{d}\tau. \tag{4}$$

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Expression $f_d(t, t_{q-1}, t_q, u_{q-1})$ is a function of four arguments: the first argument, t, refers to the independent variable inside the kernel smoothing function $G_d(t - \tau)$ and in the upper limit of the convolution transform; the second argument, t_{q-1}, and third argument t_q specify the lower and upper limits of the time interval for which the convolution is being computed and the fourth argument, u_{q-1}, specifies the latent force acting in this interval. Additionally, we define a similar function for the velocity $\dot{z}_d(t)$ as

$$\dot{z}_d(t, t_q, t_{q-1}) = \xi_d(t, t_{q-1}, t_q, u_{q-1}), \quad \text{for} \quad 1 \leq d \leq D,$$

where

$$\xi_d(t, t_{q-1}, t_q, u_{q-1}) = v_d(t)\big|_{t_{q-1}} = g_d(t - t_{q-1})y_d(t_{q-1}) + h_d(t - t_{q-1})\dot{y}_d(t_{q-1})$$
$$+ m_d(t, t, t_{q-1}, t_q, u_{q-1}),$$

and $m_d(t, t_{q-1}, t_q, u_{q-1})$ follows

$$m_d(t, t_{q-1}, t_q, u_{q-1}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t - t_{q-1}} G_d(t - t_{q-1} - \tau)u_{q-1}(\tau)\mathrm{d}\tau \right). \tag{5}$$

Again, for simplicity, we write $\dot{z}_d(t, t_q, t_{q-1})$ as $\dot{z}_d(t)$. The initial conditions $y_d(t_{q-1})$ and $\dot{y}_d(t_{q-1})$ can be defined again in terms of $z_d(t)$ and $\dot{z}_d(t)$

$$y_d(t_{q-1}) = z_d(t_{q-1}) = p_d(t_{q-1}, t_{q-2}, t_{q-1}, u_{q-2}),$$
$$\dot{y}_d(t_{q-1}) = \dot{z}_d(t_{q-1}) = \xi_d(t_{q-1}, t_{q-2}, t_{q-1}, u_{q-2}).$$

Without loss of generality, we assume that the initial conditions at $t = t_0$ for all d, are parameters of the model. This is $y_d(t_0)$ and $\dot{y}_d(t_0)$ are parameters that need to be estimated. Eventually, we might need to put a prior over them. A similar expression is obtained for the acceleration $\ddot{z}_d(t)$.

Example 1. Suppose we have $Q = 3$ as in figure 1. Then, the outputs $z_d(t)$ will be given as $z_d(t, t_0, t_1) = p_d(t, t_0, t_1, u_0)$, $z_d(t, t_1, t_2) = p_d(t, t_1, t_2, u_1)$ and $z_d(t, t_2, t_3) = p_d(t, t_2, t_3, u_2)$. Equally, the velocities $\dot{z}_d(t)$ will follow $\dot{z}_d(t, t_0, t_1) = \xi_d(t, t_0, t_1, u_0)$, $\dot{z}_d(t, t_1, t_2) = \xi_d(t, t_1, t_2, u_1)$ and $\dot{z}_d(t, t_2, t_3) = \xi_d(t, t_2, t_3, u_2)$. We also have the initial conditions. For t_0, the initial conditions are parameters $y_d(t_0)$ and $\dot{y}_d(t_0)$. For the intervals starting at t_1 and t_2, the initial conditions are given as $y_d(t_1) = p_d(t_1, t_0, t_1, u_0)$ and $y_d(t_2) = p_d(t_2, t_1, t_2, u_1)$. And for the velocities $\dot{y}_d(t_1) = \xi_d(t_1, t_1, t_0, t_1, u_0)$ and $\dot{y}_d(t_2) = \xi_d(t_2, t_1, t_2, u_1)$.

2.2 Covariance for the outputs

In general, we need to compute the covariance $\mathrm{cov}[z_d(t), z_{d'}(t')]$ for every time interval (t_{q-1}, t_q) and for intervals (t_{q-1}, t_q) and $(t_{q'-1}, t'_q)$. The covariance $\mathrm{cov}[z_d(t), z_{d'}(t')]$ for time interval (t_{q-1}, t_q) is given as

$$\mathrm{cov}[z_d(t), z_{d'}(t')] = \mathrm{cov}\left[p_d(t, t_{q-1}, t_q, u_{q-1}), p_{d'}(t', t_{q-1}, t_q, u_{q-1})\right]. \tag{6}$$

And the covariance $\mathrm{cov}[z_d(t), z_{d'}(t')]$ for time intervals (t_{q-1}, t_q) and $(t_{q'-1}, t'_q)$ is given as

$$\mathrm{cov}[z_d(t), z_{d'}(t')] = \mathrm{cov}\left[p_d(t, t, t_{q-1}, t_q, u_{q-1}), p_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\right]. \tag{7}$$

2.2.1 Covariance for interval (t_{q-1}, t_q)

The covariance in equation (6), follows

$$\mathrm{cov}\{[c_d(t - t_{q-1})y_d(t_{q-1}) + e_d(t - t_{q-1})\dot{y}_d(t_{q-1}) + f_d(t, t, t_{q-1}, t_q, u_{q-1})]$$
$$[c_{d'}(t' - t_{q-1})y_{d'}(t_{q-1}) + e_{d'}(t' - t_{q-1})\dot{y}_{d'}(t_{q-1}) + f_{d'}(t', t', t_{q-1}, t_q, u_{q-1})]\}$$
$$= c_d(t - t_{q-1})c_{d'}(t' - t_{q-1})\,\mathrm{cov}\{y_d(t_{q-1})y_{d'}(t_{q-1})\} + c_d(t - t_{q-1})e_{d'}(t' - t_{q-1})\,\mathrm{cov}\{y_d(t_{q-1})\dot{y}_{d'}(t_{q-1})\}$$
$$+ c_d(t - t_{q-1})\,\mathrm{cov}\{y_d(t_{q-1})f_{d'}(t', t', t_{q-1}, t_q, u_{q-1})\} + e_d(t - t_{q-1})c_{d'}(t' - t_{q-1})\,\mathrm{cov}\{\dot{y}_d(t_{q-1})y_{d'}(t_{q-1})\}$$
$$+ e_d(t - t_{q-1})e_{d'}(t' - t_{q-1})\,\mathrm{cov}\{\dot{y}_d(t_{q-1})\dot{y}_{d'}(t_{q-1})\} + e_d(t - t_{q-1})\,\mathrm{cov}\{\dot{y}_d(t_{q-1})f_{d'}(t', t', t_{q-1}, t_q, u_{q-1})\}$$
$$+ c_{d'}(t' - t_{q-1})\,\mathrm{cov}\{f_d(t, t, t_{q-1}, t_q, u_{q-1})y_{d'}(t_{q-1})\} + e_{d'}(t' - t_{q-1})\,\mathrm{cov}\{f_d(t, t, t_{q-1}, t_q, u_{q-1})\dot{y}_{d'}(t_{q-1})\}$$
$$+ \mathrm{cov}\{f_d(t, t, t_{q-1}, t_q, u_{q-1})f_{d'}(t', t', t_{q-1}, t_q, u_{q-1})\}.$$

4

The terms $\text{cov}\{y_d(t_{q-1})y_{d'}(t_{q-1})\}$, $\text{cov}\{y_d(t_{q-1})\dot{y}_{d'}(t_{q-1})\}$, $\text{cov}\{\dot{y}_d(t_{q-1})y_{d'}(t_{q-1})\}$ and $\text{cov}\{\dot{y}_d(t_{q-1})\dot{y}_{d'}(t_{q-1})\}$ are obtained from the covariance already computed. These terms are equivalent as $k_{z_d,z_{d'}}(t_{q-1},t_{q-1}) = \text{cov}\{y_d(t_{q-1})y_{d'}(t_{q-1})\}$, $k_{z_d,\dot{z}_{d'}}(t_{q-1},t_{q-1}) = \text{cov}\{y_d(t_{q-1})\dot{y}_{d'}(t_{q-1})\}$, $k_{\dot{z}_d,z_{d'}}(t_{q-1},t_{q-1}) = \text{cov}\{\dot{y}_d(t_{q-1})y_{d'}(t_{q-1})\}$ and $k_{\dot{z}_d,\dot{z}_{d'}}(t_{q-1},t_{q-1}) = \text{cov}\{\dot{y}_d(t_{q-1})\dot{y}_{d'}(t_{q-1})\}$. The expressions $\text{cov}\{y_d(t_{q-1})f_{d'}(t',t',t_{q-1},t_q,u_{q-1})\}$, $\text{cov}\{\dot{y}_d(t_{q-1})f_{d'}(t',t',t_{q-1},t_q,u_{q-1})\}$, $\text{cov}\{f_d(t,t,t_{q-1},t_q,u_{q-1})y_{d'}(t_{q-1})\}$ and $\text{cov}\{f_d(t,t,t_{q-1},t_q,u_{q-1})\dot{y}_{d'}(t_{q-1})\}$ are zero. This can be seen from the fact that terms like $y_d(t_{q-1})$ are a obtained as a result of terms $y_d(t_{k-1})$ and $f_d(t_{k-1},t_{k-1},t_k,u_k)$, for $k < q$, and the covariance between those terms with $f_d(t,t_{q-1},t_q,u_{q-1})$ is zero. Finally, the term $\text{cov}\{f_d(t,t_{q-1},t_q,u_{q-1})f_{d'}(t',t_{q-1},t_q,u_{q-1})\}$ is denoted as $k_{f_d,f_{d'}}^{(q-1)}(t,t')$.

