Appendices for
Online Ad Procurement in Non-stationary Autobidding Worlds

A Proofs for Section 4

A.1 Additional definitions for Section 4

Definition A.1 (Total variation between probability distributions)**.** *Consider two distributions* $\mathcal{P}, \mathcal{P}' \subseteq \Delta(\mathcal{S})$*. Then we define their total variation as* $\|\mathcal{P} - \mathcal{P}'\|_{TV} = \frac{1}{2}\int_{\mathcal{S}} |\mathcal{P}(s) - \mathcal{P}'(s)|ds$.

We also define the smoothed version of $h_t : \mathcal{X} \to \mathbb{R}$ (see Eq. (4)) for any t as followed:

$$\hat{h}_t(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{v}\sim U(\mathbb{B})}[\mathcal{L}_t(\boldsymbol{x} + \rho\boldsymbol{v}, \boldsymbol{\lambda}_t)] \tag{10}$$

where we recall the Lagrangian function $\mathcal{L}_t$ is defined in Eq. (3).

A.2 Additional lemmas for Section 4

Lemma A.1 (Lipschitz continuity)**.** *Let Assumption 2.1 hold, and recall the definitions* $h_t(\boldsymbol{x})$ *and* $\hat{h}_t(\boldsymbol{x})$ *from Eqs.* (4) *as well as* (10)*, respectively, and recall* $\boldsymbol{\lambda}_1 \ldots \boldsymbol{\lambda}_T$ *are the dual variables generated from Algorithm 1. Then for any* $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X}$*, we have* $|h_t(\boldsymbol{x}) - h_t(\boldsymbol{x}')| \le (1 + K\frac{\bar{F}}{\beta})L \cdot \|\boldsymbol{x} - \boldsymbol{x}'\|$ *and* $\left|h_t(\boldsymbol{x}) - \hat{h}_t(\boldsymbol{x})\right| \le (1 + K\frac{\bar{F}}{\beta})L\rho$*.*

Lemma A.2 (Bounding BOCO dynamic regret with surrogate loss)**.** *Recall the definition* $\hat{h}_t(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{v}\sim U(\mathbb{B})}[\mathcal{L}_t(\boldsymbol{x} + \rho\boldsymbol{v}, \boldsymbol{\lambda}_t)]$*. Then,* $\hat{h}_t(\boldsymbol{x})$ *is concave. Further, For any* $\boldsymbol{y} \in (1-\alpha)\mathcal{X}$*, we have* $\hat{h}_t(\boldsymbol{y}) - \hat{h}_t(\widetilde{\boldsymbol{x}}_t) \le \mathbb{E}_{\boldsymbol{u}_t\sim U(\mathbb{S})}[\ell_t(\widetilde{\boldsymbol{x}}_t) - \ell_t(\boldsymbol{y})]$*, where* $\widetilde{\boldsymbol{x}}_t$ *is defined in Eq.* (5)*, and the surrogate loss function* $\ell_t : \mathcal{X} \to \mathbb{R}$ *is defined in Eq.* (7)*.*

Lemma A.3 (Bounding surrogate loss for each expert)**.** *Recall the definition of individual forecasters* $\widetilde{\boldsymbol{x}}_t^i$ *defined in Eq.* (8)*, and the surrogate loss function* $\ell_t : \mathcal{X} \to \mathbb{R}$ *defined in Eq.* (7)*. Then for any* $i \in [N]$ *and any sequence* $\boldsymbol{y}_{1:T} \in \mathcal{X}^T$ *we have (i)* $\sum_{t\in[T]} \ell_t\left(\widetilde{\boldsymbol{x}}_t^i\right) - \ell_t\left((1-\alpha)\boldsymbol{y}_t\right) \le \mathcal{O}\left(\frac{1+P(\boldsymbol{y}_{1:T})}{\gamma_i} + \frac{\gamma_i}{\beta^2\rho^2}T\right)$ *and (ii)* $\sum_{t\in[T]} \ell_t\left(\widetilde{\boldsymbol{x}}_t\right) - \ell_t\left(\widetilde{\boldsymbol{x}}_t^i\right) \le \mathcal{O}(T\epsilon + \frac{1}{\epsilon})$*. where the constant* β *is specified in Algorithm 1. Here, recall* D *is the diameter of the decision set* $\mathcal{X}$*.*

The proofs of Lemmas A.1, A.2,A.3 are shown in Appendices A.9, A.10, and A.11, respectively.

A.3 Proof for Lemma 4.1

Proof. For any $k \in [K]$ we have

$$\begin{aligned}\sum_{t\in[T]} g_{k,t}(\boldsymbol{x}_t) &= \sum_{t\in[\tau_A-1]} g_{k,t}(\boldsymbol{x}_t) + \sum_{t=\tau_A}^{T} g_{k,t}(\boldsymbol{x}_t) \overset{(a)}{\ge} \sum_{t\in[\tau_A-1]} g_{k,t}(\boldsymbol{x}_t) + \bar{\beta}(T-\tau_A+1)\\ &\ge \sum_{t\in[\tau_A-1]} g_{k,t}(\boldsymbol{x}_t) + \beta(T-\tau_A) + \beta \overset{(b)}{\ge} \bar{G} + \beta > 0\end{aligned} \tag{11}$$

where in (a) we set $\boldsymbol{x}_t = \widetilde{\boldsymbol{x}}_\beta$ for all $t = \tau_A \ldots T$ and $g_{k,t}(\widetilde{\boldsymbol{x}}_\beta) \ge \bar{\beta}$ for any $k \in [K]$; (b) follows from the definition of the stopping time such that for any $t' < \tau_A$ and $k \in [K]$ we have $\sum_{t\in[t']} g_{k,t}(\boldsymbol{x}_t) - \bar{G} + \beta(T - t' - 1) \ge 0$. □

A.4 Proof for Lemma 4.4

Proof. It is easy to see $\boldsymbol{\lambda}_{t+1} = \Pi_{[\boldsymbol{0}, \frac{\bar{F}}{\beta}\boldsymbol{e}]}(\boldsymbol{\lambda}_t - \eta\nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t))_+ = \arg\min_{\boldsymbol{\lambda}\in[\boldsymbol{0}, \frac{\bar{F}}{\beta}\boldsymbol{e}]} \nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t)^\top\boldsymbol{\lambda} + \frac{1}{2\eta}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_t\|^2$. By the first-order stationary condition at $\boldsymbol{\lambda}_{t+1}$, we have for any $\boldsymbol{\lambda} \in [\mathbf{0}, \frac{\bar{F}}{\beta}\boldsymbol{e}]$

$$\left(\nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t) + \frac{1}{\eta}(\boldsymbol{\lambda}_{t+1} - \boldsymbol{\lambda}_t)\right)^\top (\boldsymbol{\lambda} - \boldsymbol{\lambda}_{t+1}) \geq 0$$

Then for all $\boldsymbol{\lambda} \in \mathbb{R}^K_{\geq 0}$, it follows that

$$\begin{aligned}
& \nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t)^\top(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}) \\
= \;& \nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t)^\top(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{t+1}) + \nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t)^\top(\boldsymbol{\lambda}_{t+1} - \boldsymbol{\lambda}) \\
\leq \;& \nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t)^\top(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{t+1}) + \frac{1}{\eta}(\boldsymbol{\lambda}_{t+1} - \boldsymbol{\lambda}_t)^\top(\boldsymbol{\lambda} - \boldsymbol{\lambda}_{t+1}) \\
\leq \;& \nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t)^\top(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{t+1}) + \frac{1}{2\eta}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_t\|^2 - \frac{1}{2\eta}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{t+1}\|^2 - \frac{1}{2\eta}\|\boldsymbol{\lambda}_{t+1} - \boldsymbol{\lambda}_t\|^2 \\
\leq \;& \frac{\eta}{2}\|\nabla_{\boldsymbol{\lambda}}\mathcal{L}_t(\boldsymbol{x}_t, \boldsymbol{\lambda}_t)\|^2 + \frac{1}{2\eta}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_t\|^2 - \frac{1}{2\eta}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{t+1}\|^2
\end{aligned}$$

By a telescoping argument, we have

$$\begin{aligned}
\sum_{\tau\in[t]} \nabla_{\boldsymbol{\lambda}}\mathcal{L}_\tau(\boldsymbol{x}_\tau, \boldsymbol{\lambda}_\tau)^\top(\boldsymbol{\lambda}_\tau - \boldsymbol{\lambda}) \;&\leq\; \frac{\eta}{2}\sum_{\tau\in[t]} \|\nabla_{\boldsymbol{\lambda}}\mathcal{L}_\tau(\boldsymbol{x}_\tau, \boldsymbol{\lambda}_\tau)\|^2 + \frac{1}{2\eta}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_1\|^2 \\
&=\; \frac{\eta}{2}\sum_{\tau\in[t]} \|\nabla_{\boldsymbol{\lambda}}\mathcal{L}_\tau(\boldsymbol{x}_\tau, \boldsymbol{\lambda}_\tau)\|^2 + \frac{1}{2\eta}\|\boldsymbol{\lambda}\|^2.
\end{aligned} \tag{12}$$

where in the final equality we used $\boldsymbol{\lambda}_1 = \mathbf{0}$. Also,

$$\|\nabla_{\boldsymbol{\lambda}}\mathcal{L}_\tau(\boldsymbol{x}_\tau, \boldsymbol{\lambda}_\tau)\|^2 \;=\; \|\boldsymbol{g}_\tau(\boldsymbol{x}_\tau)\|^2 \;\leq\; K\bar{G}^2 \tag{13}$$

