Similarity, Compression and Local Steps: Three Pillars of Efficient Communications for Distributed Variational Inequalities

Aleksandr Beznosikov
Innopolis University, Skoltech, MIPT, Yandex

Martin Takáč
MBZUAI

Alexander Gasnikov
MIPT, Skoltech, IITP RAS

Abstract

Variational inequalities are a broad and flexible class of problems that includes minimization, saddle point, and fixed point problems as special cases. Therefore, variational inequalities are used in various applications ranging from equilibrium search to adversarial learning. With the increasing size of data and models, today's instances demand parallel and distributed computing for real-world machine learning problems, most of which can be represented as variational inequalities. Meanwhile, most distributed approaches have a significant bottleneck – the cost of communications. The three main techniques to reduce the total number of communication rounds and the cost of one such round are the similarity of local functions, compression of transmitted information, and local updates. In this paper, we combine all these approaches. Such a triple synergy did not exist before for variational inequalities and saddle problems, nor even for minimization problems. The methods presented in this paper have the best theoretical guarantees of communication complexity and are significantly ahead of other methods for distributed variational inequalities. The theoretical results are confirmed by adversarial learning experiments on synthetic and real datasets.

1 Introduction

Variational Inequalities (VI) have a rather long history of research. Their first applications were the equilibrium problems in economics and game theory [72, 34, 29]. Variational inequalities were seen as a universal paradigm, involving other attractive problems, including minimization, Saddle Point Problems (SPP), fixed point problems, and others. Whereas research for minimization was actively developed separately, research for saddle point problems was often coupled with science's development around variational inequalities. This trend continues even now. This is partly related to the fact that the methods for minimization problems are not always suitable for SPP and VI. Moreover, from the theoretical point of view, these methods give relatively poor or even zero guarantees of convergence. Therefore a separate branch of algorithms has been developed for VI and SPP. Such methods include, e.g. `Extra Gradient` [50, 71, 42], Tseng's method [84] and many others [14]. Over time, VI found new applications, e.g. in signal processing [77]. In the last decade, they also attracted the interest of the machine learning community. Nowadays, VI and SPP are used in various aspects of machine learning. For example, there are many applications for the already classical learning tasks in supervised learning (with non-separable loss [40]; with non-separable regularizer [6]), unsupervised learning (discriminative clustering [89]; matrix factorization [7]), image denoising [28, 17], robust optimization [10]. It turns out that VI can be also linked with modern areas of learning science, including reinforcement learning [73, 39], adversarial training [62], and GANs [31, 23, 30, 67, 20, 57, 74] to name a few.

Machine learning's rapid development offers more possibilities for consumers, but poses challenges for engineers. To meet modern society's demands, the performance and generalization capacity of the model must be improved. This is currently done by increasing training data and model size [5].

37th Conference on Neural Information Processing Systems (NeurIPS 2023).

Hardware acceleration and most importantly distributed algorithm design are essential for effective handling of these workloads [85]. The distributed nature of the problem occurs not only due to a desire to speed up the learning process via parallelization, but because of the severe external constraints, e.g. as in Federated Learning (FL) [49, 56]. FL is a pioneering area of distributed learning, where users' personal devices are computing machines. Private data transfer from local devices is prohibited due to privacy constraints, making distributed approaches the only practical option. Most distributed algorithms have communications as a bottleneck. Communication wastes time, slowing down training/optimization and preventing effective parallelization. Different approaches can reduce communication rounds and cost per round. The first popular approach is the compression of transmitted information [2]; it helps to increase the speed of the communication round without much loss in the quality of learning. The second approach is use of local steps [64]; by increasing the number of local computations, it is sometimes possible to significantly reduce the number of communication rounds. Finally, the third basic approach relies on data similarity [79]. Local steps are often employed to help with this approach. Data similarity allows us to assume local loss functions on devices are similar, meaning local computations are close and replaceable. Both approaches and combinations have been extensively researched for minimizations, but a synergy of all three pillars of distributed algorithms has not been presented for either minimization problems or VI. Therefore, the purpose of this paper is to answer the following questions:

Table 1: Summary of the results on the number of transmitted information in different approaches to **communication bottleneck for distributed VI/SPP**.

Method	**Approach**	**Communication complexity**
`Extra Gradient` [50, 42]		$\mathcal{O}\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\right)$
`Local SGDA` [25]	local steps	$\mathcal{O}\left(\frac{L^{2/5}n^{2/5}}{\mu^{2/5}\varepsilon^{1/5}} + \frac{L\zeta}{\mu^{3/2}\sqrt{\varepsilon}}\right)$
`FedAvg-S` [36]	local steps	$\mathcal{O}\left(\frac{L^2}{\mu^2}\log\frac{1}{\varepsilon} + \frac{L\zeta}{\mu^2\sqrt{\varepsilon}}\right)$
`SCAFFOLD-S` [36]	local steps	$\mathcal{O}\left(\frac{L^2}{\mu^2}\log\frac{1}{\varepsilon}\right)$
`SCAFFOLD-Catalyst-S` [36]	local steps	$\mathcal{O}\left(\frac{L}{\mu}\log^2\frac{1}{\varepsilon}\right)$
`ESTVGM` [11]	local steps	$\mathcal{O}\left(\frac{L}{\mu}\log\frac{1}{\varepsilon} + \frac{L\zeta}{\mu^2\sqrt{\varepsilon}}\right)$
`SMMDS` [16]	similarity local steps	$\mathcal{O}\left(\left[1 + \frac{\delta}{\mu}\right]\log\frac{1}{\varepsilon}\right)$
`MASHA` [15]	compression	$\mathcal{O}\left(\frac{L}{\sqrt{n}\mu}\log\frac{1}{\varepsilon}\right)$
`Optimistic MASHA` [12]	similarity compression	$\mathcal{O}\left(\left[\frac{L}{n\mu} + \frac{\delta}{\sqrt{n}\mu}\right]\log\frac{1}{\varepsilon}\right)$
`Accelerated Extra Gradient` [51]	similarity local steps	$\mathcal{O}\left(\left[1 + \frac{\delta}{\mu}\right]\log\frac{1}{\varepsilon}\right)$
Algorithm 1 (this paper)	similarity local steps compression	$\mathcal{O}\left(\left[1 + \frac{\delta}{\sqrt{n}\mu}\right]\log\frac{1}{\varepsilon}\right)$
Algorithm 2 (this paper)	similarity local steps partial participation	$\mathcal{O}\left(\left[1 + \frac{\delta}{\sqrt{n}\mu}\right]\log\frac{1}{\varepsilon}\right)$
Lower bound [16]		$\Omega\left(\left[1 + \frac{\delta}{\mu}\right]\log\frac{1}{\varepsilon}\right)$ [(1)]
Lower bound (this paper)	similarity local steps compression	$\Omega\left(\left[1 + \frac{\delta}{\sqrt{n}\mu}\right]\log\frac{1}{\varepsilon}\right)$
Lower bound (this paper)	similarity local steps partial participation	$\Omega\left(\left[1 + \frac{\delta}{\sqrt{n}\mu}\right]\log\frac{1}{\varepsilon}\right)$