In this way the covariance $\text{cov}\left[p_d(t,t,t_{q-1},t_q,u_{q-1}),p_{d'}(t',t',t_{q-1},t_q,u_{q-1})\right]$ is equal to

$$
\begin{aligned}
c_d(t-t_{q-1})c_{d'}(t'-t_{q-1})k_{z_d,z_{d'}}(t_{q-1},t_{q-1}) &+ c_d(t-t_{q-1})e_{d'}(t'-t_{q-1})k_{z_d,\dot{z}_{d'}}(t_{q-1},t_{q-1}) \\
+e_d(t-t_{q-1})c_{d'}(t'-t_{q-1})k_{\dot{z}_d,z_d}(t_{q-1},t_{q-1}) &+ e_d(t-t_{q-1})e_{d'}(t'-t_{q-1})k_{\dot{z}_d,\dot{z}_{d'}}(t_{q-1},t_{q-1}) \\
&+ k_{f_d,f_{d'}}^{(q-1)}(t,t').
\end{aligned}
\tag{8}
$$

The term $k_{z_d,z_{d'}}(t_{q-1},t_{q-1})$ is equal to $\text{cov}[z_d(t_{q-1},t_{q-2},t_{q-1}),z_{d'}(t_{q-1},t_{q-2},t_{q-1})]$ and analog expressions are obtained for $k_{z_d,\dot{z}_{d'}}(t_{q-1},t_{q-1})$, $k_{\dot{z}_d,z_{d'}}(t_{q-1},t_{q-1})$ and $k_{\dot{z}_d,\dot{z}_{d'}}(t_{q-1},t_{q-1})$.

Example 1 (Continued). We continue with the example in figure 1. We need to compute the covariance $k_{z_d,z_{d'}}(t,t')$ in the intervals $(t_0,t_1]$, $(t_1,t_2]$ and $(t_2,t_3]$. For the covariance in the interval $(t_0,t_1]$, we have

$$
\begin{aligned}
\text{cov}[z_d(t),z_{d'}(t')] &= \text{cov}[p_d(t,t_0,t_1,u_0),p_{d'}(t,t_0,t_1,u_0)] \\
&= c_d(t-t_0)c_{d'}(t'-t_0)k_{z_d,z_{d'}}(t_0,t_0) + c_d(t-t_0)e_{d'}(t'-t_0)k_{z_d,\dot{z}_{d'}}(t_0,t_0) \\
&\quad + e_d(t-t_0)c_{d'}(t'-t_0)k_{\dot{z}_d,z_d}(t_0,t_0) + e_d(t-t_0)e_{d'}(t'-t_0)k_{\dot{z}_d,\dot{z}_{d'}}(t_0,t_0) \\
&\quad + k_{f_d,f_{d'}}^{(0)}(t,t').
\end{aligned}
$$

We assume the terms $k_{z_d,z_{d'}}(t_0,t_0)$, $k_{z_d,\dot{z}_{d'}}(t_0,t_0)$, $k_{\dot{z}_d,z_d}(t_0,t_0)$ and $k_{\dot{z}_d,\dot{z}_{d'}}(t_0,t_0)$ are parameters that have to be estimated in the inference process. We also have access to $\text{cov}[z_d(t),\dot{z}_{d'}(t')]$, $\text{cov}[\dot{z}_d(t),z_{d'}(t')]$ and $\text{cov}[\dot{z}_d(t),\dot{z}_{d'}(t')]$. With these expressions we compute $k_{z_d,z_{d'}}(t_1,t_1) = \text{cov}[z_d(t_1),z_{d'}(t_1)]$, $k_{z_d,\dot{z}_{d'}}(t_1,t_1) = \text{cov}[z_d(t_1),\dot{z}_{d'}(t_1)]$, $k_{\dot{z}_d,z_d}(t_1,t_1) = \text{cov}[\dot{z}_d(t_1),z_{d'}(t_1)]$ and $k_{\dot{z}_d,\dot{z}_{d'}}(t_1,t_1) = \text{cov}[\dot{z}_d(t_1),\dot{z}_{d'}(t_1)]$, that are needed to compute the covariance in the next interval.

For the covariance in the interval $(t_1,t_2]$, we have

$$
\text{cov}[z_d(t),z_{d'}(t')] = \text{cov}[p_d(t,t_1,t_2,u_1),p_{d'}(t',t_1,t_2,u_1)],
\tag{9}
$$

which follows the same form that equation (8)

$$
\begin{aligned}
c_d(t-t_1)c_{d'}(t'-t_1)k_{z_d,z_{d'}}(t_1,t_1) &+ c_d(t-t_1)e_{d'}(t'-t_1)k_{z_d,\dot{z}_{d'}}(t_1,t_1) \\
+ e_d(t-t_1)c_{d'}(t'-t_1)k_{\dot{z}_d,z_d}(t_1,t_1) &+ e_d(t-t_1)e_{d'}(t'-t_1)k_{\dot{z}_d,\dot{z}_{d'}}(t_1,t_1) + k_{f_d,f_{d'}}^{(1)}(t,t').
\end{aligned}
$$

With the final expression for $\text{cov}[z_d(t,t_1,t_2),z_{d'}(t',t_1,t_2)]$, we compute $k_{z_d,z_{d'}}(t_2,t_2) = \text{cov}[z_d(t_2),z_{d'}(t_2)]$, $k_{z_d,\dot{z}_{d'}}(t_2,t_2) = \text{cov}[z_d(t_2),\dot{z}_{d'}(t_2)]$, $k_{\dot{z}_d,z_d}(t_2,t_2) = \text{cov}[\dot{z}_d(t_2),z_{d'}(t_2)]$ and $k_{\dot{z}_d,\dot{z}_{d'}}(t_2,t_2) = \text{cov}[\dot{z}_d(t_2),\dot{z}_{d'}(t_2)]$, that are needed to compute the covariance in the next interval.

We finally need the covariance for the interval $(t_2,t_3]$. This covariance is computed as

$$
\text{cov}[z_d(t),z_{d'}(t')] = \text{cov}[p_d(t,t_2,t_3,u_2),p_{d'}(t',t_2,t_3,u_2)],
\tag{10}
$$

given as

$$
\begin{aligned}
c_d(t-t_2)c_{d'}(t'-t_2)k_{z_d,z_{d'}}(t_2,t_2) &+ c_d(t-t_2)e_{d'}(t'-t_2)k_{z_d,\dot{z}_{d'}}(t_2,t_2) \\
+ e_d(t-t_2)c_{d'}(t'-t_2)k_{\dot{z}_d,z_d}(t_2,t_2) &+ e_d(t-t_2)e_{d'}(t'-t_2)k_{\dot{z}_d,\dot{z}_{d'}}(t_2,t_2) + k_{f_d,f_{d'}}^{(2)}(t,t').
\end{aligned}
$$

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2.2.2 Covariance for intervals (t_{q-1}, t_q) and $(t_{q'-1}, t'_q)$

For the covariance in equation (7), we have two regimes

1. $q > q'$.
2. $q < q'$.