Hence, combining Eqs. (12) and (13), we get the desired bound. □

A.5 Proof of Lemma 4.5

Proof. If $\tau_A = T$, taking $\boldsymbol{\lambda} = 0$ in Lemma 4.4 yields $\sum_{t\in[T]} \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\boldsymbol{x}_t) \;\leq\; \frac{\eta}{2}TK\bar{G}^2$ and thus the desired inequality holds. If $\tau_A < T$, then there exists some $k \in [K]$ such that $\sum_{t\in[\tau_A]} g_{k,t}(\boldsymbol{x}_t) - \bar{G} + \beta(T - \tau_A - 1) < 0$, so by taking $\boldsymbol{\lambda} = \frac{\bar{F}}{\beta}\boldsymbol{e}_k$ ($\boldsymbol{e}_k \in \mathbb{R}^K$ is the unit vector whose kth entry is 1) in Lemma 4.4 yields

$$\begin{aligned}
& \sum_{t\in[\tau_A]} \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\boldsymbol{x}_t) \\
\leq \;& \sum_{t\in[\tau_A]} \boldsymbol{\lambda}^\top \boldsymbol{g}_t(\boldsymbol{x}_t) + \frac{\eta}{2}TK\bar{G}^2 + \frac{1}{2\eta}\|\boldsymbol{\lambda}\|^2 \\
= \;& \frac{\bar{F}}{\beta}\sum_{t\in[\tau_A]} g_{k,t}(\boldsymbol{x}_t) + \frac{\eta}{2}TK\bar{G}^2 + \frac{1}{2\eta}\Big(\frac{\bar{F}}{\beta}\Big)^2 \\
\leq \;& -\frac{\bar{F}}{\beta}\cdot\beta(T - \tau_A - 1) + \frac{\bar{F}}{\beta}\bar{G} + \frac{\eta}{2}TK\bar{G}^2 + \frac{1}{2\eta}\Big(\frac{\bar{F}}{\beta}\Big)^2 \\
= \;& -\bar{F}(T - \tau_A) + \bar{F} + \frac{\bar{F}}{\beta}\bar{G} + \frac{\eta}{2}TK\bar{G}^2 + \frac{1}{2\eta}\Big(\frac{\bar{F}}{\beta}\Big)^2
\end{aligned}$$

Summing with $\bar{F}(T - \tau_A)$ yields the desired result. □

A.6 Proof of Lemma 4.6

Proof. Recall the definition of $\hat{h}_t(\boldsymbol{x})$ in Eq. (4). Then, we have

$$\begin{aligned}
& \sum_{\tau\in[t]} h_\tau(\boldsymbol{y}_\tau) - \sum_{\tau\in[t]} h_\tau(\boldsymbol{x}_\tau) \\
= \;& \sum_{\tau\in[t]} \Big(\underbrace{h_\tau(\boldsymbol{y}_\tau) - \hat{h}_\tau((1-\alpha)\boldsymbol{y}_\tau)}_{A} + \underbrace{\hat{h}_\tau((1-\alpha)\boldsymbol{y}_\tau) - \hat{h}_\tau(\widetilde{\boldsymbol{x}}_\tau)}_{B} + \underbrace{\hat{h}_\tau(\widetilde{\boldsymbol{x}}_\tau) - h_\tau(\boldsymbol{x}_\tau)}_{C}\Big)
\end{aligned} \tag{14}$$

Bounding A.

$$\begin{aligned}
h_\tau(\boldsymbol{y}_\tau) - \hat{h}_\tau((1-\alpha)\boldsymbol{y}_\tau) &= h_\tau(\boldsymbol{y}_\tau) - h_\tau((1-\alpha)\boldsymbol{y}_\tau) + h_\tau((1-\alpha)\boldsymbol{y}_\tau) - \hat{h}_\tau((1-\alpha)\boldsymbol{y}_\tau) \\
&\overset{(a)}{\leq} (1+K\frac{\bar{F}}{\beta})L\alpha\|\boldsymbol{y}_\tau\| + (1+K\frac{\bar{F}}{\beta})L\rho \\
&\overset{(b)}{\leq} (1+K\frac{\bar{F}}{\beta})L\alpha D + (1+K\frac{\bar{F}}{\beta})L\rho
\end{aligned} \tag{15}$$

where (a) follows from Lemma A.1; (b) follows from $\|\boldsymbol{y}_\tau\| = \|\boldsymbol{y}_\tau - \mathbf{0}\| \leq D$ since we assumed $\mathbf{0} \in \mathcal{X}$.

Bounding B.

$$\begin{aligned}
&\sum_{t\in[T]} \hat{h}_\tau((1-\alpha)\boldsymbol{y}_\tau) - \hat{h}_\tau(\widetilde{\boldsymbol{x}}_\tau) \\
\overset{(a)}{\leq}\ & \sum_{t\in[T]} \mathbb{E}_{\boldsymbol{u}_\tau\sim U(\mathbb{S})}\left[\ell_\tau(\widetilde{\boldsymbol{x}}_\tau) - \ell_\tau((1-\alpha)\boldsymbol{y}_\tau)\right] \\
=\ & \sum_{t\in[T]} \mathbb{E}_{\boldsymbol{u}_\tau\sim U(\mathbb{S})}\left[\ell_\tau(\widetilde{\boldsymbol{x}}_\tau) - \ell_\tau(\widetilde{\boldsymbol{x}}^i_\tau) + \ell_\tau(\widetilde{\boldsymbol{x}}^i_\tau) - \ell_\tau((1-\alpha)\boldsymbol{y}_\tau)\right] \\
\overset{(b)}{\leq}\ & \mathcal{O}\left(\frac{P(\boldsymbol{y}_{1:T})}{\gamma_i} + \frac{\gamma_i K\frac{\bar{F}}{\beta}T}{\rho^2} + T\epsilon + \frac{1}{\epsilon}\right)
\end{aligned} \tag{16}$$

where (a) follows from Lemma A.2 and (b) follows from Lemma A.3 (i) and (ii).

Bounding C.

$$\begin{aligned}
\hat{h}_\tau(\widetilde{\boldsymbol{x}}_\tau) - h_\tau(\boldsymbol{x}_\tau) &= \hat{h}_\tau(\widetilde{\boldsymbol{x}}_\tau) - h_\tau(\widetilde{\boldsymbol{x}}_\tau) + h_\tau(\widetilde{\boldsymbol{x}}_\tau) - h_\tau(\boldsymbol{x}_\tau) \\
&\overset{(a)}{\leq} (1+K\frac{\bar{F}}{\beta})L\rho + (1+K\frac{\bar{F}}{\beta})L\cdot\|\widetilde{\boldsymbol{x}}_\tau - \boldsymbol{x}_\tau\| \\
&\overset{(b)}{=} (1+K\frac{\bar{F}}{\beta})L\rho + (1+K\frac{\bar{F}}{\beta})L\cdot\|\rho\boldsymbol{u}_\tau\| \\
&\leq 2\rho(1+K\frac{\bar{F}}{\beta})L
\end{aligned} \tag{17}$$

where (a) follows from Lemma A.1; (b) follows from the definition $\boldsymbol{x}_\tau = \widetilde{\boldsymbol{x}}_\tau + \rho\boldsymbol{u}_\tau$ in Algorithm 1. □

A.7 Proof of Lemma 4.3

Stochastic.