[(1)] lower bound is deterministic and does not take into account the possibility of compression and partial participation. *Notation:* μ = constant of strong monotonicity of the operator F (strong convexity – strong concavity of the corresponding saddle point problem), L = Lipschitz constant of F, δ = similarity parameter (typically, $\delta \ll L$), ζ = heterogeneity parameter, n = number of devices, b = local data size, ε = precision of the solution.

Is it possible to combine the three main techniques for dealing with the comm. bottleneck into one algorithm? Does the new method have better convergence in both theory and practice?

2 Related works

Recent years have seen a surge of research into effective communication techniques for distributed algorithms. Much of this work focuses on minimization problems, but there are also methods for SPP and VI that address the communication bottleneck issue. We summarize these approaches in Table 1.

• **Local methods.** The use of local steps is probably the most popular approach to the communication bottleneck. Methods with local updates are also widely used in general distributed and FL. The idea of distributed minimization using methods similar to Local SGD is not an achievement of recent years and comes back to [64, 65, 91]. The development of theoretical analysis and new modifications for the original methods can be traced in the following works [58, 80, 46, 45, 87, 86, 48]. With the appearance of [69], local methods for distributed minimization problems get a new boost. The first local methods for VI and SPP were simply adaptations of local algorithms for the minimization problem. In particular, the `Local SGD/FedAvg` method was adapted in [25, 36]. [36] also analysed the Scaffold method [45] for SPP. In [11], the `Extra Gradient` method, a key method for SPP and VI, was modified by the use of local steps.

• **Compression.** Methods with compression also have a good background. Here we can highlight the basic methods that add compression to GD or SGD [2], and modifications of this approach using memory [68]. Also of interest are works investigating biased compression operators [13] and error-compensation techniques [3, 81, 76]. For VI and SPP, methods with different compressions were proposed [15]. These algorithms are based on the variance reduction method. See also [76] and [15] for a more detailed survey of compressed methods for distributed minimization problems.

• **Similarity.** This paper examines the similarity of Hessians in distributed minimization problems, building on established theory and practice (Assumption 4.3). The first method for distributed minimization problems with data similarity was `DANE` [79]. The lower bounds on communications under similarity assumption was discussed in [4]. The optimal method was obtained in [90] (for quadratic problems) and [51] (for general case). For details and a complete picture of the methods for the distributed minimization problems with similarity, see [51] and [35]. The SPP were also considered under similarity assumption. In [16], both the lower bounds and the optimal method were obtained. In the current work, we "break down" these lower estimates by using compression (see Table 1 with footnote 1). Our current study advances these concepts by integrating compression techniques, significantly outperforming previous estimates (see Table 1, footnote 1). Most related methods (for the distributed problem with similarity) incorporate local steps.

• **Local methods with compression.** The combination of local updates and compression for minimization problems has been presented in [9, 70] (note that the theoretical rates are not optimal and used restrictive assumptions). Using the same local approach as in [69], in papers [63, 21], authors combine it with compression technique and get good convergence rates. For VI and SPP, there are no methods with both local steps and compression.

• **Compression and similarity.** Recently, a new class of compressions called permutation compressors was introduced for the distributed minimization problem under the data similarity assumption [82]. However, due to not using local steps technique, the gain from combining these two approaches was slight. For VI the close results were presented in [12].

3 Our contributions

We answer in the affirmative to the questions posed at the end of Section 1. In particular, our contributions are as follows.

• **Three Pillars Algorithm.** We present a new method `Three Pillars Algorithm` (Algorithm 1) that combines three approaches: compression, similarity, and local steps for the efficient solution of distributed VI and SPP.

• **The best communication complexity.** We analyze the convergence of the new distributed algorithm for smooth strongly monotone VI and strongly convex-strongly concave SPP under similarity assumption. As can be seen in Table 1, our algorithm has better communication complexity than all available competitors. Since $\delta \leq L$, it is easy to see that our algorithms always outperform all methods except `Optimistic MASHA` [12]. One can note that in the worst case with $\delta \sim L$ we repeat the results of `Optimistic MASHA`. But if the data are uniformly distributed, then $\delta \sim \frac{L}{\sqrt{b}}$ or even $\frac{L}{b}$, where b is the size of the local sample. Assume that $N = bn$ is the size of the whole data, then our methods outperform `Optimistic MASHA` when $L > \delta\sqrt{n}$, i.e. when $N > n^2$. Such a ratio is typical because already classic modern datasets contain hundreds of thousands [54, 53] or even millions [24] of samples, and the number of devices rarely exceeds 100.