The case for which $q = q'$ was analized in the subsection before this one. We are interested in computing the term $\text{cov}\left[p_d(t, t, t_{q-1}, t_q, u_{q-1}), p_{d'}(t', t', t_{q'-1}, t_{q'}, u_{q'-1})\right]$, for $q > q'$ and $q < q'$. For $q > q'$, we have

$$c_d(t - t_{q-1})c_{d'}(t' - t_{q'-1}) \operatorname{cov}\{y_d(t_{q-1})y_{d'}(t_{q'-1})\} + c_d(t - t_{q-1})e_{d'}(t' - t_{q'-1}) \operatorname{cov}\{y_d(t_{q-1})\dot{y}_{d'}(t_{q'-1})\}$$

$$+c_d(t - t_{q-1}) \operatorname{cov}\{y_d(t_{q-1})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\} + e_d(t - t_{q-1})c_{d'}(t' - t_{q'-1}) \operatorname{cov}\{\dot{y}_d(t_{q-1})y_{d'}(t_{q'-1})\}$$

$$+e_d(t - t_{q-1})e_{d'}(t' - t_{q'-1}) \operatorname{cov}\{\dot{y}_d(t_{q-1})\dot{y}_{d'}(t_{q'-1})\} + e_d(t - t_{q-1}) \operatorname{cov}\{\dot{y}_d(t_{q-1})f_{d'}(t', t_{q'-1}, t_q, u_{q'-1})\}$$

$$+c_{d'}(t' - t_{q-1}) \operatorname{cov}\{f_d(t, t_{q-1}, t_q, u_{q-1})y_{d'}(t_{q'-1})\} + e_{d'}(t' - t_{q-1}) \operatorname{cov}\{f_d(t, t_{q-1}, t_q, u_{q-1})\dot{y}_{d'}(t_{q'-1})\}$$

$$+ \operatorname{cov}\{f_d(t, t_{q-1}, t_q, u_{q-1})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}.$$

The terms $\operatorname{cov}\{y_d(t_{q-1})y_{d'}(t_{q'-1})\}$, $\operatorname{cov}\{y_d(t_{q-1})\dot{y}_{d'}(t_{q'-1})\}$, $\operatorname{cov}\{\dot{y}_d(t_{q-1})y_{d'}(t_{q'-1})\}$ and $\operatorname{cov}\{\dot{y}_d(t_{q-1})\dot{y}_{d'}(t_{q'-1})\}$ are obtained from the covariance already computed. The term $\operatorname{cov}\{f_d(t, t_{q-1}, t_q, u_{q-1})f_{d'}(t', t_{q'-1}, t'_q, u_{q'-1})\}$ is equal to zero, because there is no correlation between u_{q-1} and $u_{q'-1}$. Also, the covariances $c_{d'}(t' - t_{q-1}) \operatorname{cov}\{f_d(t, t_{q-1}, t_q, u_{q-1})y_{d'}(t_{q'-1})\}$ and $e_{d'}(t' - t_{q-1}) \operatorname{cov}\{f_d(t, t_{q-1}, t_q, u_{q-1})\dot{y}_{d'}(t_{q'-1})\}$ are zero, since $q > q'$, there is no correlation between force u_{q-1} and any force u_{k-1} for $k <= q' - 2$. We can rewrite the above expression as

$$c_d(t - t_{q-1})c_{d'}(t' - t_{q'-1}) \operatorname{cov}\{y_d(t_{q-1})y_{d'}(t_{q'-1})\} + c_d(t - t_{q-1})e_{d'}(t' - t_{q'-1}) \operatorname{cov}\{y_d(t_{q-1})\dot{y}_{d'}(t_{q'-1})\}$$

$$+e_d(t - t_{q-1})c_{d'}(t' - t_{q'-1}) \operatorname{cov}\{\dot{y}_d(t_{q-1})y_{d'}(t_{q'-1})\} + e_d(t - t_{q-1})e_{d'}(t' - t_{q'-1}) \operatorname{cov}\{\dot{y}_d(t_{q-1})\dot{y}_{d'}(t_{q'-1})\}$$

$$+c_d(t - t_{q-1}) \operatorname{cov}\{y_d(t_{q-1})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\} + e_d(t - t_{q-1}) \operatorname{cov}\{\dot{y}_d(t_{q-1})f_{d'}(t', t_{q'-1}, t_q, u_{q'-1})\}$$

Terms like $\operatorname{cov}\{y_d(t_{q-1})f_{d'}(t', t', t_{q'-1}, t_{q'}, u_{q'-1})\}$ and $\operatorname{cov}\{\dot{y}_d(t_{q-1})f_{d'}(t', t', t_{q'-1}, t_q, u_{q'-1})\}$ requiere further analysis.

Let's look in detail the term $\operatorname{cov}\{y_d(t_{q-1})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}$. This term is equal to

$$\begin{aligned}
\operatorname{cov}\{y_d(t_{q-1})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\} &= \operatorname{cov}\left\{p_d(t_{q-1}, t_{q-2}, t_{q-1}, u_{q-2})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\right\} \\
&= \operatorname{cov}\left\{\left[c_d(t_{q-1} - t_{q-2})y_d(t_{q-2}) + e_d(t_{q-1} - t_{q-2})\dot{y}_d(t_{q-2})\right.\right. \\
&\quad \left.\left. + f_d(t_{q-1}, t_{q-2}, t_{q-1}, u_{q-2})\right]f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\right\} \\
&= \underbrace{c_d(t_{q-1} - t_{q-2}) \operatorname{cov}\{y_d(t_{q-2})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}}_{A} \\
&\quad + \underbrace{e_d(t_{q-1} - t_{q-2}) \operatorname{cov}\{\dot{y}_d(t_{q-2})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}}_{B} \\
&\quad + \operatorname{cov}\{f_d(t_{q-1}, t_{q-2}, t_{q-1}, u_{q-2})f_{d'}(t', t', t_{q'-1}, t_{q'}, u_{q'-1})\}.
\end{aligned}$$

The term $\operatorname{cov}\{f_d(t_{q-1}, t_{q-2}, t_{q-1}, u_{q-2})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}$ is only different from zero for $q = q' + 1$ and it would reduce to $\widehat{k}_{f_d, f_{d'}}^{(q'-1)}(t_{q-1}, t')$. For A and B, if $q < q' + 1$, the terms in the are zero because there is no correlation between forces $u_{q'-1}$ and forces u_{q-2}, for $q < q' + 1$. For $q > q' + 1$, the term in A is equal to

$$\begin{aligned}
c_d(t_{q-1} - t_{q-2}) \operatorname{cov}&\left\{\left[c_d(t_{q-2} - t_{q-3})y_d(t_{q-3}) + e_d(t_{q-2} - t_{q-3})\dot{y}_d(t_{q-3})\right.\right. \\
&\quad \left.\left. + f_d(t_{q-2}, t_{q-3}, t_{q-2}, u_{q-3})\right]f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\right\} \\
&= \underbrace{c_d(t_{q-1} - t_{q-2})c_d(t_{q-2} - t_{q-3}) \operatorname{cov}\{y_d(t_{q-3})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}}_{A'} \\
&\quad + \underbrace{c_d(t_{q-1} - t_{q-2})e_d(t_{q-2} - t_{q-3}) \operatorname{cov}\{\dot{y}_d(t_{q-3})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}}_{B'} \\
&\quad + c_d(t_{q-1} - t_{q-2}) \operatorname{cov}\{f_d(t_{q-2}, t_{q-3}, t_{q-2}, u_{q-3})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}.
\end{aligned}$$