Proof. In the stochastic regime, we have $\mathcal{P} = \mathcal{P}_1 = \cdots = \mathcal{P}_T$ for some $\mathcal{P}$, and therefore we can rewrite $\text{OPT}(\mathcal{P}_{1:T})$ in Eq. (1) as followed

$$\text{OPT}(\mathcal{P}_{1:T}) = \max_{\boldsymbol{x}_{1:T}\in\mathcal{X}^T} \sum_{t\in[T]} F(\boldsymbol{x}_t) \quad \text{s.t.} \sum_{t\in[T]} \boldsymbol{G}(\boldsymbol{x}_t) \geq \mathbf{0}\,.$$

where we defined $F(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}}[f(\boldsymbol{x})]$, and $\boldsymbol{G}(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}}[\boldsymbol{g}(\boldsymbol{x})]$ for any $\boldsymbol{x}\in\mathcal{X}$. Hence, for any $\boldsymbol{\lambda}\geq\mathbf{0}$ we have

$$\begin{aligned}
\text{OPT}(\mathcal{P}_{1:T}) &= \frac{T-\tau_A}{T}\text{OPT}(\mathcal{P}_{1:T}) + \frac{\tau_A}{T}\text{OPT}(\mathcal{P}_{1:T}) \\
&\leq (T-\tau_A)\bar{F} + \frac{\tau_A}{T}\max_{\boldsymbol{x}_{1:T}\in\mathcal{X}^T}\sum_{t\in[T]}\left(F(\boldsymbol{x}_t) + \boldsymbol{\lambda}^\top\boldsymbol{G}(\boldsymbol{x}_t)\right) \\
&= (T-\tau_A)\bar{F} + \frac{\tau_A}{T}\max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[T]}\left(F(\boldsymbol{x}) + \boldsymbol{\lambda}^\top\boldsymbol{G}(\boldsymbol{x})\right) \\
&= (T-\tau_A)\bar{F} + \tau_A\max_{\boldsymbol{x}\in\mathcal{X}}\left(F(\boldsymbol{x}) + \boldsymbol{\lambda}^\top\boldsymbol{G}(\boldsymbol{x})\right)
\end{aligned} \tag{18}$$

where in the inequality we applied Assumption 2.1 which states $\max_{\boldsymbol{x}\in\mathcal{X}} f(\boldsymbol{x})$ for all $(f,\boldsymbol{g})\in\mathcal{S}$. Choosing $\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}_{\tau_A} := \frac{1}{\tau_A}\sum_{t\in[\tau_A]}\boldsymbol{\lambda}_t$ we have

$$\begin{aligned}
\mathrm{OPT}(\mathcal{P}_{1:T}) \;&\le\; \mathbb{E}\Big[(T-\tau_A)\bar{F} + \tau_A \max_{\boldsymbol{x}\in\mathcal{X}}\big(F(\boldsymbol{x})+\boldsymbol{\lambda}^\top\boldsymbol{G}(\boldsymbol{x})\big)\Big] \\
&\le\; \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}\big(F(\boldsymbol{x})+\boldsymbol{\lambda}_t^\top\boldsymbol{G}(\boldsymbol{x})\big)\Big] \\
&\overset{(a)}{\le}\; \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}\mathbb{E}\left[f_t(\boldsymbol{x})+\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x})\,\middle|\,\sigma(\mathcal{H}_{t-1})\right]\Big] \\
&\overset{(b)}{=}\; \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}\mathbb{E}\left[h_t(\boldsymbol{x})\,\middle|\,\sigma(\mathcal{H}_{t-1})\right]\Big] \\
&\le\; \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}h_t(\boldsymbol{x})\Big]
\end{aligned} \tag{19}$$

where in (a) we used the fact that $\boldsymbol{\lambda}_t$ is $\mathcal{H}_{t-1}$-measurable; in (b) we used definitions $h_t(\boldsymbol{x}) = \mathcal{L}_t(\boldsymbol{x};\boldsymbol{\lambda}_t)$ and $\mathcal{L}_t(\boldsymbol{x};\boldsymbol{\lambda}) = f_t(\boldsymbol{x})+\boldsymbol{\lambda}^\top\boldsymbol{g}_t(\boldsymbol{x})$ in Eqs. (3) and (4) respectively.

On the other hand, we have

$$f_t(\boldsymbol{x}_t) = h_t(\boldsymbol{x}_t) - \boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_t), \tag{20}$$

so combining this with Eq. (19) we have

$$\mathrm{OPT}(\mathcal{P}_{1:T}) - \sum_{t\in[T]}\mathbb{E}[f_t(\boldsymbol{x}_t)] \;\le\; \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}\Big(h_t(\boldsymbol{x})-h_t(\boldsymbol{x}_t)\Big) + \sum_{t\in\tau_A}\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_t)\Big] \tag{21}$$

where we also used the fact that $f_t(\boldsymbol{x})\ge 0$ for all $t=\tau_A+1\ldots T$ and $\boldsymbol{x}\in\mathcal{X}$. □

Adversarial.

Proof. Recall the definition of ξ is Theorem 4.2:

$$\xi = 1 - \frac{\min_{(f,\boldsymbol{g})\in\mathcal{S}}\min_{k\in[K],\boldsymbol{x}\in\mathcal{X}} g_k(\boldsymbol{x})}{\bar{\beta}} > 1 \tag{22}$$

For any $t\in[T]$, define $\widetilde{\boldsymbol{y}}_t = \arg\max_{\boldsymbol{x}} f_t(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x})$.

By comparing to the safety action $\boldsymbol{x}_\beta\in\mathcal{X}$ which ensures $g_k(\boldsymbol{x}_\beta)\ge\bar{\beta}$ for any $k\in[K]$ and $(f,\boldsymbol{g})\in\mathcal{S}$, as well as the optimal hindsight action $\boldsymbol{x}_t^*\in\mathcal{X}$ (i.e. $\boldsymbol{x}_1^*\ldots\boldsymbol{x}_T^*$ is the optimal decision sequence to $\mathrm{OPT}(\mathcal{P}_{1:T})$), we have

$$\begin{aligned}
f_t(\widetilde{\boldsymbol{y}}_t)+\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t) \;&\ge\; f_t(\boldsymbol{x}_\beta)+\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_\beta) \ge \bar{\beta}\boldsymbol{\lambda}_t^\top\boldsymbol{e} \\
f_t(\widetilde{\boldsymbol{y}}_t)+\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t) \;&\ge\; f_t(\boldsymbol{x}_t^*)+\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_t^*).
\end{aligned} \tag{23}$$

We further have

$$\begin{aligned}
\xi f_t(\widetilde{\boldsymbol{y}}_t) \;&=\; f_t(\widetilde{\boldsymbol{y}}_t) + (\xi-1)f_t(\widetilde{\boldsymbol{y}}_t) \\
&\overset{(a)}{\ge}\; f_t(\boldsymbol{x}_t^*)+\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_t^*) - \boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t) + (\xi-1)\left(-\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t)+\bar{\beta}\boldsymbol{\lambda}_t^\top\boldsymbol{e}\right) \\
&=\; f_t(\boldsymbol{x}_t^*)+\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_t^*) - \xi\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t) + (\xi-1)\bar{\beta}\boldsymbol{\lambda}_t^\top\boldsymbol{e} \\
&\overset{(b)}{\ge}\; f_t(\boldsymbol{x}_t^*) - \xi\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t)
\end{aligned} \tag{24}$$

where (a) follows Eq.(23); in (b) we used the fact that $g_{k,t}(\boldsymbol{x}_t^*) + (\xi - 1)\bar{\beta} \geq 0$ since we have $\min_{(f,\boldsymbol{g})\in\mathcal{S}} \min_{k\in[K],\boldsymbol{x}\in\mathcal{X}}(g_{k,t}(\boldsymbol{x}) + (\xi-1)\bar{\beta}) \geq 0$ (see Eq. (22)). Hence we have

$$\begin{aligned} &\mathrm{OPT}(\mathcal{P}_{1:T}) - \sum_{t\in[T]} \mathbb{E}[f_t(\boldsymbol{x}_t)] \\ =\ & \Big(1 - \frac{1}{\xi}\Big)\mathrm{OPT}(\mathcal{P}_{1:T}) + \sum_{t\in[T]} \mathbb{E}\Big[\frac{1}{\xi} f_t(\boldsymbol{x}_t^*) - f_t(\boldsymbol{x}_t)\Big] \\ \leq\ & \Big(1 - \frac{1}{\xi}\Big)\mathrm{OPT}(\mathcal{P}_{1:T}) + \sum_{t\in[T]} \mathbb{E}\Big[f_t(\widetilde{\boldsymbol{y}}_t) - f_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t)\Big] \\ \leq\ & \Big(1 - \frac{1}{\xi}\Big)\mathrm{OPT}(\mathcal{P}_{1:T}) + \mathbb{E}\Big[(T - \tau_A)\bar{F} + \sum_{t\in\tau_A} \Big(f_t(\widetilde{\boldsymbol{y}}_t) - f_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\widetilde{\boldsymbol{y}}_t)\Big)\Big] \end{aligned} \tag{25}$$

□

δ-corrupted.