• **Main calculations on server.** Many distributed approaches, such as one of the most popular from [66], use the power of all computing devices evenly. But in terms of FL, we prefer to have the main computational workload on the server, rather than on the local devices of the users [16, 51]. Our algorithm has this feature.

• **Extension with partial participation.** We present a modified version of our first method – Algorithm 2 (see Appendix A). Instead of compression, one device is selected to send an uncompressed message. This reduces the local complexity on devices compared to Algorithm 1, while keeping the overall communication complexity the same.

• **Lower bounds.** To demonstrate the optimality of our proposed approaches, we establish lower bounds for the communication complexities. These estimates enhance those from [16], which did not consider the potential for compression and partial participation. Our results are shown to be unsurpassable in both scenarios involving compression and partial participation (see Table 1).

• **Extension with stochastic local computations.** Motivated by applications where computation is expensive, we present a modification of Algorithm 1 that emphasizes the use of stochastic operators in local computations, termed Algorithm 3 (see Appendix A).

• **Non-monotone analysis.** Although we focus primarily on strongly monotone VI, many machine learning problems are non-monotone, and hence we extend the convergence analysis to this case. Our algorithm outperforms competitors in terms of communication complexity for both cases and is the first to gain acceleration in the non-monotone case.

• **Experimental confirmation.** Experimental results on adversarial training of regression models confirm the theoretical conclusion that both of our algorithms outperform their competitors.

4 Problem and assumptions

We denote standard inner product of $x, y \in \mathbb{R}^d$ as $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$, where x_i corresponds to the i-th component of x in the standard basis in $\mathbb{R}^d$. It induces ℓ_2-norm in $\mathbb{R}^d$ in the following way $\|x\| = \sqrt{\langle x, x\rangle}$. Operator $\mathbb{E}[\cdot]$ denotes full mathematical expectation and $\mathbb{E}_\xi[\cdot]$ introduces the conditional mathematical expectation on ξ.

Formally, the problem of finding a solution for the variation inequality can be stated as follows:

$$\text{Find } z^* \in \mathcal{Z} \text{ such that } \langle F(z^*), z - z^*\rangle \geq 0, \ \forall z \in \mathcal{Z}, \tag{1}$$

where $F : \mathcal{Z} \to \mathbb{R}^d$ is an operator, and $\mathcal{Z} \subseteq \mathbb{R}^d$ is a convex set. We assume that the training data describing F is distributed across n devices/nodes: $F(z) = \frac{1}{n}\sum_{i=1}^n F_i(z)$, where $F_n : \mathcal{Z} \to \mathbb{R}^d$ for all $i \in [n] := \{1, 2, \ldots, n\}$. To emphasize the extensiveness of the formalism (1), we give some examples of variational inequalities.

• **Minimization.** Consider the minimization problem:

$$\min_{z \in \mathcal{Z}} f(z). \tag{2}$$

Suppose that $F(z) = \nabla f(z)$. Then, if f is convex, it can be proved that $z^* \in \mathcal{Z}$ is a solution for (1) if and only if $z^* \in \mathcal{Z}$ is a solution for (2).

• **Saddle point problem.** Consider the saddle point problem:

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} g(x, y). \tag{3}$$

Suppose that $F(z) = F(x, y) = [\nabla_x g(x, y), -\nabla_y g(x, y)]$ and $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ with $\mathcal{X} \subseteq \mathbb{R}^{d_x}$, $\mathcal{Y} \subseteq \mathbb{R}^{d_y}$. Then, if g is convex-concave, it can be proved that $z^* \in \mathcal{Z}$ is a solution for (1) if and only if $z^* \in \mathcal{Z}$ is a solution for (3).

• **Fixed point problem.** Consider the fixed point problem:

$$\text{Find } z^* \in \mathbb{R}^d \text{ such that } T(z^*) = z^*, \tag{4}$$

where $T : \mathbb{R}^d \to \mathbb{R}^d$ is an operator. With $F(z) = z - T(z)$, it can be proved that $z^* \in \mathcal{Z} = \mathbb{R}^d$ is a solution for (1) if and only if $F(z^*) = 0$, i.e. $z^* \in \mathbb{R}^d$ is a solution for (4).

In order to prove convergence results, the following assumptions are introduced on the problem (1).

Assumption 4.1 (Lipschitzness)**.** Each operator F_i is L-Lipschitz continuous on $\mathcal{Z}$, i.e. for all $u, v \in \mathcal{Z}$ we have $\|F_i(u) - F_i(v)\| \leq L\|u - v\|$.

For problems (2) and (3), L-Lipschitzness of the operator means that the functions $f(z)$ and $f(x, y)$ are L-smooth.

Assumption 4.2 (Strong monotonicity)**.** The operator F is μ-strongly monotone on $\mathcal{Z}$, i.e. for all $u, v \in \mathcal{Z}$ we have $\langle F(u) - F(v), u - v\rangle \geq \mu\|u - v\|^2$. Each operator F_i is monotone on $\mathcal{Z}$, i.e. strongly monotone with $\mu = 0$.

For problems (2) and (3), strong monotonicity of F means strong convexity of $f(z)$ and strong convexity-strong concavity of $f(x, y)$.

Assumption 4.3 (δ-relatedness in mean)**.** The operators $\{F_i\}$ is δ-related in mean on $\mathcal{Z}$. It means that for any j operators $\{F_i - F_j\}$ is δ-Lipschitz continuous in mean on $\mathcal{Z}$, i.e. for all $u, v \in \mathcal{Z}$ we have $\frac{1}{n}\sum_{i=1}^n \|F_i(u) - F_j(u) - F_i(v) + F_j(v)\|^2 \leq \delta^2\|u - v\|^2$.