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The last term in the above equation is different from zero for $q = q' + 2$. Thus, this last term follows

$$c_d(t_{q-1} - t_{q-2})k_{f_d,f_{d'}}^{(q'-1)}(t_{q-2}, t').$$

The terms A' and B' follow the same form that the terms A and B. Again, if $q < q' + 2$, then the particular terms in are zeros. If, $q > q' + 2$, the recursion repeats until the most inner term in $\mathrm{cov}\{y_d(t_{q-n})f_{d'}(t', t_{q'-1}, t_{q'}, u_{q'-1})\}$ is such that $q = q' + n$. A similar expression can analysis can be made for the term B. The final covariance would then be equal to

$$c_d(t - t_{q-1})c_{d'}(t' - t_{q'-1})k_{z_d,z_{d'}}(t_{q-1}, t_{q'-1}) + c_d(t - t_{q-1})e_{d'}(t' - t_{q'-1})k_{z_d,\dot{z}_{d'}}(t_{q-1}, t_{q'-1})$$
$$+e_d(t - t_{q-1})c_{d'}(t' - t_{q'-1})k_{\dot{z}_d,z_{d'}}(t_{q-1}, t_{q'-1}) + e_d(t - t_{q-1})e_{d'}(t' - t_{q'-1})k_{\dot{z}_d,\dot{z}_{d'}}(t_{q-1}, t_{q'-1})$$
$$+c_d(t - t_{q-1})f_1(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k_{f_d,f_d'}^{(q'-1)}(t_{q-n}, t')$$
$$+c_d(t - t_{q-1})f_2(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k_{m_d,f_d'}^{(q'-1)}(t_{q-n}, t')$$
$$+e_d(t - t_{q-1})f_3(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k_{f_d,f_d'}^{(q'-1)}(t_{q-n}, t')$$
$$+e_d(t - t_{q-1})f_4(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k_{m_d,f_d'}^{(q'-1)}(t_{q-n}, t'),$$

where $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$ and $f_4(\cdot)$ are functions of the form

$$\sum x(t_{q-1} - t_{q-2})x(t_{q-2} - t_{q-3}) \ldots x(t_{q-n+1} - t_{q-n}),$$

with x being equal to c_d, e_d, g_d or h_d, depending on the case. To compute the exact form of the expression $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$ and $f_4(\cdot)$ we use the following set of rules

- After a $c_d(\cdot)$ term, only $c_d(\cdot)$ and $e_d(\cdot)$ terms follow.
- After a $e_d(\cdot)$ term, only $g_d(\cdot)$ and $h_d(\cdot)$ terms follow.
- After a $g_d(\cdot)$ term, only $c_d(\cdot)$ and $e_d(\cdot)$ terms follow.
- After a $h_d(\cdot)$ term, only $h_d(\cdot)$ and $g_d(\cdot)$ terms follow.

Figures 2, 3, 4 and 5 show examples of the kind of recursions that are generated. In all figures, red indicates a term like $c_d(\cdot)$, blue indicates a term like $e_d(\cdot)$, green indicates a term like $g_d(\cdot)$ and purple indicates $h_d(\cdot)$.
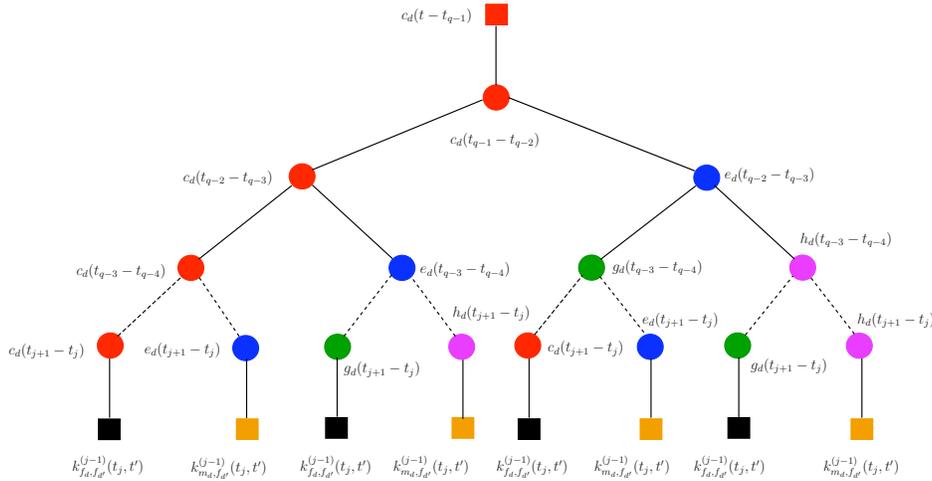
Figure 2: This figure represents the innermost covariances involved when computing the term A'

For $q' > q$ we can make a similar analysis (not presented here).

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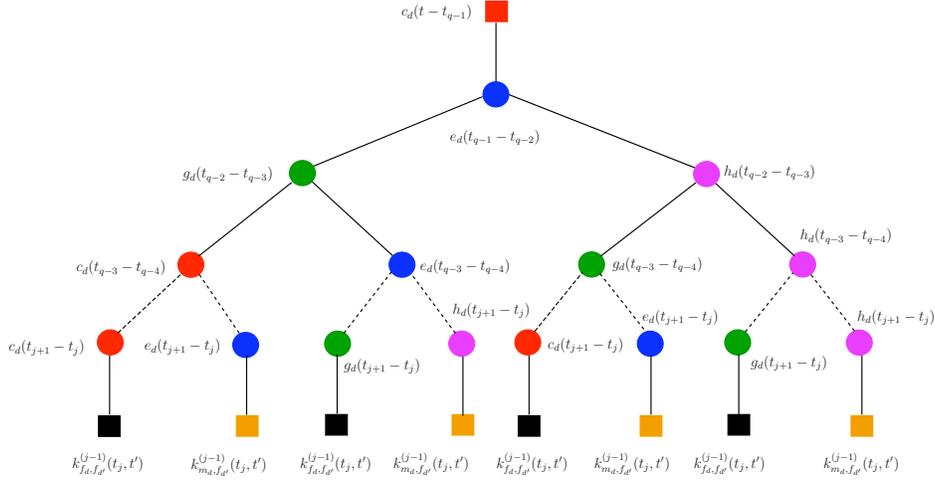
Figure 3: This figure represents the innermost covariances involved when computing the term B'
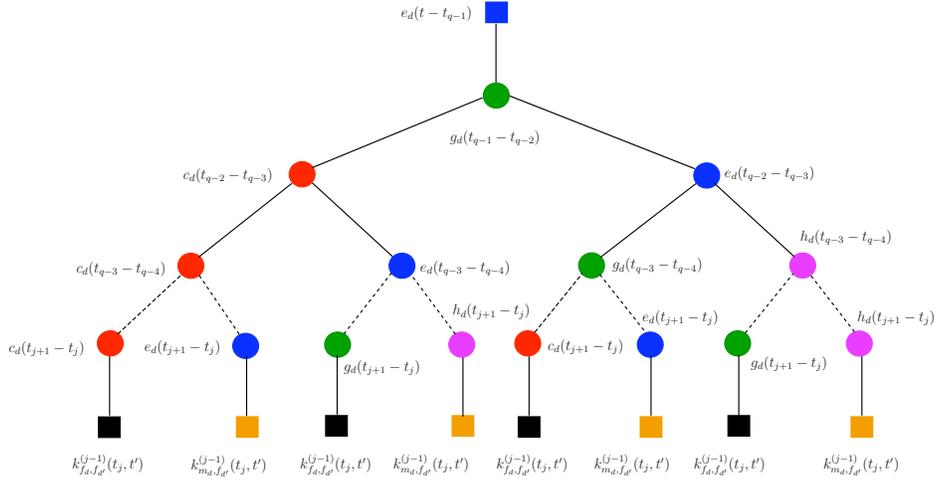
Figure 4: This figure represents the innermost covariances involved when computing the term C'

Example 1 (Continued). We continue with the example in figure 1. First, we compute the covariance between intervals $(t_1, t_2]$ and $(t_0, t_1]$. For this covariance we have

$$\text{cov}[z_d(t), z_{d'}(t')] = \text{cov}\left[p_d(t, t_1, t_2, u_1), p_{d'}(t', t_0, t_1, u_0)\right].$$

This covariance is equal to

$$c_d(t - t_1)c_{d'}(t' - t_0)\,\text{cov}\{y_d(t_1)y_{d'}(t_0)\} + c_d(t - t_1)e_{d'}(t' - t_0)\,\text{cov}\{y_d(t_1)\dot{y}_{d'}(t_0)\}$$
$$+e_d(t - t_1)c_{d'}(t' - t_0)\,\text{cov}\{\dot{y}_d(t_1)y_{d'}(t_0)\} + e_d(t - t_1)e_{d'}(t' - t_0)\,\text{cov}\{\dot{y}_d(t_1)\dot{y}_{d'}(t_0)\}$$
$$+c_d(t - t_1)\,\text{cov}\{y_d(t_1)f_{d'}(t', t_0, t_1, u_0)\} + e_d(t - t_1)\,\text{cov}\{\dot{y}_d(t_1)f_{d'}(t', t_0, t_1, u_0)\},$$

which reduces to

$$c_d(t - t_1)c_{d'}(t' - t_0)k_{z_d, z_{d'}}(t_1, t_0) + c_d(t - t_1)e_{d'}(t' - t_0)k_{z_d, \dot{z}_{d'}}(t_1, t_0)$$
$$+e_d(t - t_1)c_{d'}(t' - t_0)k_{\dot{z}_d, z_{d'}}(t_1, t_0) + e_d(t - t_1)e_{d'}(t' - t_0)k_{\dot{z}_d, \dot{z}_{d'}}(t_1, t_0)$$
$$+c_d(t - t_1)k^{(0)}_{f_d, f_{d'}}(t_1, t') + e_d(t - t_1)k^{(0)}_{m_d, f_{d'}}(t_1, t').$$