Here, we will prove a more general δ-corrupted model where the input distribution sequence $\mathcal{P}_{1:T}$ satisfies the following:

$$\sum_{t\in[T]} \|\mathcal{P}_t - \frac{1}{T}\sum_{s\in[T]} \mathcal{P}_s\|_{TV} \leq \delta \tag{26}$$

where the total variation norm is defined in Definition A.1. In fact, the definition in Section 2.2 for the δ-corrupted regime satisfies the above property: recall in the definition of Section 2.2, there exists $\mathcal{P} \in \Delta(\mathcal{S})$ as well as $\delta \in \mathbb{N}$ periods $\mathcal{T} = \{\tau_1 \ldots \tau_\delta\} \subset [T]$ such that $\mathcal{P}_t = \mathcal{P}$ for all $t \notin \mathcal{T}$, hence for any $t \notin \mathcal{T}$, we have

$$\begin{aligned} \|\mathcal{P}_t - \frac{1}{T}\sum_{s\in[T]} \mathcal{P}_s\|_{TV} &= \|\mathcal{P} - \frac{1}{T}\Big(T\mathcal{P} + \sum_{s\in\mathcal{T}}(\mathcal{P} - \mathcal{P}_s)\Big)\|_{TV} \\ &= \|\frac{1}{T}\sum_{s\in\mathcal{T}}(\mathcal{P} - \mathcal{P}_s)\|_{TV} \\ &\leq \frac{\delta}{2T} \end{aligned} \tag{27}$$

On the other hand, we have for any $\tau \in \mathcal{T}$, $\|\mathcal{P}_t - \frac{1}{T}\sum_{s\in[T]} \mathcal{P}_s\|_{TV} \leq \frac{1}{2}$. Hence, summing up we get

$$\begin{aligned} \sum_{t\in[T]} \|\mathcal{P}_t - \frac{1}{T}\sum_{s\in[T]} \mathcal{P}_s\|_{TV} &= \sum_{t\in\mathcal{T}} \|\mathcal{P}_t - \frac{1}{T}\sum_{s\in[T]} \mathcal{P}_s\|_{TV} + \sum_{t\notin\mathcal{T}} \|\mathcal{P}_t - \frac{1}{T}\sum_{s\in[T]} \mathcal{P}_s\|_{TV} \\ &\leq \frac{\delta}{2} + (T - \delta)\frac{\delta}{2T} \leq \delta \end{aligned}$$

which coincides with our general definition of δ-corruption in Eq. (26).

We now prove the δ-corruption regime under the general definition in Eq. (26). Define $\widetilde{\mathcal{P}} = \frac{1}{T}\sum_{s\in[T]} \mathcal{P}_s$, $\widetilde{F}(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\widetilde{\mathcal{P}}}[f(\boldsymbol{x})]$, $\widetilde{\boldsymbol{G}}(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\widetilde{\mathcal{P}}}[\boldsymbol{g}(\boldsymbol{x})]$, $F_t(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_t}[f(\boldsymbol{x})]$ and $\boldsymbol{G}_t(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_t}[\boldsymbol{g}(\boldsymbol{x})]$ for all $t \in [T]$ and any $x \in \mathcal{X}$. Then for any $\boldsymbol{\lambda} \in [\boldsymbol{0}, \frac{\bar{F}}{\beta}\boldsymbol{e}]$, we have

$$\begin{aligned} \mathrm{OPT}(\mathcal{P}_{1:T}) &\leq \max_{\boldsymbol{x}_{1:T}\in\mathcal{X}^T} \sum_{t\in[T]} \big(F_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}^\top \boldsymbol{G}_t(\boldsymbol{x}_t)\big) \\ &\leq \max_{\boldsymbol{x}_{1:T}\in\mathcal{X}} \sum_{t\in[T]} (\widetilde{F}(\boldsymbol{x}_t) + \boldsymbol{\lambda}^\top \widetilde{\boldsymbol{G}}(\boldsymbol{x}_t)) + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta \\ &= T \cdot \max_{\boldsymbol{x}\in\mathcal{X}} (\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}^\top \widetilde{\boldsymbol{G}}(\boldsymbol{x})) + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta, \end{aligned} \tag{28}$$

where the last inequality follows the definitions of $(\widetilde{F}, \widetilde{\boldsymbol{G}})$, Assumption 2.1, and the general definition of δ-corruption in Eq. (26). After choosing $\bar{\boldsymbol{\lambda}} = \frac{1}{\tau_A}\sum_{t\in[\tau_A]} \boldsymbol{\lambda}_t$, similar to our proof in Eq. (19) for the stochastic case we have

$$
\begin{aligned}
&\mathrm{OPT}(\mathcal{P}_{1:T})\\
=\ & \mathbb{E}\Big[\frac{T-\tau_A}{T}\mathrm{OPT}(\mathcal{P}_{1:T}) + \frac{\tau_A}{T}\mathrm{OPT}(\mathcal{P}_{1:T})\Big]\\
\overset{(a)}{\leq}\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \tau_A\cdot\max_{\boldsymbol{x}\in\mathcal{X}}(\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})) + \frac{\tau_A}{T}(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta\Big]\\
=\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \frac{\tau_A}{T}(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta + \max_{\boldsymbol{x}\in\mathcal{X}}(\sum_{t\in[\tau_A]}\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x}))\Big]\\
\overset{(b)}{\leq}\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \Big(1+\frac{\tau_A}{T}\Big)(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}(F_t(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top\boldsymbol{G}_t(\boldsymbol{x}))\Big]\\
\leq\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \Big(1+\frac{\tau_A}{T}\Big)(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}\mathbb{E}\Big[f_t(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x})\ \Big|\ \sigma(\mathcal{H}_{t-1})\Big]\Big]\\
\leq\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}h_t(\boldsymbol{x})\Big],
\end{aligned}
\tag{29}
$$

where (a) follows from Eq. (28); (b) follows from the definition of general δ-corruption in Eq. (26). Finally, we complete the proof by using the definition $f_t(\boldsymbol{x}_t) = h_t(\boldsymbol{x}_t) - \boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_t)$ and following the same argument as in Eq. (21) for the stochastic regime.

Periodic.

Recall in Section 2.2 that in the periodic regime, there exists cycle length $q\in\mathbb{N}$ such that $T = cq$ for some integer $c\geq 2$ with $\mathcal{P}_{1:T}$ as $\mathcal{P}_{1:q} = \mathcal{P}_{q+1:2q} = \cdots = \mathcal{P}_{(c-1)q+1:T}$. For any $t\in[T]$, define $c_t\in[c]$ such that $(c_t-1)q+1\leq t\leq c_t q$. After denoting $\widetilde{\mathcal{P}} = \frac{1}{q}\sum_{t\in[q]}\mathcal{P}_t$, we define the mean deviation within a single cycle of length q as

$$
MD(\mathcal{P}_{1:q}) = \sum_{1\leq t\leq q}\|\mathcal{P}_t - \widetilde{\mathcal{P}}\|_{TV} \quad \text{and} \quad \delta = c\cdot MD(\mathcal{P}_{1:q}). \tag{30}
$$

We define $\widetilde{F}(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\widetilde{\mathcal{P}}}[f(\boldsymbol{x})]$, $\widetilde{\boldsymbol{G}}(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\widetilde{\mathcal{P}}}[\boldsymbol{g}(\boldsymbol{x})]$, $F_t(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_t}[f(\boldsymbol{x})]$ and $\boldsymbol{G}_t(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_t}[\boldsymbol{g}(\boldsymbol{x})]$ for all $t\in[T]$ and any $x\in\mathcal{X}$. Then for any $\boldsymbol{\lambda}\in[\boldsymbol{0}, \frac{\bar{F}}{\beta}\boldsymbol{e}]$, we have

$$
\begin{aligned}
\mathrm{OPT}(\mathcal{P}_{1:T}) \ \leq\ & \max_{\boldsymbol{x}_{1:T}\in\mathcal{X}^T}\sum_{t\in[T]}\big(F_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}^\top\boldsymbol{G}_t(\boldsymbol{x}_t)\big)\\
=\ & c\cdot\max_{\boldsymbol{x}_{1:q}\in\mathcal{X}^q}\sum_{t\in[q]}\big(F_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}^\top\boldsymbol{G}_t(\boldsymbol{x}_t)\big)\\
\leq\ & cq\cdot\max_{\boldsymbol{x}\in\mathcal{X}}(\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})) + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})c\cdot MD(\mathcal{P}_{1:q})\\
\leq\ & cq\cdot\max_{\boldsymbol{x}\in\mathcal{X}}(\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})) + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\delta,
\end{aligned}
$$

where the equality follows the nature of periodic setting and the last inequality follows the definitions of $(\widetilde{F}, \widetilde{\boldsymbol{G}})$, Assumption 2.1, and (30). After choosing $\boldsymbol{\lambda} = \sum_{\hat{c}\in[c_{\tau_A}-1]}\frac{q}{\tau_A}\boldsymbol{\lambda}_{(\hat{c}-1)q+1} +$