While Assumptions 4.1 and 4.2 are basic and widely known, Assumption 4.3 requires further remarks. This assumption goes back to the conditions of data similarity (as discussed in Section 2). In more detail, we consider distributed minimization (2) and saddle point (3) problems $f(z) = \frac{1}{n}\sum_{i=1}^n f_i(z)$, $f(x, y) = \frac{1}{n}\sum_{i=1}^n f_i(x, y)$, and assume that for minimization local and global hessians are δ-similar [79, 4, 90, 35, 51], i.e., $\|\nabla^2 f_j(z) - \nabla^2 f_i(z)\| \leq \delta$, and for SPP, second derivatives differ by δ [16, 51] $\|\nabla^2_{xx} f_j(x, y) - \nabla^2_{xx} f_i(x, y)\| \leq \delta$, $\|\nabla^2_{xy} f_j(x, y) - \nabla^2_{xy} f_i(x, y)\| \leq \delta$, $\|\nabla^2_{yy} f_j(x, y) - \nabla^2_{yy} f_i(x, y)\| \leq \delta$. In the worst case $\delta \sim L$, but often for practical machine learning problems $\delta < L$. In particular, if we assume that the data is uniformly distributed between devices, it can be proven (see Theorem 1.3 from [83], Theorem 2 from [35]) that $\delta = \tilde{\mathcal{O}}(L/\sqrt{b})$ or even $\delta = \tilde{\mathcal{O}}(L/b)$, where b is the number of local data points on each of the devices.

Let us note some of the terms we use in the paper. By local steps/updates, we mean that some device i, using only local information about operator F_i, changes the local value of variable z. Local computations/calculations includes all computations of F_i on a local device: local updates/steps may take place, or the value of the operator is accumulated and is sent to the other device without changes

Algorithm 1 `Three Pillars Algorithm`

```
Parameters: stepsizes $\gamma$ and $\eta$, momentum $\tau$, probability $p \in (0;1]$, number of local steps $H$;
Initialization: Choose $z^0 = m^0 = (x^0, y^0) \in \mathcal{Z}$;
for $k = 0, 1, \dots, K-1$ do
    Server takes $u_0^k = z^k$;
    for $t = 0, 1, \dots, H-1$ do
        Server computes $u_{t+1/2}^k = \text{proj}_{\mathcal{Z}}[u_t^k - \eta(F_1(u_t^k) - F_1(m^k) + F(m^k) + \frac{1}{\gamma}(u_t^k - z^k - \tau(m^k - z^k)))]$;
        Server updates $u_{t+1}^k = \text{proj}_{\mathcal{Z}}[u_t^k - \eta(F_1(u_{t+1/2}^k) - F_1(m^k) + F(m^k) + \frac{1}{\gamma}(u_{t+1/2}^k - z^k - \tau(m^k - z^k)))]$;
    end for
    Server broadcasts $u_H^k$ and $F_1(u_H^k)$ to devices;
    Devices in parallel compute $Q_i(F_i(m^k) - F_1(m^k) - F_i(u_H^k) + F_1(u_H^k))$ and send to server;
    Server updates $z^{k+1} = \text{proj}_{\mathcal{Z}}[u_H^k + \gamma \cdot \frac{1}{n}\sum_{i=1}^n Q_i(F_i(m^k) - F_1(m^k) - F_i(u_H^k) + F_1(u_H^k))]$;
    Server updates $m^{k+1} = \begin{cases} z^k, & \text{with probability } p, \\ m^k, & \text{with probability } 1-p, \end{cases}$;
    if $m^{k+1} = z^k$ then
        Server broadcasts $m^{k+1}$ to devices;
        Devices in parallel compute $F_i(m^k)$ and send to server;
        Server computes $F(m^{k+1}) = \frac{1}{n}\sum_{i=1}^n F_i(m^{k+1})$;
    end if
end for
```

in local variables z. The local device i complexity we mean the number of local computations of F_i during the entire operation of algorithms, and by the total local complexity – the total number of computations of F_i for all i. A communication round refers to establishing a connection between two devices and transmitting information in one or both directions. When multiple devices communicate in parallel—for instance, if all nodes send information to a central device—it is counted as a single communication round. Since we employ sparsifiers as compressors, we quantify the transmitted information by the number of coordinates sent. Communication complexity is the total volume of information exchanged between all devices during the execution of the algorithm. Communication time, on the other hand, is the duration spent on all such communication rounds.

5 Three pillars algorithm

5.1 The algorithm

We consider a centralized communication architecture where the main device is the server (node with number 1, which stores data associated with F_1). All other devices can send information to the server and receive responses from it. We follow the standard assumption that forwarding from devices to the server takes considerably longer than broadcasting from the server to the devices (see Section 3.5 from [43] and Section M.3 from [68]), hence we consider one-side compression from devices to the server. Our approach can be generalized to two-sided compression.

We split the idea of our new method `Three Pillars Algorithm` (Algorithm 1) into five parts.