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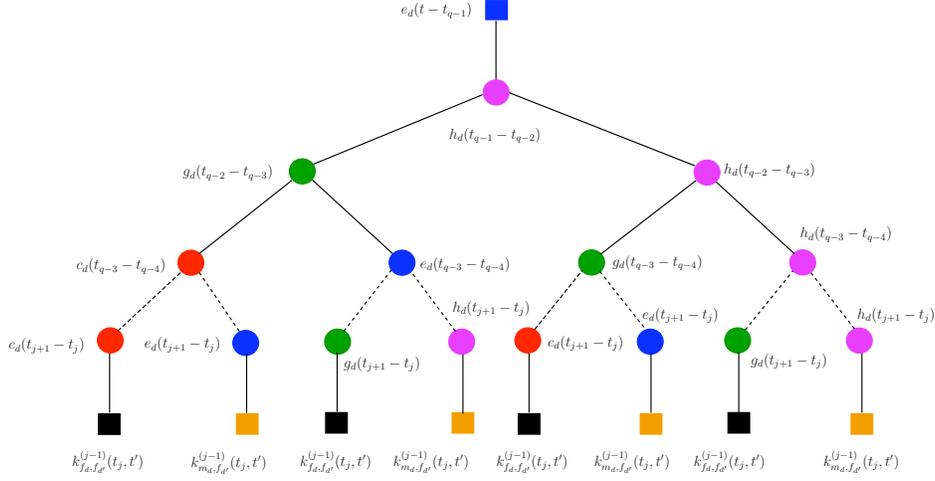
Figure 5: This figure represents the innermost covariances involved when computing the term D'

Now we compute the covariance between intervals $(t_2, t_3]$ and $(t_0, t_1]$. For this covariance we have

$$\text{cov}[z_d(t), z_{d'}(t')] = \text{cov}\left[p_d(t, t, t_2, t_3, u_2), p_{d'}(t', t', t_0, t_1, u_0)\right].$$

We then have

$$c_d(t - t_2)c_{d'}(t' - t_0)\,\text{cov}\{y_d(t_2)y_{d'}(t_0)\} + c_d(t - t_2)e_{d'}(t' - t_0)\,\text{cov}\{y_d(t_2)\dot{y}_{d'}(t_0)\}$$
$$+e_d(t - t_2)c_{d'}(t' - t_0)\,\text{cov}\{\dot{y}_d(t_2)y_{d'}(t_0)\} + e_d(t - t_2)e_{d'}(t' - t_0)\,\text{cov}\{\dot{y}_d(t_2)\dot{y}_{d'}(t_0)\}$$
$$+c_d(t - t_2)\,\text{cov}\{y_d(t_2)f_{d'}(t', t_0, t_1, u_0)\} + e_d(t - t_2)\,\text{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_0, t_1, u_0)\}.$$

Using $y_d(t_2) = z_d(t_2, t_1, t_2)$ and $\dot{y}_d(t_2) = \dot{z}_d(t_2.t_1, t_2)$,

$$y_d(t_2) = z_d(t_2, t_1, t_2) = p_d(t_2, t_1, t_2, u_1) = c_d(t_2 - t_1)y_d(t_1) + e_d(t_2 - t_1)\dot{y}_d(t_1) + f_d(t_2, t_1, t_2, u_1)$$
$$\dot{y}_d(t_2) = \dot{z}_d(t_2, t_1, t_2) = \xi_d(t_2, t_1, t_2, u_1) = g_d(t_2 - t_1)y_d(t_1) + h_d(t_2 - t_1)\dot{y}_d(t_1) + m_d(t_2, t_1, t_2, u_1),$$

we have for $\text{cov}\{y_d(t_2)f_{d'}(t', t', t_0, t_1, u_0)\}$ and $\text{cov}\{\dot{y}_d(t_2)f_{d'}(t', t', t_0, t_1, u_0)\}$

$$\text{cov}\{y_d(t_2)f_{d'}(t', t_0, t_1, u_0)\} = c_d(t_2 - t_1)\,\text{cov}\{y_d(t_1)f_{d'}(t', t_0, t_1, u_0)\}$$
$$+ e_d(t_2 - t_1)\,\text{cov}\{\dot{y}_d(t_1)f_{d'}(t', t_0, t_1, u_0)\}$$
$$\text{cov}\{\dot{y}_d(t_2)f_{d'}(t', t', t_0, t_1, u_0)\} = g_d(t_2 - t_1)\,\text{cov}\{y_d(t_1)f_{d'}(t', t_0, t_1, u_0)\}$$
$$+ h_d(t_2 - t_1)\,\text{cov}\{\dot{y}_d(t_1)f_{d'}(t', t_0, t_1, u_0)\}.$$

Furthermore,

$$y_d(t_1) = z_d(t_1, t_0, t_1) = p_d(t_1, t_0, t_1, u_0) = c_d(t_1 - t_0)y_d(t_0) + e_d(t_1 - t_0)\dot{y}_d(t_0) + f_d(t_1, t_0, t_1, u_0)$$
$$\dot{y}_d(t_1) = \dot{z}_d(t_2, t_1, t_2) = \xi_d(t_1, t_0, t_1, u_0) = g_d(t_1 - t_0)y_d(t_0) + h_d(t_1 - t_0)\dot{y}_d(t_0) + m_d(t_1, t_0, t_1, u_0).$$

Then we get $\text{cov}\{y_d(t_1)f_{d'}(t', t_0, t_1, u_0)\} = k^{(0)}_{f_d, f_{d'}}(t_1, t')$ and $\text{cov}\{\dot{y}_d(t_1)f_{d'}(t', t_0, t_1, u_0)\} = k^{(0)}_{m_d, f_{d'}}(t_1, t')$. Putting all these expressions together, we get

$$c_d(t - t_2)c_{d'}(t' - t_0)k_{z_d, z_{d'}}(t_2, t_0) + c_d(t - t_2)e_{d'}(t' - t_0)k_{z_d, \dot{z}_{d'}}(t_2, t_0)$$
$$+e_d(t - t_2)c_{d'}(t' - t_0)k_{\dot{z}_d, z_{d'}}(t_2, t_0) + e_d(t - t_2)e_{d'}(t' - t_0)k_{\dot{z}_d, \dot{z}_{d'}}$$
$$+c_d(t - t_2)c_d(t_2 - t_1)k^{(0)}_{f_d, f_{d'}}(t_1, t') + c_d(t - t_2)e_d(t_2 - t_1)k^{(0)}_{m_d, f_{d'}}(t_1, t')$$
$$+e_d(t - t_2)g_d(t_2 - t_1)k^{(0)}_{f_d, f_{d'}}(t_1, t') + e_d(t - t_2)h_d(t_2 - t_1)k^{(0)}_{m_d, f_{d'}}(t_1, t')$$

Next we compute the covariance between intervals $(t_2, t_3]$ and $(t_1, t_2]$. For this covariance we have

$$\text{cov}[z_d(t), z_{d'}(t')] = \text{cov}\left[p_d(t, t_2, t_3, u_2), p_{d'}(t', t_1, t_2, u_1)\right]$$

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We have

$$c_d(t - t_2)c_{d'}(t' - t_1)k_{z_d,z_{d'}}(t_2, t_1) + c_d(t - t_2)e_{d'}(t' - t_1)k_{z_d,\dot{z}_{d'}}(t_2, t_1)$$
$$+ e_d(t - t_2)c_{d'}(t' - t_1)k_{\dot{z}_d,z_{d'}}(t_2, t_1) + e_d(t - t_2)e_{d'}(t' - t_1)k_{\dot{z}_d,\dot{z}_{d'}}(t_2, t_1)$$
$$+ c_d(t - t_2)\,\mathrm{cov}\{y_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} + e_d(t - t_2)\,\mathrm{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_1, t_2, u_1)\}.$$