$\frac{\tau_A-(c_{\tau_A}-1)q}{\tau_A}\boldsymbol{\lambda}_{(c_{\tau_A}-1)q+1}$, we further have that

$$
\begin{aligned}
&\mathrm{OPT}(\mathcal{P}_{1:T})\\
=\;& \frac{T-\tau_A}{T}\mathrm{OPT}(\mathcal{P}_{1:T})+\frac{\tau_A}{T}\mathrm{OPT}(\mathcal{P}_{1:T})\\
\le\;& (T-\tau_A)\bar{F}+\tau_A\cdot\max_{\boldsymbol{x}\in\mathcal{X}}(\widetilde{F}(\boldsymbol{x})+\boldsymbol{\lambda}^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x}))+\frac{\tau_A}{T}(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta\\
=\;& (T-\tau_A)\bar{F}+\max_{\boldsymbol{x}\in\mathcal{X}}\left(\tau_A\widetilde{F}(\boldsymbol{x})+\left(\sum_{\hat{c}\in[c_{\tau_A}-1]}q\boldsymbol{\lambda}_{(\hat{c}-1)q+1}+(\tau_A-(c_{\tau_A}-1)q)\boldsymbol{\lambda}_{(c_{\tau_A}-1)q+1}\right)^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})\right)\\
&+\frac{\tau_A}{T}(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta\\
=\;& (T-\tau_A)\bar{F}+\max_{\boldsymbol{x}\in\mathcal{X}}\Big(q\cdot\sum_{\hat{c}\in[c_{\tau_A}-1]}\Big(\widetilde{F}(\boldsymbol{x})+\boldsymbol{\lambda}_{(\hat{c}-1)q+1}^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})\Big)\\
&+(\tau_A-(c_{\tau_A}-1)q)\cdot\Big(\widetilde{F}(\boldsymbol{x})+\boldsymbol{\lambda}_{(c_{\tau_A}-1)q+1}^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})\Big)\Big)+\frac{\tau_A}{T}(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta\\
\le\;& (T-\tau_A)\bar{F}+\max_{x\in\mathcal{X}}\sum_{t\in[\tau_A]}\Big(\widetilde{F}(\boldsymbol{x})+\boldsymbol{\lambda}_t^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})\Big)+\bar{G}\cdot\sum_{t\in[\tau_A]}\|\boldsymbol{\lambda}_t-\boldsymbol{\lambda}_{(c_t-1)q+1}\|_1+\frac{\tau_A}{T}(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta.
\end{aligned}
$$

From (8) in Algorithm 1, we know that $\|\boldsymbol{\lambda}_{t+1}-\boldsymbol{\lambda}_t\|_1\le\eta\bar{G}K$, which further implies $\|\boldsymbol{\lambda}_{t+i}-\boldsymbol{\lambda}_t\|_1\le\eta\bar{G}Ki$ for any $i\in[q-1]$ and thus

$$\sum_{t\in[\tau_A]}\|\boldsymbol{\lambda}_t-\boldsymbol{\lambda}_{(c_t-1)q+1}\|_1\le c_{\tau_A}\eta\bar{G}K\sum_{i\in[q-1]}i\le\frac{1}{2}\bar{G}K\eta c_{\tau_A}q^2. \tag{31}$$

After combining the two equations above, it follows that

$$
\begin{aligned}
&\mathrm{OPT}(\mathcal{P}_{1:T})\\
\le\;& \mathbb{E}\Big[(T-\tau_A)\bar{F}+\max_{x\in\mathcal{X}}\sum_{t\in[\tau_A]}\Big(\widetilde{F}(\boldsymbol{x})+\boldsymbol{\lambda}_t^\top\widetilde{\boldsymbol{G}}(\boldsymbol{x})\Big)+\frac{1}{2}\bar{G}^2K\eta c_{\tau_A}q^2+\frac{\tau_A}{T}(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta\Big]\\
\le\;& \mathbb{E}\Big[(T-\tau_A)\bar{F}+\max_{x\in\mathcal{X}}\sum_{t\in[\tau_A]}\big(F_t(\boldsymbol{x})+\boldsymbol{\lambda}_t^\top\boldsymbol{G}_t(\boldsymbol{x})\big)+\frac{1}{2}\bar{G}^2K\eta c_{\tau_A}q^2+2(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta\Big]\\
\le\;& \mathbb{E}\Big[(T-\tau_A)\bar{F}+2(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta+\frac{1}{2}\bar{G}^2K\eta qT+\max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}\mathbb{E}\Big[h_t(\boldsymbol{x})\;\Big|\;\sigma(\mathcal{H}_{t-1})\Big]\Big]\\
\le\;& \mathbb{E}\Big[(T-\tau_A)\bar{F}+2(\bar{F}+\bar{G}K\frac{\bar{F}}{\beta})\delta+\frac{1}{2}\bar{G}^2K\eta qT+\max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]}h_t(\boldsymbol{x})\Big]
\end{aligned}
$$

where the second last inequality follows from $c_{\tau_A}q\le cq=T$.

Finally, we complete the proof by using the definition $f_t(\boldsymbol{x}_t)=h_t(\boldsymbol{x}_t)-\boldsymbol{\lambda}_t^\top\boldsymbol{g}_t(\boldsymbol{x}_t)$ and following the same argument as in Eq. (21) for the stochastic regime.

Ergodic.

Consider some $\kappa\ge\log(T)$. Given the input distribution sequence $\mathcal{P}_{1:T}$, denote $\mathcal{P}_{(t+\kappa)|[t-1]}$ as the conditional distribution of $(f_{t+\kappa},\boldsymbol{g}_{t+\kappa})$ conditioned on the $\{(f_\tau,\boldsymbol{g}_\tau)\}_{\tau\in[t]}$. Then, in the ergodic regime, there exists a stationary distribution $\widetilde{\mathcal{P}}\in\Delta(\mathcal{S})$ and absolute constant $R>0$ such that

$$\sup_{\{(f_t,\boldsymbol{g}_t)\}_{t\in[T]}\in\mathcal{S}^T}\;\sup_{t\in[T-\kappa]}\|\mathcal{P}_{(t+\kappa)|[t-1]}-\widetilde{\mathcal{P}}\|_{TV}\le\delta:=R\exp(-\kappa) \tag{32}$$

By defining $\widetilde{F}(\boldsymbol{x})=\mathbb{E}_{(f,\boldsymbol{g})\sim\widetilde{\mathcal{P}}}[f(\boldsymbol{x})]$, $\widetilde{\boldsymbol{G}}(\boldsymbol{x})=\mathbb{E}_{(f,\boldsymbol{g})\sim\widetilde{\mathcal{P}}}[\boldsymbol{g}(\boldsymbol{x})]$, $\hat{F}_{t+\kappa}(\boldsymbol{x})=\mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_{(t+\kappa)|[t-1]}}[f(\boldsymbol{x})]$, $\hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x})=\mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_{(t+\kappa)|[t-1]}}[\boldsymbol{g}(\boldsymbol{x})]$, $F_t(\boldsymbol{x})=\mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_t}[f(\boldsymbol{x})]$

and $\boldsymbol{G}_t(\boldsymbol{x}) = \mathbb{E}_{(f,\boldsymbol{g})\sim\mathcal{P}_t}[\boldsymbol{g}(\boldsymbol{x})]$ for all $t \in [T]$ and any $x \in \mathcal{X}$, we know that for any $\boldsymbol{\lambda} \in [\boldsymbol{0}, \frac{\bar{F}}{\beta}\boldsymbol{e}]$, it follows that

$$
\begin{aligned}
& \mathrm{OPT}(\mathcal{P}_{1:T}) \\
\leq\ & \max_{\boldsymbol{x}_{1:T}\in\mathcal{X}^T} \mathbb{E}\left[\sum_{t\in[T]} \left(F_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}^\top \boldsymbol{G}_t(\boldsymbol{x}_t)\right)\right] \\
=\ & \max_{\boldsymbol{x}_{1:\kappa}\in\mathcal{X}^\kappa} \mathbb{E}\left[\sum_{t\in[\kappa]} \left(F_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}^\top \boldsymbol{G}_t(\boldsymbol{x}_t)\right)\right] + \max_{\boldsymbol{x}_{\kappa+1:T}\in\mathcal{X}^{T-\kappa}} \mathbb{E}\left[\sum_{t=1}^{T-\kappa}(\hat{F}_{t+\kappa}(\boldsymbol{x}_{t+\kappa}) + \boldsymbol{\lambda}^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x}_{t+\kappa}))\right] \\
\leq\ & (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + \max_{\boldsymbol{x}_{\kappa+1:T}\in\mathcal{X}^{T-k}} \sum_{t=1}^{T-\kappa}(\widetilde{F}(\boldsymbol{x}_{t+\kappa}) + \boldsymbol{\lambda}^\top \widetilde{\boldsymbol{G}}(\boldsymbol{x}_{t+\kappa})) + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot (T-\kappa)\delta \\
\leq\ & T \cdot \max_{x\in\mathcal{X}}(\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}^\top \widetilde{\boldsymbol{G}}(\boldsymbol{x})) + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta
\end{aligned} \tag{33}
$$