• **Local problem.** The idea of the simplest local method is that the devices make a lot of independent gradient steps, using only their local operators/gradients and sometimes averaging the current value of the optimization variables [64, 66]. With this approach, between communications, the devices minimize their local functions and can move quite far away if the number of local steps is large [47]. Therefore, the estimates for `Local SGDA`, `FedAvg-S`, `ESTVGM` from Table 1 depend on the data heterogeneity parameter. This problem can be solved by modifying original local operators/functions via regularization. For example, we can ask to locally minimize not the local $f_i(z)$, but $f_i(z) + \lambda\|z - z_s\|^2$, where x_s is the last point synchronized with all devices [45]. Statistical preconditioning, the popular trick for most methods under similarity conditions, is also a regularization of the local problem [79, 90]. It also can be considered as Mirror Descent with special Bregman divergence (see Section 1.1. of [35] for more details). The papers [51, 16] suggest looking at the stochastic preconditioning in distributed optimization from a different perspective. The authors propose to rewrite the original distributed problem as a composite problem of two terms: $\frac{1}{n}\sum_{i=1}^n f_i(x) = r(x) + q(x) = [f_1(x)] + [\frac{1}{n}\sum_{i=1}^n f_i(x) - f_1(x)]$. The proximal method works well for composite problems with proximal-friendly functions (e.g, simple ℓ_2 regularizer) where the calculation of the proximal operator is for free, but, in the general case, the proximal operator is calculated by an auxiliary method with another iteration loop inside the main algorithm. The key idea in [51, 16] is to use $r(x) = f_1(x)$ for the proximal operator. This implies the need of additional solving $\arg\min_x\{f_1(x) + \lambda\|x - x_s\|\}$, where x_s is some point. Note that no commutation is needed to calculate the proximal operator as to solution needs access only to f_1 and can be solved locally. This reasoning shows that the idea with composite reformulation is also related to the regularization of the local problem. To extend the idea with composite reformulation into VI (see (2) for connections between min. prob. and VIs), we use the following proximal operator (see the loop in line 3 in Alg. 1):

$$\text{Find } \hat{u}^k \in \mathcal{Z} \text{ such that } \langle G(\hat{u}^k), z - \hat{u}^k\rangle \geq 0, \ \forall z \in \mathcal{Z}$$

with $G(z) = F_1(z) + \frac{1}{\gamma}(z - v^k)$ and $v^k = z^k + \tau(m^k - z^k) - \gamma \cdot (F(m^k) - F_1(m^k))$, where the essence of m^k will be explained later. An important issue to be addressed is the selection of the basic method that should be used for an inexact calculation of the proximal operator (we will discuss it in the next paragraphs). We also need to underline one more point. Our approach solves the local problem (proximal operator) only on the server that owns F_1. It is possible to solve different local problems on all devices in parallel as proposed in [79], but this lead to non-optimal convergence guarantees. The method from [79] was revised in [90], which proposed local steps on one device and improved convergence rates. In [45], an algorithm was constructed that uses local updates on all devices and achieves the same convergence results as in [90] (only in the quadratic case). If no improvement is gained from running local updates on all devices, why should we do it?

• **Basic outer method.** We need a basic method for solving the composite VI. Gradient Descent is one option for minimization problems, but VI and SPP have their own optimal methods. Therefore, the basic outer method for Algorithm 1 also needs to be specialized for VI. In [51, 16], the autors used `Extra Gradient` [50, 71, 42]. In our work, we selected Tseng's method [84] for this task.

• **Local method.** Since the subproblem in the inner loop from line 3 in Algorithm 1 is a VI with a strongly monotone and Lipschitz continuous operator, then it can be solved using the `Extra Gradient` method [50]. It is optimal for this class of problems.

• **Use of compression.** In order to use compression in the method, an additional technique is also required. Following [75, 32, 15], we take the idea of basing on the variance reduction technique from [1]. We introduce a reference sequence $\{m^k\}_{k\geq 0}$ as in the classical variance reduction method [41]. At point m^k, we need to know the full values of operator F. When m^k is updated (line 10), we transfer the full operators without compression (lines 12 and 13). If probability p is small, m^k is rarely modified and hence condition 11 is satisfied with low probability. It turns out that in Algorithm 1 at each iteration all nodes send a compressed package and very rarely transmit a full operator. If p is chosen correctly, the effect of this is unnoticeable. But simply introducing an additional m^k sequence is not enough, we also need to add a negative momentum: $\tau(m^k - z^k)$ (lines 4 and 5). This is dictated by the nature of the variance reduction method for VI.

• **Compression operators.** In Algorithm 1 (lines 8, 9) any unbiased compressor can be used, but to reveal the δ-similarity (δ-relatedness), we apply the so-called permutation compressors [82].

Definition 5.1 (Permutation compressors [82])**.** For different cases of n and d, we define

• **for** $d \geq n$**.** Assume that $d \geq n$ and $d = qn$, where $q \geq 1$ is an integer. Let $\pi = (\pi_1, \ldots, \pi_d)$ be a random permutation of $\{1, \ldots, d\}$. Then for all $u \in \mathbb{R}^d$ and each $i \in \{1, 2, \ldots, n\}$ we define $Q_i(u) = n \cdot \sum_{j=q(i-1)+1}^{qi} u_{\pi_j} e_{\pi_j}$.

• **for** $d \leq n$**.** Assume that $n \geq d$, $n > 1$ and $n = qd$, where $q \geq 1$ is an integer. Define the multiset $S := \{1, \ldots, 1, 2, \ldots, 2, \ldots, d, \ldots, d\}$, where each number occurs precisely q times. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a random permutation of S. Then for all $u \in \mathbb{R}^d$ and each $i \in \{1, 2, \ldots, n\}$ we define $Q_i(u) = du_{\pi_i} e_{\pi_i}$.

In practice, the case of $d \geq n$ is more relevant. Permutation compressors require generating a new permutation of numbers from 1 to d each iteration. To ensure the same permutations are generated on all devices, a random seed can be defined in each iteration in a predetermined way (e.g. iteration counter). The permutation organizes how compression operators correspond. Take $d = 12$ and $n = 3$: a permutation $(5, 2, 10, 7, 4, 12, 1, 9, 3, 8, 11, 6)$ directs the first device to send coordinates $5, 2, 10, 7$; the second $4, 12, 1, 9$; the third $3, 8, 11, 6$. This system ensures non-overlapping data transmission, reducing the data sent per device by a factor of n and avoiding the zeroed "gradient" average that random selection risks. Permutation compressors thus correct redundancy and omission errors. They also capitalize on data similarity, ensuring the "gradient" at the server closely mirrors the true average. The original paper [82] recognized the benefits of this compressor with similar data but didn't fully leverage it due to its focus on non-convex settings without local steps.