The covariance $\mathrm{cov}\{y_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} = k^{(1)}_{f_d,f'_d}(t_2, t')$ and $\mathrm{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} = k^{(1)}_{m_d,f'_d}(t_2, t')$. Then, the complete covariance would be equal to

$$c_d(t - t_2)c_{d'}(t' - t_1)k_{z_d,z_{d'}}(t_2, t_1) + c_d(t - t_2)e_{d'}(t' - t_1)k_{z_d,\dot{z}_{d'}}(t_2, t_1)$$
$$+ e_d(t - t_2)c_{d'}(t' - t_1)k_{\dot{z}_d,z_{d'}}(t_2, t_1) + e_d(t - t_2)e_{d'}(t' - t_1)k_{\dot{z}_d,\dot{z}_{d'}}(t_2, t_1)$$
$$+ c_d(t - t_2)k^{(1)}_{f_d,f'_d}(t_2, t') + e_d(t - t_2)k^{(1)}_{m_d,f'_d}(t_2, t').$$

Suppose we need to compute the covariance between the intervals $(t_4, t_5]$ and $(t_1, t_2]$. For this $q = 4$ and $q' = 1$. The covariance is given as

$$\mathrm{cov}\{[c_d(t - t_4)y_d(t_4) + e_d(t - t_4)\dot{y}_d(t_4) + f_d(t, t_4, t_5, u_4)]$$
$$[c_{d'}(t' - t_1)y_{d'}(t_1) + e_{d'}(t' - t_1)\dot{y}_{d'}(t_1) + f_{d'}(t', t_1, t_2, u_1)\}$$
$$= c_d(t - t_4)c_{d'}(t' - t_1)\,\mathrm{cov}\{y_d(t_4)y_{d'}(t_1)\} + c_d(t - t_4)e_{d'}(t' - t_1)\,\mathrm{cov}\{y_d(t_4)\dot{y}_{d'}(t_1)\}$$
$$+ e_d(t - t_4)c_{d'}(t' - t_1)\,\mathrm{cov}\{\dot{y}_d(t_4)y_{d'}(t_1)\} + e_d(t - t_4)e_{d'}(t' - t_1)\,\mathrm{cov}\{\dot{y}_d(t_4)\dot{y}_{d'}(t_1)\}$$
$$+ c_d(t - t_4)\,\mathrm{cov}\{y_d(t_4)f_{d'}(t', t_1, t_2, u_1)\} + e_d(t - t_4)\,\mathrm{cov}\{\dot{y}_d(t_4)f_{d'}(t', t_1, t_2, u_1)\}$$

We need to compute the covariances $\mathrm{cov}\{y_d(t_4)f_{d'}(t', t_1, t_2, u_1)\}$ and $\mathrm{cov}\{\dot{y}_d(t_4)f_{d'}(t', t_1, t_2, u_1)\}$. The expression for $y_d(t_4)$ is

$$y_d(t_4) = z_d(t_4, t_3, t_4) = p_d(t_4, t_3, t_4, u_3) = c_d(t_4 - t_3)y_d(t_3) + e_d(t_4 - t_3)\dot{y}_d(t_3) + f_d(t_4, t_3, t_4, u_3)$$
$$\dot{y}_d(t_4) = \dot{z}_d(t_4, t_3, t_4) = \xi_d(t_4, t_3, t_4, u_3) = g_d(t_4 - t_3)y_d(t_3) + h_d(t_4 - t_3)\dot{y}_d(t_3) + m_d(t_4, t_3, t_4, u_3)$$

Then the covariances $\mathrm{cov}\{y_d(t_4)f_{d'}(t', t_1, t_2, u_1)\}$ and $\mathrm{cov}\{\dot{y}_d(t_4)f_{d'}(t', t_1, t_2, u_1)\}$ are equal to

$$c_d(t_4 - t_3)\,\mathrm{cov}\{y_d(t_3)f_{d'}(t', t_1, t_2, u_1)\} + e_d(t_4 - t_3)\,\mathrm{cov}\{\dot{y}_d(t_3)f_{d'}(t', t_1, t_2, u_1)\},$$
$$g_d(t_4 - t_3)\,\mathrm{cov}\{y_d(t_3)f_{d'}(t', t_1, t_2, u_1)\} + h_d(t_4 - t_3)\,\mathrm{cov}\{\dot{y}_d(t_3)f_{d'}(t', t_1, t_2, u_1)\}.$$

At the same time, in the above expression, we have that $y_d(t_2)$ and $\dot{y}_d(t_2)$ follow

$$y_d(t_3) = c_d(t_3 - t_2)y_d(t_2) + e_d(t_3 - t_2)\dot{y}_d(t_2) + f_d(t_3, t_2, t_3, u_2)$$
$$\dot{y}_d(t_3) = g_d(t_3 - t_2)y_d(t_2) + h_d(t_3 - t_2)\dot{y}_d(t_2) + m_d(t_3, t_2, t_3, u_2)$$

Then, we can write the expression for $\mathrm{cov}\{y_d(t_4)f_{d'}(t', t_1, t_2, u_1)\}$ as

$$c_d(t_4 - t_3)\big[c_d(t_3 - t_2)\,\mathrm{cov}\{y_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} + e_d(t_3 - t_2)\,\mathrm{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_1, t_2, u_1)\}\big]$$
$$+ e_d(t_4 - t_3)\big[g_d(t_3 - t_2)\,\mathrm{cov}\{y_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} + h_d(t_3 - t_2)\,\mathrm{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_1, t_2, u_1)\}\big].$$

The expression for $\mathrm{cov}\{\dot{y}_d(t_4)f_{d'}(t', t_1, t_2, u_1)\}$ would follow

$$g_d(t_4 - t_3)\big[c_d(t_3 - t_2)\,\mathrm{cov}\{y_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} + e_d(t_3 - t_2)\,\mathrm{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_1, t_2, u_1)\}\big]$$
$$+ h_d(t_4 - t_3)\big[g_d(t_3 - t_2)\,\mathrm{cov}\{y_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} + h_d(t_3 - t_2)\,\mathrm{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_1, t_2, u_1)\}\big].$$

From the expression for $y_d(t_2)$ and $\dot{y}_d(t_2)$, we get $\mathrm{cov}\{y_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} = k^{(1)}_{f_d,f_{d'}}(t_2, t')$ and $\mathrm{cov}\{\dot{y}_d(t_2)f_{d'}(t', t_1, t_2, u_1)\} = k^{(1)}_{m_d,f_{d'}}(t_2, t')$. The total covariance then would be equal to

$$c_d(t - t_4)c_{d'}(t' - t_1)k_{z_d,z_{d'}}(t_4, t_1) + c_d(t - t_4)e_{d'}(t' - t_1)k_{z_d,\dot{z}_{d'}}(t_4, t_1)$$
$$+ e_d(t - t_4)c_{d'}(t' - t_1)k_{\dot{z}_d,z_{d'}}(t_4, t_1) + e_d(t - t_4)e_{d'}(t' - t_1)k_{\dot{z}_d,\dot{z}_{d'}}(t_4, t_1)$$
$$+ c_d(t - t_4)\big[c_d(t_4 - t_3)\big[c_d(t_3 - t_2)k^{(1)}_{f_d,f_{d'}}(t_2, t') + e_d(t_3 - t_2)k^{(1)}_{m_d,f_{d'}}(t_2, t')\big]$$
$$+ e_d(t_4 - t_3)\big[g_d(t_3 - t_2)k^{(1)}_{f_d,f_{d'}}(t_2, t') + h_d(t_3 - t_2)k^{(1)}_{m_d,f_{d'}}(t_2, t')\big]\big]$$
$$+ e_d(t - t_4)\big[g_d(t_4 - t_3)\big[c_d(t_3 - t_2)k^{(1)}_{f_d,f_{d'}}(t_2, t') + e_d(t_3 - t_2)k^{(1)}_{m_d,f_{d'}}(t_2, t')\big]$$
$$+ h_d(t_4 - t_3)\big[g_d(t_3 - t_2)k^{(1)}_{f_d,f_{d'}}(t_2, t') + h_d(t_3 - t_2)k^{(1)}_{m_d,f_{d'}}(t_2, t')\big]\big].$$