By choosing $\boldsymbol{\lambda} = \frac{1}{\tau_A}\sum_{t\in[\tau_A]}\boldsymbol{\lambda}_t$, we further have

$$
\begin{aligned}
& \mathrm{OPT}(\mathcal{P}_{1:T}) \\
=\ & \mathbb{E}\Big[\frac{T-\tau_A}{T}\mathrm{OPT}(\mathcal{P}_{1:T}) + \frac{\tau_A}{T}\mathrm{OPT}(\mathcal{P}_{1:T})\Big] \\
\leq\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \tau_A \cdot \max_{x\in\mathcal{X}}\Big(\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}^\top \widetilde{\boldsymbol{G}}(\boldsymbol{x})\Big)\Big] + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta \\
=\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{x\in\mathcal{X}}\sum_{t\in[\tau_A]}\Big(\widetilde{F}(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top \widetilde{\boldsymbol{G}}(\boldsymbol{x})\Big)\Big] + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta \\
\leq\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{x\in\mathcal{X}}\sum_{t\in[\tau_A]}\Big(\hat{F}_{t+\kappa}(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x})\Big)\Big] + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta \\
=\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{x\in\mathcal{X}}\mathbb{E}\sum_{t\in[\tau_A]}(\hat{F}_{t+\kappa}(\boldsymbol{x}) + \boldsymbol{\lambda}_{t+\kappa}^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x}) + (\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{t+\kappa})^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x}_t))\Big] \\
& + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta \\
\overset{(a)}{\leq}\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{x\in\mathcal{X}}\sum_{t\in[\tau_A]}(\hat{F}_{t+\kappa}(\boldsymbol{x}) + \boldsymbol{\lambda}_{t+\kappa}^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x}_t))\Big] + \kappa\eta TK\bar{G}^2 \\
& + (\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta \\
\leq\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{x\in\mathcal{X}}\sum_{t\in[\tau_A-\kappa]}(\hat{F}_{t+\kappa}(\boldsymbol{x}) + \boldsymbol{\lambda}_{t+\kappa}^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x}_t))\Big] + \kappa\eta TK\bar{G}^2 \\
& + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{K}\frac{\bar{F}}{\beta}) \cdot T\delta \\
\overset{(b)}{\leq}\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t=\kappa+1}^{\tau_A} h_t(\boldsymbol{x})\Big] + \kappa\eta TK\bar{G}^2 + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta \\
\leq\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]} h_t(\boldsymbol{x})\Big] + \kappa\eta TK\bar{G}^2 + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}) \cdot T\delta \\
\overset{(c)}{\leq}\ & \mathbb{E}\Big[(T-\tau_A)\bar{F} + \max_{\boldsymbol{x}\in\mathcal{X}}\sum_{t\in[\tau_A]} h_t(\boldsymbol{x})\Big] + \kappa\eta TK\bar{G}^2 + 2(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta})\kappa + 2R(\bar{F} + \bar{G}K\frac{\bar{F}}{\beta}),
\end{aligned} \tag{34}
$$

where in (a), from (8) in Algorithm 1, we know that $\|\boldsymbol{\lambda}_{t+1} - \boldsymbol{\lambda}_t\|_1 \le \eta \bar{G} K$, which further implies $\|\boldsymbol{\lambda}_{t+\kappa} - \boldsymbol{\lambda}_t\|_1 \le \kappa\eta\bar{G}K$ and thus

$$(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{t+\kappa})^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x}_t) \le \kappa\eta K \bar{G}^2 \tag{35}$$

In (b), we used the fact that for any $t \ge \kappa + 1$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\max_{x\in\mathcal{X}} \sum_{t\in[\tau_A-\kappa]} (\hat{F}_{t+\kappa}(\boldsymbol{x}) + \boldsymbol{\lambda}_{t+\kappa}^\top \hat{\boldsymbol{G}}_{t+\kappa}(\boldsymbol{x}_t))\Big] \\
=\ &\mathbb{E}\Big[\max_{x\in\mathcal{X}} \sum_{t\in[\tau_A-\kappa]} \mathbb{E}\Big[h_{t+\kappa}(\boldsymbol{x}) \mid (f_\tau, \boldsymbol{g}_\tau)_{\tau\in[t-1]}\Big]\Big] \\
\le\ &\mathbb{E}\Big[\max_{x\in\mathcal{X}} \sum_{t\in[\tau_A-\kappa]} h_{t+\kappa}(\boldsymbol{x})\Big]
\end{aligned} \tag{36}$$

In (c) we used the fact that $\kappa \ge \log(T)$, so $\delta = R\exp(-\kappa) \ge R$.

Finally, we complete the proof by using the definition $f_t(\boldsymbol{x}_t) = h_t(\boldsymbol{x}_t) - \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\boldsymbol{x}_t)$ and following the same argument as in Eq. (21) for the stochastic regime.

A.8 Proof of Theorem 4.2

Proof. We bound the regret in every world as followed

$$\begin{aligned}
\mathcal{R}_T\ &=\ \mathrm{OPT}(\mathcal{P}_{1:T}) - \sum_{t\in[T]} \mathbb{E}\left[f_t(\boldsymbol{x}_t)\right] \\
&\overset{(a)}{\le}\ \mathbb{E}\Big[\bar{F}(T-\tau_A) + \sum_{t\in[\tau_A]} \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\boldsymbol{x}_t) + \mathcal{R}_{\mathrm{BOCO}}(\tau_A)\Big] \\
&\overset{(b)}{\le}\ \mathbb{E}\Big[\mathcal{R}_{\mathrm{BOCO}}(\tau_A)\Big]
\end{aligned}$$

where (a) follows from Lemma 4.3, and (b) follows from Lemma 4.5. Recall $\mathcal{R}_{\mathrm{BOCO}}(\tau_A)$ is specified in Lemma 4.3 for each world.

In the following we bound $\mathcal{R}_{\mathrm{BOCO}}(\tau_A)$ for each world.

Stochastic.

$$\mathbb{E}\Big[\mathcal{R}_{\mathrm{BOCO}}(\tau_A)\Big] = \mathbb{E}\Big[\max_{\boldsymbol{x}\in\mathcal{X}} \sum_{t\in[\tau_A]} h_t(\boldsymbol{x}) - h_t(\boldsymbol{x}_t)\Big] \overset{(a)}{\le} \mathcal{O}\Big(\frac{\rho T}{\beta} + \frac{1}{\gamma_i} + \frac{\gamma_i K T}{\beta^2\rho^2} + T\epsilon + \frac{1}{\epsilon}\Big) \overset{(b)}{=} \mathcal{O}\Big(T^{\frac{3}{4}}\Big) \tag{37}$$

where (a) follows from Lemma 4.6 by taking the comparator sequence $\boldsymbol{y}_t = \arg\max_{\boldsymbol{x}\in\mathcal{X}} \sum_{t\in[\tau_A]} h_t(\boldsymbol{x})$ for all $t \in [\tau_A]$ such that $P(\boldsymbol{y}_{1:T}) = 1$, as well as any primal ascent expert $i \in [N]$; (b) follows from taking $\eta = \frac{1}{\sqrt{KT}}$, $\rho = K^{\frac{1}{3}} T^{-\frac{1}{4}}$, $\epsilon = T^{-\frac{1}{2}}$, $\beta = \frac{1}{\log(T)}$, and finally choosing $\gamma_i = K^{-\frac{1}{6}}(1+DT)^{\frac{1}{2}} T^{-\frac{3}{4}}$. Recall all primal ascent expert stepsizes arer $\{\gamma_1 \dots \gamma_N\} = \{2^{-i} K^{-\frac{1}{6}} (1+DT)^{\frac{1}{2}} T^{-\frac{3}{4}} : i = 0 \dots N\}$.

δ-corrupted, Periodic, and Ergodic. The proof is nearly identical with that of the stochastic world in Eq. (37) given that we still consider the comparator sequence $\boldsymbol{y}_t = \arg\max_{\boldsymbol{x}\in\mathcal{X}} \sum_{t\in[\tau_A]} h_t(\boldsymbol{x})$ for all $t \in [\tau_A]$ such that $P(\boldsymbol{y}_{1:T}) = 1$. Hence we will omit the proof.