• **Summary.** We use Tseng's method [84] for the composite VI, inaccurately computing the resolvent (proximal operator) with the `Extra Gradient` method [50]. Our convergence analysis is novel for Tseng's method and can be regarded as a byproduct and secondary contribution. We use a variance reduction technique [1] for Tseng's method, remade into a compression technique in distributed settings [75, 32, 15], and choose a special compression for similarity setting [82].

5.2 Convergence

This section presents the convergence results of Algorithm 1 and its analysis.

Theorem 5.2. *Let $\{z^k\}_{k\geq 0}$ denote the iterates of Algorithm 1 with compressors from Definition 5.1 for solving problem (1), which satisfies Assumptions 4.1, 4.2, 4.3. Then, if we choose the stepsizes $\gamma = \tilde{\mathcal{O}}(\min\{\frac{p}{\mu}, \frac{\sqrt{p}}{\delta}, \frac{H}{L}\})$, $\eta = \mathcal{O}((L + \frac{1}{\gamma})^{-1})$ and the momentum $\tau = p$ then we have the convergence guarantee $\mathbb{E}[\left\|z^K - z^*\right\|^2] \leq (1 - \frac{\gamma\mu}{2})^K \cdot 2\left\|z^0 - z^*\right\|^2$.*

In Algorithm 1, one mandatory communication round with compression occurs (lines 7 and 8) and possibly one more (without compression) with probability p (lines 12 and 13). Permutation compressor compress by a factor of n; hence each iteration requires $\mathcal{O}\left(\frac{1}{n} + p\right)$ data transfers from devices to the server on average: in line 8, we use compression and forward $\frac{d}{n}$ coordinates from the devices to the server, in line 13, with probability p we transfer the full vector of d coordinates.

If p is close to 1, Theorem 5.2 gives faster convergence, but more data transfer is needed. If p tends to 0, the transmitted information complexity per iteration decreases but the iterative convergence rate drops. The optimal choice of p is $\frac{1}{n}$, as stated in the corollary.

Corollary 5.3. *Under the conditions of Theorem 5.2, the following number of the outer iterations is needed to achieve the accuracy ε (in terms of $\mathbb{E}\left[\|z - z^*\|^2\right] \lesssim \varepsilon$) by Algorithm 1 with $p = \frac{1}{n}$:*

$$\mathcal{O}([n + \frac{\delta\sqrt{n}}{\mu} + \frac{L}{\mu H}] \log \frac{\|z^0 - z^*\|^2}{\varepsilon}). \tag{5}$$

Next, let us give a brief discussion of the previous estimate.

- **Communications and local calculations.** The iteration of Algorithm 1 has $(1 + p)$ communication rounds and forwards $\mathcal{O}\left(\frac{1}{n} + p\right)$ information on average. With $p = \frac{1}{n}$, the estimate in equation (5) is valid for both the number of iterations and communication rounds. To get an upper bound for the amount of information transmitted, divide (5) by n. The bound from Corollary 5.3 reflects the local complexity on all devices except the first (the server). To get an estimate for the first device, we need to multiply (5) by H.
- **Optimal choice of H.** It is clear from (5) that the number of local H steps can improve the number of communications, but there is a limit beyond which more iterations are not useful. Taking $H = \lceil \frac{L}{\delta\sqrt{n}} \rceil$ is optimal. If there is no data similarity ($\delta \sim L$), then one can choose $H = 1$, and Algorithm 1 communication complexity estimate becomes similar to those of `MASHA` and `Optimistic MASHA`. Choosing $H = 1$ implies that local updates are meaningless when $\delta \sim L$. Papers [45, 88] on distributed minimization problems show that collecting large batches works just as well as methods that perform many local updates. Our paper considers deterministic methods, so we collect full batches and need local steps only within the presence of data similarity.
- **"Break" lower bounds.** H from the previous paragraph yields estimates of $\tilde{\mathcal{O}}(n + \delta\sqrt{n}/\mu)$ and $\tilde{\mathcal{O}}(1 + \delta/\sqrt{n}\mu)$ for the number of communications and transmitted information, respectively. These results are better than any existing methods and even superior to lower bounds for deterministic methods. Note that our method uses randomized compression and hence lower bounds for deterministic methods does not apply for Algorithm 1 (see Table 1). To close this gap in Section 5.4 we provide new lower bound estimates that show the optimality of our approach.
- **Variable server.** In Algorithm 1, the server can be reselected deterministically or randomly at each outer iteration. Communication with the server can be decentralized using reduce and gather procedures [18], for this kind of procedures compression also greatly speeds up communication both theoretically and in practice [55]. Although the server can change, and decentralized communication networks can be used, it must still receive data from all devices. Asynchronous communication is not supported, the lack of communication with the device is also a problem for Algorithm 1. This is a potential problem that could slow down the method. This issue is partially solved by Algorithm 2 in the next section, which assumes communication with only 1 device per iteration.

5.3 Extension with partial participation

We also propose a modification of Algorithm 1 with partial participation (see Algorithm 2 in Appendix A). The essence of this method is to replace lines 8, 9 of Algorithm 1 with the following steps

```
8: Generate uniformly $i_k$ from $[n]$, device $i_k$ computes $F_{i_k}(u_H^k)$ and sends to server;
9: Server updates $z^{k+1} = \text{proj}_{\mathcal{Z}}[u_H^k + \gamma \cdot (F_{i_k}(m^k) - F_1(m^k) - F_{i_k}(u_H^k) + F_1(u_H^k))]$;
```

The following theorem is valid for Algorithm 2.