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Reorganizing, we get

$$c_d(t-t_4)c_{d'}(t'-t_1)k_{z_d,z_{d'}}(t_4,t_1) + c_d(t-t_4)e_{d'}(t'-t_1)k_{z_d,\dot{z}_{d'}}(t_4,t_1)$$

$$+ e_d(t-t_4)c_{d'}(t'-t_1)k_{\dot{z}_d,z_{d'}}(t_4,t_1) + e_d(t-t_4)e_{d'}(t'-t_1)k_{\dot{z}_d,\dot{z}_{d'}}(t_4,t_1)$$

$$+ c_d(t-t_4)\big[c_d(t_4-t_3)c_d(t_3-t_2) + e_d(t_4-t_3)g_d(t_3-t_2)\big]k_{f_d,f_{d'}}^{(1)}(t_2,t')$$

$$+ c_d(t-t_4)\big[c_d(t_4-t_3)e_d(t_3-t_2) + e_d(t_4-t_3)h_d(t_3-t_2)\big]k_{m_d,f_{d'}}^{(1)}(t_2,t')$$

$$+ e_d(t-t_4)\big[g_d(t_4-t_3)c_d(t_3-t_2) + h_d(t_4-t_3)g_d(t_3-t_2)\big]k_{f_d,f_{d'}}^{(1)}(t_2,t')$$

$$+ e_d(t-t_4)\big[g_d(t_4-t_3)e_d(t_3-t_2) + h_d(t_4-t_3)h_d(t_3-t_2)\big]k_{m_d,f_{d'}}^{(1)}(t_2,t').$$

Or in a more familiar expression,

$$c_d(t-t_4)c_{d'}(t'-t_1)k_{z_d,z_{d'}}(t_4,t_1) + c_d(t-t_4)e_{d'}(t'-t_1)k_{z_d,\dot{z}_{d'}}(t_4,t_1)$$

$$+ e_d(t-t_4)c_{d'}(t'-t_1)k_{\dot{z}_d,z_{d'}}(t_4,t_1) + e_d(t-t_4)e_{d'}(t'-t_1)k_{\dot{z}_d,\dot{z}_{d'}}(t_4,t_1)$$

$$+ c_d(t-t_4)f_1(t_4,t_3,t_2)k_{f_d,f_{d'}}^{(1)}(t_2,t') + c_d(t-t_4)f_2(t_4,t_3,t_2)k_{m_d,f_{d'}}^{(1)}(t_2,t')$$

$$+ e_d(t-t_4)f_3(t_4,t_3,t_2)k_{f_d,f_{d'}}^{(1)}(t_2,t') + e_d(t-t_4)f_4(t_4,t_3,t_2)k_{m_d,f_{d'}}^{(1)}(t_2,t').$$

where, $f_1(t_4,t_3,t_2) = c_d(t_4-t_3)c_d(t_3-t_2) + e_d(t_4-t_3)g_d(t_3-t_2)$, $f_2(t_4,t_3,t_2) = c_d(t_4-t_3)e_d(t_3-t_2)+e_d(t_4-t_3)h_d(t_3-t_2)$, $f_3(t_4,t_3,t_2) = g_d(t_4-t_3)c_d(t_3-t_2)+h_d(t_4-t_3)g_d(t_3-t_2)$ and $f_4(t_4,t_3,t_2) = g_d(t_4-t_3)e_d(t_3-t_2) + h_d(t_4-t_3)h_d(t_3-t_2)$.

2.3 Covariances between outputs and latent functions

For inference purposes, we'll also need the cross-covariances between the ouputs $z_d(t,t_{q-1},t_q)$ and the latent forces $u_{q'-1}(t')$. If $q' > q$, then this covariance is zero. We are left with the cases $q' = q$ and $q' < q$.

2.3.1 Covariance between $z_d(t,t_{q-1},t_q)$ and $u_{q'-1}(t')$, with $q' = q$

We have

$$\mathrm{cov}[z_d(t,t_{q-1},t_q),u_{q-1}(t')] = \mathrm{cov}\left[p_d(t,t_{q-1},t_q,u_{q-1})u_{q-1}(t')\right],$$

which is given as

$$c_d(t-t_{q-1})\,\mathrm{cov}\left[y_d(t_{q-1})u_{q-1}(t)\right] + e_d(t-t_{q-1})\,\mathrm{cov}\left[\dot{y}_d(t_{q-1})u_{q-1}(t)\right]$$
$$+ \mathrm{cov}\left[f_d(t,t_{q-1},t_q,u_{q-1})u_{q-1}(t')\right].$$

From the above equation, the only term different from zero is $\mathrm{cov}\left[f_d(t,t_{q-1},t_q,u_{q-1})u_{q-1}(t')\right] = k_{f_d,u_{q-1}}(t,t')$. Then, we have $\mathrm{cov}\left[p_d(t,t_{q-1},t_q,u_{q-1})u_{q-1}(t')\right] = k_{f_d,u_{q-1}}(t,t')$.

2.3.2 Covariance between $z_d(t,t_{q-1},t_q)$ and $u_{q'-1}(t')$, with $q' < q$

We have

$$\mathrm{cov}[z_d(t,t_{q-1},t_q),u_{q'-1}(t')] = \mathrm{cov}\left[p_d(t,t_{q-1},t_q,u_{q-1})u_{q'-1}(t')\right].$$

It would be

$$c_d(t-t_{q-1})\,\mathrm{cov}\left[y_d(t_{q-1})u_{q'-1}(t')\right] + e_d(t-t_{q-1})\,\mathrm{cov}\left[\dot{y}_d(t_{q-1})u_{q'-1}(t')\right]$$
$$+ \mathrm{cov}\left[f_d(t,t_{q-1},t_q,u_{q-1})u_{q'-1}\right].$$

Being q strictly greater than q', we only need to compute $\mathrm{cov}\left[y_d(t_{q-1})u_{q'-1}(t')\right]$ and $\mathrm{cov}\left[\dot{y}_d(t_{q-1})u_{q'-1}(t')\right]$. For the first term, we have

$$\mathrm{cov}\left[y_d(t_{q-1})u_{q'-1}(t')\right] = \mathrm{cov}\Big[\big(c_d(t_{q-1}-t_{q-2})y_d(t_{q-2}) + e_d(t_{q-1}-t_{q-2})\dot{y}_d(t_{q-2})$$
$$+ f_d(t_{q-1},t_{q-2},t_{q-1},u_{q-2})\big)u_{q'-1}(t')\Big]$$
$$= \underbrace{c_d(t_{q-1}-t_{q-2})\,\mathrm{cov}\left[y_d(t_{q-2})u_{q'-1}(t')\right]}_{A}$$
$$+ \underbrace{e_d(t_{q-1}-t_{q-2})\,\mathrm{cov}\left[\dot{y}_d(t_{q-2})u_{q'-1}(t')\right]}_{B}$$
$$+ \mathrm{cov}\left[f_d(t_{q-1},t_{q-2},t_{q-1},u_{q-2})u_{q'-1}(t')\right].$$

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The terms A and B, repeat again in a recursion similar to the ones in section 2.2.2. The final expression is then equal to

$$c_d(t - t_{q-1})f_1(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k^{(q'-1)}_{f_d, u_{q'-1}}(t_{q-n}, t')$$

$$+c_d(t - t_{q-1})f_2(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k^{(q'-1)}_{m_d, u_{q'-1}}(t_{q-n}, t')$$

$$+e_d(t - t_{q-1})f_3(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k^{(q'-1)}_{f_d, u_{q'-1}}(t_{q-n}, t')$$

$$+e_d(t - t_{q-1})f_4(t_{q-1}, t_{q-1}, \ldots, t_{q-n})k^{(q'-1)}_{m_d, u_{q'-1}}(t_{q-n}, t'),$$

where $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$ and $f_4(\cdot)$ are again functions of the form

$$\sum x(t_{q-1} - t_{q-2})x(t_{q-2} - t_{q-3}) \ldots x(t_{q-n+1} - t_{q-n}),$$

with x being equal to c_d, e_d, g_d or h_d, depending on the case.