Adversarial. Recall the definition $\widetilde{\boldsymbol{y}}_t = \arg\max_{\boldsymbol{x}\in\mathcal{X}} f_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}_t^\top \boldsymbol{g}(\boldsymbol{x}_t)$. Then we have

$$\begin{aligned}
\mathbb{E}\Big[\mathcal{R}_{\mathrm{BOCO}}(\tau_A)\Big]\ &=\ \Big(1 - \frac{1}{\xi}\Big)\mathrm{OPT}(\mathcal{P}_{1:T}) + \sum_{t\in[\tau_A]} \mathbb{E}\Big[h_t(\widetilde{\boldsymbol{y}}_t) - h_t(\boldsymbol{x}_t)\Big] \\
&\le\ \mathcal{O}\Big(\frac{\rho T}{\beta} + \frac{1 + P(\widetilde{\boldsymbol{y}}_{1:T})}{\gamma_i} + \frac{\gamma_i K T}{\beta^2\rho^2} + T\epsilon + \frac{1}{\epsilon}\Big) = \Big(1 - \frac{1}{\xi}\Big)\mathrm{OPT}(\mathcal{P}_{1:T}) + o(T)
\end{aligned} \tag{38}$$

where we chose the primal ascent stepsize γ_i s.t.

$$\frac{1}{2}K^{-\frac{1}{6}}(1+P(\widetilde{\boldsymbol{y}}_{1:T}))^{\frac{1}{2}}T^{-\frac{3}{4}} \le \gamma_i \le K^{-\frac{1}{6}}(1+P(\widetilde{\boldsymbol{y}}_{1:T}))^{\frac{1}{2}}T^{-\frac{3}{4}} \tag{39}$$

We note that such a γ_i must exist because $P(\widetilde{\boldsymbol{y}}_{1:T}) \le DT$ given all $\widetilde{\boldsymbol{y}}_t \in \mathcal{X}$, so that the largest element in the primal ascent stepsize set, namely $K^{-\frac{1}{6}}(1+DT)^{\frac{1}{2}}T^{-\frac{3}{4}}$ is larger than the upper bound above, namely $K^{-\frac{1}{6}}(1+P(\widetilde{\boldsymbol{y}}_{1:T}))^{\frac{1}{2}}T^{-\frac{3}{4}}$.

□

A.9 Proof for Lemma A.1

Proof. Recall the definition $h_t(\boldsymbol{x}) = f_t(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\boldsymbol{x})$ in Eq. (4). Then we have

$$\begin{aligned} \Big|h_t(\boldsymbol{x}) - h_t(\boldsymbol{x}')\Big| &\le \Big|f_t(\boldsymbol{x}) - f_t(\boldsymbol{x}')\Big| + \|\boldsymbol{\lambda}_t\| \cdot \|\boldsymbol{g}_t(\boldsymbol{x}) - \boldsymbol{g}_t(\boldsymbol{x}')\| \\ &\overset{(a)}{\le} L\|\boldsymbol{x}-\boldsymbol{x}'\| + K\frac{\bar{F}}{\beta}L\|\boldsymbol{x}-\boldsymbol{x}'\| = (1+K\frac{\bar{F}}{\beta})L \cdot \|\boldsymbol{x}-\boldsymbol{x}'\| \end{aligned} \tag{40}$$

where (a) follows from the fact that any $(f, \boldsymbol{g}) \in \mathcal{S}$ are L-lipschitz under Assumption 2.1.

On the other hand, recall the definition $\hat{h}_t(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{v}\sim U(\mathbb{B})}[\mathcal{L}_t(\boldsymbol{x}+\rho\boldsymbol{v}, \boldsymbol{\lambda}_t)]$ in Eq. (10). Then we have

$$\Big|h_t(\boldsymbol{x}) - \hat{h}_t(\boldsymbol{x})\Big| = \mathbb{E}_{\boldsymbol{v}\sim U(\mathbb{B})}\Big[h_t(\boldsymbol{x}) - h_t(\boldsymbol{x}+\rho\boldsymbol{v})\Big] \le (1+K\frac{\bar{F}}{\beta})L\rho \cdot E_{\boldsymbol{v}\sim U(\mathbb{B})}\Big[\boldsymbol{v}\Big] = (1+K\frac{\bar{F}}{\beta})L\rho \tag{41}$$

where the inequality follows from the first part of this lemma. □

A.10 Proof of Lemma A.2

Proof. Recall the definitions $h_t(\boldsymbol{x}) = f_t(\boldsymbol{x}) + \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\boldsymbol{x})$ in Eq. (4), and $\hat{h}_t(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{v}\sim U(\mathbb{B})}[\mathcal{L}_t(\boldsymbol{x} + \rho\boldsymbol{v}, \boldsymbol{\lambda}_t)]$ in Eq. (10). Then, we have

$$\begin{aligned} \hat{h}_t(\boldsymbol{y}) - \hat{h}_t(\widetilde{\boldsymbol{x}}_t) &\overset{(a)}{\le} \langle \nabla \hat{h}_t(\widetilde{\boldsymbol{x}}_t), \boldsymbol{y} - \widetilde{\boldsymbol{x}}_t\rangle \\ &\overset{(b)}{=} \Big\langle \frac{d}{\rho} \cdot \mathbb{E}_{\boldsymbol{u}\sim U(\mathbb{S})}\left[h_t(\widetilde{\boldsymbol{x}}_t + \rho\boldsymbol{u}) \cdot \boldsymbol{u}\right] , \ \boldsymbol{y} - \widetilde{\boldsymbol{x}}_t \Big\rangle \\ &= \mathbb{E}_{\boldsymbol{u}_t\sim U(\mathbb{S})}\left[\Big\langle \frac{d}{\rho} \cdot h_t(\widetilde{\boldsymbol{x}}_t + \rho\boldsymbol{u}_t) \cdot \boldsymbol{u}_t \ , \ \boldsymbol{y} - \widetilde{\boldsymbol{x}}_t\Big\rangle\right] \\ &\overset{(c)}{=} \mathbb{E}_{\boldsymbol{u}_t\sim U(\mathbb{S})}\left[\langle \nabla_t, \boldsymbol{y} - \widetilde{\boldsymbol{x}}_t\rangle\right] \\ &\overset{(d)}{=} \mathbb{E}_{\boldsymbol{u}_t\sim U(\mathbb{S})}\left[\ell_t(\widetilde{\boldsymbol{x}}_t) - \ell_t(\boldsymbol{y})\right] \end{aligned} \tag{42}$$

where (a) follows from concavity of $\hat{h}_t(\cdot)$; (b) follows from Lemma B.2 by taking $h = -h_t$, so that in the lemma $-\nabla_{\boldsymbol{x}}\mathbb{E}_{\boldsymbol{v}\sim U(\mathbb{B})}[h(\boldsymbol{x}+\rho\boldsymbol{v})] = \nabla\hat{h}_t(\boldsymbol{x})$ and $-\mathbb{E}_{\boldsymbol{u}\sim U(\mathbb{S})}\left[h(\boldsymbol{x}+\rho\boldsymbol{u})\cdot\boldsymbol{u}\right] = \mathbb{E}_{\boldsymbol{u}\sim U(\mathbb{S})}\left[h_t(\boldsymbol{x}+\rho\boldsymbol{u})\cdot\boldsymbol{u}\right]$; (c) follows from the gradient estimate in Eq. (6) where

$$\nabla_t = \frac{d}{\rho}\left(f_t(\boldsymbol{x}_t) + \boldsymbol{\lambda}_t^\top \boldsymbol{g}_t(\boldsymbol{x}_t)\right) \cdot \boldsymbol{u}_t = \frac{d}{\rho} \cdot h_t(\boldsymbol{x}_t) \cdot \boldsymbol{u}_t = \frac{d}{\rho} \cdot h_t(\widetilde{\boldsymbol{x}}_t + \rho\boldsymbol{u}_t) \cdot \boldsymbol{u}_t$$

Finally, (d) follows from the definition of surrogate loss functions in Eq. (7). □

A.11 Proof of Lemma A.3

Proving (i):

Proof. Since $\widetilde{\boldsymbol{x}}_{t+1}^i = \Pi_{(1-\alpha)\mathcal{X}}(\widetilde{\boldsymbol{x}}_t^i + \gamma_i\boldsymbol{\nabla}_t)$ we have $\|\boldsymbol{y} - \widetilde{\boldsymbol{x}}_{t+1}^i\| \le \|\boldsymbol{y} - (\widetilde{\boldsymbol{x}}_t^i + \gamma_i\boldsymbol{\nabla}_t)\|$ for any $\boldsymbol{y} \in (1-\alpha)\mathcal{X}$. Then

$$\begin{aligned} &\|\boldsymbol{y} - \widetilde{\boldsymbol{x}}_{t+1}^i\|^2 \le \|\boldsymbol{y} - \widetilde{\boldsymbol{x}}_t^i\|^2 - 2\gamma_i\nabla_t^\top(\boldsymbol{y} - \widetilde{\boldsymbol{x}}_t^i) + \gamma_i^2\nabla_t^2 \\ \implies \ &\|\widetilde{\boldsymbol{x}}_{t+1}^i\|^2 \le \|\widetilde{\boldsymbol{x}}_t^i\|^2 + 2\boldsymbol{y}^\top(\widetilde{\boldsymbol{x}}_{t+1}^i - \widetilde{\boldsymbol{x}}_{t+}^i) - 2\gamma_i\nabla_t^\top(\boldsymbol{y} - \widetilde{\boldsymbol{x}}_t^i) + \gamma_i^2\nabla_t^2 \end{aligned}$$