Theorem 5.4. *Let $\{z^k\}_{k\geq 0}$ denote the iterates of Algorithm 2 for solving problem (1), which satisfies Assumptions 4.1, 4.2, 4.3. Then, if we choose the stepsizes $\gamma = \tilde{\mathcal{O}}(\min\{\frac{p}{\mu}, \frac{\sqrt{p}}{\delta}, \frac{H}{L}\})$, $\eta = \mathcal{O}((L + \frac{1}{\gamma})^{-1})$ and the momentum $\tau = p$ then we have the convergence $\mathbb{E}[\left\|z^K - z^*\right\|^2] \leq \left(1 - \frac{\gamma\mu}{2}\right)^K \cdot 2\left\|z^0 - z^*\right\|^2$.*

While in lines 8 and 9 of Algorithm 1 all devices compute the full local operators but forward only $\mathcal{O}\left(\frac{1}{n}\right)$ information, in Algorithm 2 only one local device computes its operator and forwards this full operator. Hence, the total amount of transmitted information from all devices is equivalent for Algorithms 1 and 2. Therefore the optimal choice of p for Algorithm 2 is the same as for Algorithm 1.

Corollary 5.5. *Under the conditions of Theorem 5.4, $\mathcal{O}([n + \frac{\delta\sqrt{n}}{\mu} + \frac{L}{\mu H}]\log\frac{\|z^0 - z^*\|^2}{\varepsilon})$ outer iterations are needed for the accuracy ε (in terms of $\mathbb{E}\left[\|z - z^*\|^2\right] \lesssim \varepsilon$) by Algorithm 2 with $p = \frac{1}{n}$.*

Despite the fact that the estimates from Corollary 5.3 and 5.5 are the same. Algorithm 2 is better than Algorithm 1 in terms of local devices (but not the server) complexity. In line 8 of Algorithm 1 all devices collect full operator, while in Algorithm 2 only one node does it, then local complexity (not for first node) of Algorithm 2 is n times better. But while Algorithm 1 and Algorithm 2 are the same in terms of the number of transmitted information, Algorithm 2 is n times worse in terms of communication time. This is due to the fact that in line 8 of Algorithm 1, all devices in parallel send information using $\mathcal{O}\left(\frac{1}{n}\right)$ units of time, while in Algorithm 2, one node need $\mathcal{O}\left(1\right)$ units of time.

5.4 Lower bounds

To derive lower bounds, we define a "full package" as the amount of information transmitted in one complete communication round from all devices to the server without compression. We start from the lower bounds for algorithms with partial participation, as in the previous section.

Theorem 5.6. *For any $\delta, \mu > 0$ and $n \in \mathbb{N}$ and $K \in \mathbb{N}$, there exists a distirbuted variational inequality (satisfying Assumptions 4.2 and 4.3) on $\mathbb{R}^d$ (where d is sufficiently large) with $z^* \neq 0$, such that for any output $\hat{z}$ of any distributed algorithm with partial participation after sending K full packages from the devices to the server, it holds that $\mathbb{E}[\|\hat{z} - z^*\|^2]$ is $\Omega(\exp(-16/(1 + \sqrt{\frac{\delta^2 n}{16\mu^2} + 1}) \cdot K)\|z^0 - z^*\|^2)$.*

The lower bounds confirm that results of the form $(1 + \delta/(\sqrt{n}\mu))$ are optimal for distributed methods with partial participation; thus, our algorithm achieves this optimality. Lower bounds serve to present a "worst-case" distributed VI where any algorithm with partial participation cannot outperform the established bounds. For a formal classification of distributed algorithms with partial participation, an exemplification of a problematic case, and proofs of the lower bounds, see Appendix C.3. For the methods with compression, we can also show that our result is optimal. But here we have to restrict the class of compression operators and consider only random sparsifiers – random choice of coordinates for forwarding (Permutation compressors are suitable). See Appendix C.4. How to obtain lower guarantees for arbitrary compressors remains an open question.

5.5 Extension with stochastic local computations

In this section, we consider a stochastic formulation of (1), namely, we assume that each local term from (1) also has the form of a finite sum: $F_i(z) = \frac{1}{M}\sum_{j=1}^M F_{i,j}(z)$. Although we can compute F_i locally on each ith device, we would like to avoid this due to computational cost. The stochastic finite sum setting is common for machine learning and refers to Monte Carlo approach and empirical risk minimization [78]. Moreover, such a setup is more than natural especially in the context of similarity. For example, if we initially have a large dataset $N = nMb$ and distribute it to n devices, then each device has Mb samples: $F = \frac{1}{n}\sum_{i=1}^n F_i = \frac{1}{n}\sum_{i=1}^n[\frac{1}{M}\sum_{j=1}^M F_{i,j}] = \frac{1}{n}\sum_{i=1}^n[\frac{1}{M}\sum_{j=1}^M(\frac{1}{b}\sum_{l=1}^b F_{i,j,l})]$. The nature of similarity goes precisely that F_i have the form of a sub-sum taken from a whole large sum. As we noted in Section 4, the larger the sample size Mb for F_i, the smaller the similarity parameter δ. Now in the stochastic case we consider $F_{i,j}$ and they also have similarity, but with a larger $\tilde{\delta}$, since Mb is larger than b. To work with new setting we introduce two assumptions.

Assumption 5.7 (Lipschitzness in mean)**.** Each operator F_i is $\tilde{L}$-Lipschitz continuous in mean on $\mathcal{Z}$, i.e. for all $u, v \in \mathcal{Z}$ we have $\frac{1}{M}\sum_{j=1}^M \|F_{i,j}(u) - F_{i,j}(v)\|^2 \leq \tilde{L}^2\|u - v\|^2$.

Assumption 5.8. The operators $\{F_{i,j}\}$ is $\tilde{\delta}$-related in mean on $\mathcal{Z}$. For any i_1 operators $\{F_{i_1,j_1} - F_{i_2,j_2}\}$ is $\tilde{\delta}$-Lipschitz continuous in mean on $\mathcal{Z}$, i.e. for all $u, v \in \mathcal{Z}$ we have $\frac{1}{nM^2}\sum_{i_2=1}^{n}\sum_{j_1,j_2=1}^{M}\|F_{i_1,j_1}(u) - F_{i_2,j_2}(u) - F_{i_1,j_1}(v) + F_{i_2,j_2}(v)\|^2 \leq \tilde{\delta}^2\|u-v\|^2$.