Example 1 (continued). We continue with the example. We want to compute the following terms

$$\mathrm{cov}[z_d(t, t_0, t_1), u_0(t')] \qquad \mathrm{cov}[z_d(t, t_0, t_1), u_1(t')] \qquad \mathrm{cov}[z_d(t, t_0, t_1), u_2(t')]$$
$$\mathrm{cov}[z_d(t, t_1, t_2), u_0(t')] \qquad \mathrm{cov}[z_d(t, t_1, t_2), u_1(t')] \qquad \mathrm{cov}[z_d(t, t_1, t_2), u_2(t')]$$
$$\mathrm{cov}[z_d(t, t_2, t_3), u_0(t')] \qquad \mathrm{cov}[z_d(t, t_2, t_3), u_1(t')] \qquad \mathrm{cov}[z_d(t, t_2, t_3), u_2(t')]$$

From the above analysis, the terms $\mathrm{cov}[z_d(t, t_0, t_1), u_1(t')]$, $\mathrm{cov}[z_d(t, t_0, t_1), u_2(t')]$ and $\mathrm{cov}[z_d(t, t_1, t_2), u_2(t')]$ are zero. Furthermore, the terms $\mathrm{cov}[z_d(t, t_0, t_1), u_0(t')]$, $\mathrm{cov}[z_d(t, t_1, t_2), u_1(t')]$ and $\mathrm{cov}[z_d(t, t_2, t_3), u_2(t')]$ are

$$\mathrm{cov}[z_d(t, t_0, t_1), u_0(t')] = k_{f_d, u_0}(t, t')$$
$$\mathrm{cov}[z_d(t, t_1, t_2), u_1(t')] = k_{f_d, u_1}(t, t')$$
$$\mathrm{cov}[z_d(t, t_2, t_3), u_2(t')] = k_{f_d, u_2}(t, t').$$

We are left with the terms $\mathrm{cov}[z_d(t, t_1, t_2), u_0(t')]$, $\mathrm{cov}[z_d(t, t_2, t_3), u_0(t')]$ and $\mathrm{cov}[z_d(t, t_2, t_3), u_1(t')]$. The term $\mathrm{cov}[z_d(t, t_1, t_2), u_0(t')]$ follows as

$$\begin{aligned}
\mathrm{cov}[z_d(t, t_1, t_2), u_0(t')] &= \mathrm{cov}\{[p_d(t, t_1, t_2, u_1)]u_0(t')\} \\
&= \mathrm{cov}\{[c_d(t - t_1)y_d(t_1) + e_d(t - t_1)\dot{y}_d(t_1) + f_d(t, t_1, t_2, u_1)]u_0(t')\} \\
&= c_d(t - t_1)\,\mathrm{cov}\,[y_d(t_1)u_0(t')] + e_d(t - t_1)\,\mathrm{cov}\,[\dot{y}_d(t_1)u_0(t')].
\end{aligned}$$

The terms $\mathrm{cov}\,[y_d(t_1)u_0(t')]$ and $\mathrm{cov}\,[\dot{y}_d(t_1)u_0(t')]$ are

$$\begin{aligned}
\mathrm{cov}\,[y_d(t_1)u_0(t')] &= \mathrm{cov}\,[(c_d(t_1 - t_0)y_d(t_0) + e_d(t_1 - t_0)\dot{y}_d(t_0) + f_d(t_1, t_0, t_1, u_0))u_0(t')] \\
&= k_{f_d, u_0}(t_1, t') \\
\mathrm{cov}\,[\dot{y}_d(t_1)u_0(t')] &= \mathrm{cov}\,[(g_d(t_1 - t_0)y_d(t_0) + h_d(t_1 - t_0)\dot{y}_d(t_0) + m_d(t_1, t_0, t_1, u_0))u_0(t')] \\
&= k_{m_d, u_0}(t_1, t').
\end{aligned}$$

The final covariance is then

$$\mathrm{cov}[z_d(t, t_1, t_2), u_0(t')] = c_d(t - t_1)k_{f_d, u_0}(t_1, t') + e_d(t - t_1)k_{m_d, u_0}(t_1, t').$$

Now, we compute the term $\mathrm{cov}[z_d(t, t_2, t_3), u_0(t')]$, which will be given as

$$\begin{aligned}
\mathrm{cov}[z_d(t, t_2, t_3), u_0(t')] &= \mathrm{cov}\{[p_d(t, t, t_2, t_3, u_2)]u_0(t')\} \\
&= \mathrm{cov}\{[c_d(t - t_2)y_d(t_2) + e_d(t - t_2)\dot{y}_d(t_2) + f_d(t, t_2, t_3, u_2)]u_0(t')\} \\
&= c_d(t - t_2)\,\mathrm{cov}\,[y_d(t_2)u_0(t')] + e_d(t - t_2)\,\mathrm{cov}\,[\dot{y}_d(t_2)u_0(t')].
\end{aligned}$$

The term $\mathrm{cov}\,[y_d(t_2)u_0(t')]$ follows

$$\begin{aligned}
\mathrm{cov}[y_d(t_2), u_0(t')] &= \mathrm{cov}\{p_d(t_2, t_1, t_2, u_1)u_0(t')\} \\
&= c_d(t_2 - t_1)\,\mathrm{cov}[y_d(t_1)u_0(t')] + e_d(t_2 - t_1)\,\mathrm{cov}[\dot{y}_d(t_1)u_0(t')].
\end{aligned}$$

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The term $\mathrm{cov}\left[\dot{y}_d(t_2)u_0(t')\right]$ follows

$$\mathrm{cov}[\dot{y}_d(t_2), u_0(t')] = \mathrm{cov}\{\xi_d(t_2, t_1, t_2, u_1)u_0(t')\}$$
$$= g_d(t_2 - t_1)\,\mathrm{cov}[y_d(t_1)u_0(t')] + h_d(t_2 - t_1)\,\mathrm{cov}[\dot{y}_d(t_1)u_0(t')]$$

Putting together all these terms, the covariance $\mathrm{cov}[z_d(t, t_2, t_3), u_0(t')]$ is given as

$$\mathrm{cov}[z_d(t, t_2, t_3), u_0(t')] = c_d(t - t_2)\left[c_d(t_2 - t_1)k_{f_d, u_0}(t_1, t') + e_d(t_2 - t_1)k_{m_d, u_0}(t_1, t')\right]$$
$$+ e_d(t - t_2)\left[g_d(t_2 - t_1)k_{f_d, u_0}(t_1, t') + h_d(t_2 - t_1)k_{m_d, u_0}(t_1, t')\right].$$

Or in a more familiar form

$$\mathrm{cov}[z_d(t, t_2, t_3), u_0(t')] = c_d(t - t_2)f_1(t_2, t_1)k_{f_d, u_0}(t_1, t') + c_d(t - t_2)f_2(t_2, t_1)k_{m_d, u_0}(t_1, t')$$
$$+ e_d(t - t_2)f_3(t_2, t_1)k_{f_d, u_0}(t_1, t') + e_d(t - t_2)f_4(t_2, t_1)k_{m_d, u_0}(t_1, t'),$$

where $f_1(t_2, t_1) = c_d(t_2 - t_1)$, $f_2(t_2, t_1) = e_d(t_2 - t_1)$, $f_3(t_2, t_1) = g_d(t_2 - t_1)$ and $f_4(t_2, t_1) = h_d(t_2 - t_1)$.
Finally, we compute $\mathrm{cov}[z_d(t, t_2, t_3), u_1(t')]$ as

$$\mathrm{cov}[z_d(t, t_2, t_3), u_1(t')] = \mathrm{cov}\{p_d(t, t_2, t_3, u_2)u_1(t')\}$$
$$= \mathrm{cov}\{[c_d(t - t_2)y_d(t_2) + e_d(t - t_2)\dot{y}_d(t_2) + f_d(t, t_2, t_3, u_2)]u_1(t')\}$$
$$= c_d(t - t_2)k_{f_d, u_1}(t_2, t') + e_d(t - t_2)k_{m_d, u_1}(t_2, t').$$

3 Covariance for the velocities and accelerations

To get expressions for the covariances $\mathrm{cov}\left[z_d(t), \dot{z}_{d'}(t')\right]$ (Position - Velocity), $\mathrm{cov}\left[\dot{z}_d(t), z_{d'}(t')\right]$ (Velocity - Position), $\mathrm{cov}\left[\dot{z}_d(t), \dot{z}_{d'}(t')\right]$ (Velocity - Velocity), $\mathrm{cov}\left[z_d(t), \ddot{z}_{d'}\right]$ (Position - Acceleration), $\mathrm{cov}\left[\ddot{z}_d(t), z_{d'}(t')\right]$ (Acceleration - Position), $\mathrm{cov}\left[\dot{z}_d(t), \ddot{z}_{d'}\right]$ (Velocity - Acceleration), $\mathrm{cov}\left[\ddot{z}_d(t), \dot{z}_{d'}(t')\right]$ (Acceleration - Velocity) and $\mathrm{cov}\left[\ddot{z}_d(t), \ddot{z}_{d'}(t')\right]$ (Acceleration - Acceleration), we take the appropiate number of derivatives with respect to t and t' [3].

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