Hence by taking $\boldsymbol{y} = (1-\alpha)\boldsymbol{y}_t \in (1-\alpha)\mathcal{X}$ and rearranging we get

$$\begin{aligned} & 2\gamma_i \left(\ell_t\left(\widetilde{\boldsymbol{x}}_t^i\right) - \ell_t\left((1-\alpha)\boldsymbol{y}_t\right)\right) \\ = & \; 2\gamma_i \nabla_t^\top ((1-\alpha)\boldsymbol{y}_t - \widetilde{\boldsymbol{x}}_t^i) \\ \leq & \; \|\widetilde{\boldsymbol{x}}_t^i\|^2 - \|\widetilde{\boldsymbol{x}}_{t+1}^i\|^2 + 2(1-\alpha)\boldsymbol{y}_t^\top(\widetilde{\boldsymbol{x}}_{t+1}^i - \widetilde{\boldsymbol{x}}_t^i) + \gamma_i^2 \nabla_t^2 \end{aligned} \tag{43}$$

Telescoping with $\tau = 1 \ldots t$ we get

$$\begin{aligned} & \sum_{\tau\in[t]} \ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau^i\right) - \sum_{\tau\in[t]} \ell_\tau\left((1-\alpha)\boldsymbol{y}_\tau\right) \\ = & \; \frac{1}{2\gamma_i}\|\widetilde{\boldsymbol{x}}_1^i\|^2 + \frac{1-\alpha}{\gamma_i}\sum_{t\in[T]} \boldsymbol{y}_\tau^\top(\widetilde{\boldsymbol{x}}_{\tau+1}^i - \widetilde{\boldsymbol{x}}_\tau^i) + \frac{\gamma_i}{2}\sum_{\tau\in[t]}\nabla_\tau^2 \\ = & \; \frac{1}{2\gamma_i}\|\widetilde{\boldsymbol{x}}_1^i\|^2 + \frac{1-\alpha}{\gamma_i}\left(\sum_{\tau\in[t-1]} \left(\boldsymbol{y}_\tau - \boldsymbol{y}_{\tau+1}\right)^\top \widetilde{\boldsymbol{x}}_{\tau+1}^i + \boldsymbol{y}_\tau^\top \widetilde{\boldsymbol{x}}_{\tau+1}^i\right) + \frac{\gamma_i}{2}\sum_{\tau\in[t]}\nabla_\tau^2 \\ \leq & \; \frac{1}{2\gamma_i}\|\widetilde{\boldsymbol{x}}_1^i\|^2 + \frac{1-\alpha}{\gamma_i}\sum_{\tau\in[t-1]}\left(\|\boldsymbol{y}_\tau - \boldsymbol{y}_{\tau+1}\|\cdot\|\widetilde{\boldsymbol{x}}_{\tau+1}^i\| + \|\boldsymbol{y}_\tau\|\cdot\|\widetilde{\boldsymbol{x}}_{\tau+1}^i\|\right) + \frac{\gamma_i}{2}\sum_{\tau\in[t]}\nabla_\tau^2 \\ \leq & \; \frac{(1-\alpha)^2 D^2}{2\gamma_i} + \frac{(1-\alpha)^2 D}{\gamma_i}\left(P(\boldsymbol{y}_{1:T}) + D\right) + \frac{\gamma_i d^2}{2\rho^2}\left(\bar{F} + K\frac{\bar{F}}{\beta}\bar{G}\right)^2 t \end{aligned} \tag{44}$$

□

Proving (ii):

Proof. First, we have for any $t \in [T]$, $i \in [N]$

$$\left|\ell_t(\widetilde{\boldsymbol{x}}_t^i)\right| \;=\; \left|\boldsymbol{\nabla}_t^T(\widetilde{\boldsymbol{x}}_t - \widetilde{\boldsymbol{x}}_t^i)\right| \;\leq\; \|\boldsymbol{\nabla}_t\|\cdot\|\widetilde{\boldsymbol{x}}_t^i - \widetilde{\boldsymbol{x}}_t\| \;\leq\; \frac{d}{\rho}\left(\bar{F} + K\frac{\bar{F}}{\beta}\bar{G}\right)\cdot(1-\alpha)D \tag{45}$$

where we recall $D = \sup_{\boldsymbol{x},\boldsymbol{x}'\in\mathcal{X}}\|\boldsymbol{x} - \boldsymbol{x}'\|$ is the diameter of $\mathcal{X}$, and both $\widetilde{\boldsymbol{x}}_t^i, \widetilde{\boldsymbol{x}}_t \in (1-\alpha)\mathcal{X}$.

Define $W_t = \sum_{i\in[N]} w_{i,t}$ for all $t \in [T]$, then

$$\begin{aligned} \log\left(\frac{W_{t+1}}{W_t}\right) \;&=\; \log\left(\sum_{i\in[N]}\frac{w_{i,t}\exp\left(-\epsilon\ell_t(\widetilde{\boldsymbol{x}}_t^i)\right)}{W_t}\right) \\ &=\; \log\left(\mathbb{E}_{I_t\sim\boldsymbol{w}_t/W_t}\left[\exp\left(-\epsilon\ell_t(\widetilde{\boldsymbol{x}}_t^{I_t})\right)\right]\right) \\ &\overset{(a)}{\leq}\; -\epsilon\mathbb{E}_{I_t\sim\boldsymbol{w}_t/W_t}\left[\ell_t(\widetilde{\boldsymbol{x}}_t^{I_t})\right] + \frac{\epsilon^2}{8} \\ &\overset{(b)}{=}\; -\epsilon\ell_t\left(\mathbb{E}_{I_t\sim\boldsymbol{w}_t/W_t}\left[\widetilde{\boldsymbol{x}}_t^{I_t}\right]\right) + \frac{\epsilon^2}{8} \\ &\overset{(c)}{=}\; -\epsilon\ell_t\left(\widetilde{\boldsymbol{x}}_t\right) + \frac{\epsilon^2}{8} \end{aligned} \tag{46}$$

Here (a) follows from Hoeffding's Lemma as described in Lemma B.1 where we take $X = \ell_t(\widetilde{\boldsymbol{x}}_t^{I_t})$, $a =$ and $b =$; (b) follows from the definition that $\ell_t(\widetilde{\boldsymbol{x}}) = \boldsymbol{\nabla}_t^T(\widetilde{\boldsymbol{x}} - \widetilde{\boldsymbol{x}}_t)$ is a linear function in $\widetilde{\boldsymbol{x}}$; (c) follows from Eq.(5).

Hence, telescoping the above we get

$$\log\left(\frac{W_{t+1}}{W_1}\right) \;\leq\; -\epsilon\sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau\right) + \frac{t\epsilon^2}{8} \tag{47}$$

On the other hand, we have

$$\begin{aligned}
\log\left(\frac{W_{t+1}}{W_1}\right) &= \log(W_{t+1}) - \log(W_1) \\
&\geq \log(\max_{i\in[N]} w_{i,t}) - \log(N) \\
&= \max_{i\in[N]} \log(w_{i,t}) - \log(N) \\
&\overset{(a)}{=} \max_{i\in[N]} \log\left(w_{i,1}\exp\left(-\epsilon\sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau^i\right)\right)\right) - \log(N) \\
&= -\epsilon \min_{i\in[N]}\sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau^i\right) - \log(N)
\end{aligned} \tag{48}$$

Hence, combining Eqs.(47) and (48), and dividing both sides by $\epsilon > 0$ we get

$$\begin{aligned}
-\sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau\right) + \frac{t\epsilon}{8} \geq -\min_{i\in[N]}\sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau^i\right) - \frac{\log(N)}{\epsilon} \\
\Longrightarrow \sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau\right) - \min_{i\in[N]}\sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau^i\right) &\leq \frac{t\epsilon}{8} + \frac{\log(N)}{\epsilon} \\
\Longrightarrow \sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau\right) - \sum_{\tau\in[t]}\ell_\tau\left(\widetilde{\boldsymbol{x}}_\tau^i\right) &\leq \frac{t\epsilon}{8} + \frac{\log(N)}{\epsilon}, \quad \forall i\in[N]
\end{aligned} \tag{49}$$

□

B Supplementary lemmas

Lemma B.1 (Hoeffding's lemma)**.** *Let X be some random variable such that $a \leq X \leq b$ almost surely for some $a, b \in \mathbb{R}$. Then for any $\epsilon \in \mathbb{R}$, we have $\mathbb{E}\left[\exp(-\epsilon X)\right] \leq \exp\left(-\epsilon\mathbb{E}\left[X\right] + \frac{\epsilon^2(b-a)^2}{8}\right)$.*

Lemma B.2 ([25] Lemma 2.1)**.** *Let $h : \mathcal{X} \to \mathbb{R}$ be some convex function (not necessarily differentiable). Then for any $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^d$ and $\delta > 0$ we have*

$$\nabla_{\boldsymbol{x}}\mathbb{E}_{\boldsymbol{v}\sim U(\mathbb{B})}[h(\boldsymbol{x}+\delta\boldsymbol{v})] = \frac{d}{\delta}\cdot\mathbb{E}_{\boldsymbol{u}\sim U(\mathbb{S})}\left[h(\boldsymbol{x}+\delta\boldsymbol{u})\cdot\boldsymbol{u}\right] \tag{50}$$