The modified version of Algorithm 1 for the new setting looks as follows (see the full version in Appendix A – Algorithm 3):

```
2: Server takes $u_0^k = w_0^k = z^k$;
4: Server computes
       $u_{t+1/2}^k = \text{proj}_{\mathcal{Z}}[\alpha u_t^k + (1-\alpha)w_t^k - \eta(F_1(w_t^k) - F_1(m^k) + F(m^k) + \frac{1}{\gamma}(w_t^k - z^k - \tau(m^k - z^k)))]$;
5: Server updates
       $u_{t+1}^k = \text{proj}_{\mathcal{Z}}[\alpha u_t^k + (1-\alpha)w_t^k - \eta(F_{1,j_t}(u_{t+1/2}^k) - F_{1,j_t}(w_t^k) + F_1(w_t^k) - F_1(m^k) + F(m^k)$
           $+\frac{1}{\gamma}(u_{t+1/2}^k - z^k - \tau(m^k - z^k)))]$, where $j_t$ is generated uniformly from $[M]$;
   Server updates $w_{t+1}^k = \begin{cases} u_{t+1}^k, & \text{with probability } q, \\ w_t^k, & \text{with probability } 1-q, \end{cases}$;
7: Server broadcasts $u_H^k$ and $F_{1,j_k^1}(u_H^k) - F_{1,j_k^1}(m^k)$ to devices, where $j_k^1$ is generated uniformly from $[M]$;
8: Devices in parallel compute $Q_i(F_{i,j_k^i}(m^k) - F_{1,j_k^1}(m^k) - F_{i,j_k^i}(u_H^k) + F_{1,j_k^1}(u_H^k))$ and send to server, where $j_k^i$ is generated uniformly from $[M]$;
9: Server updates
       $z^{k+1} = \text{proj}_{\mathcal{Z}}[u_H^k + \gamma \cdot \frac{1}{n}\sum_{i=1}^{n} Q_i(F_{i,j_k^i}(m^k) - F_{1,j_k^1}(m^k) - F_{i,j_k^i}(u_H^k) + F_{1,j_k^1}(u_H^k))]$;
```

To deal with stochasticity we make two modifications in Algorithm 1: in lines 2–5 we switch the local algorithm from classical deterministic `Extra Gradient` to stochastic `Extra Gradient` from [1] (Algorithm 1), in lines 7–9 we use stochastic operators for all $\{F_i\}$ instead of deterministic ones. The next theorem and corollary provides the convergence.

Theorem 5.9. *Let $\{z^k\}_{k\geq 0}$ denote the iterates of Algorithm 3 with compressors from Definition 5.1 for solving problem (1), which satisfies Assumptions 4.2, 5.7, 5.8. Then, if we choose the stepsizes $\gamma = \tilde{\mathcal{O}}(\min\{\frac{p}{\mu}, \frac{\sqrt{p}}{\tilde{\delta}}, \frac{H}{\sqrt{M}\tilde{L}}\})$, $\eta = \mathcal{O}(M^{-1/2}(\tilde{L} + \frac{1}{\gamma})^{-1})$, the momentums $\tau = p$, $\alpha = 1 - q$ and the probability $q = \frac{1}{M}$ then we have the convergence guarantee $\mathbb{E}[\left\|z^K - z^*\right\|^2] \leq (1 - \frac{\gamma\mu}{2})^K \cdot 2\left\|z^0 - z^*\right\|^2$.*

Corollary 5.10. *Under the conditions of Theorem 5.9, $\mathcal{O}([n + \frac{\tilde{\delta}\sqrt{n}}{\mu} + \frac{\sqrt{M}\tilde{L}}{\mu H}]\log\frac{\|z^0 - z^*\|^2}{\varepsilon})$ outer iterations are needed for the accuracy ε (in terms of $\mathbb{E}\left[\|z - z^*\|^2\right] \lesssim \varepsilon$) by Alg.3 with $p = \frac{1}{n}$.*

5.6 Non-monotone case

In this section, we abandon the monotonicity assumption, but introduce the following assumption.

Assumption 5.11 (Non-monotonicity)**.** The operator F is non-monotone (minty), if and only if there exists $z^* \in \mathbb{R}^d$ such that for all $z \in \mathbb{R}^d$ we have $\langle F(z), z - z^*\rangle \geq 0$.

This assumption is called the minty or variational stability condition. It is not a general non-monotonicity, but is already associated in the community with non-monotonicity [22, 38, 67, 61, 44, 37, 26], particularly with the setup, which is somewhat appropriate for GANS [59, 60, 27, 8].

Theorem 5.12. *Let $\{z^k\}_{k\geq 0}$ denote the iterates of Algorithm 1 for solving problem (1) with $\mathcal{Z} = \mathbb{R}^d$, which satisfies Assumptions 4.1, 5.11, 4.3. Then, if we additionally assume that $\|z^k\|, \|u_H^k\|, \|z^*\| \leq \Omega$ and choose the stepsizes $\gamma = \frac{\sqrt{p}}{6\delta}$, $\eta = \frac{1}{2(L+1/\gamma)}$, the momentum $\tau = p \leq \frac{1}{2}$ then $\frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}[\|F(u_H^k)\|^2] \lesssim \frac{\delta^2\Omega^2}{p}\big(\frac{1}{K} + \frac{1}{\sqrt{H}}\big(\frac{L\sqrt{p}}{\delta} + 1\big) + \frac{1}{H}\big(\frac{L\sqrt{p}}{\delta} + 1\big)^2\big)$.*

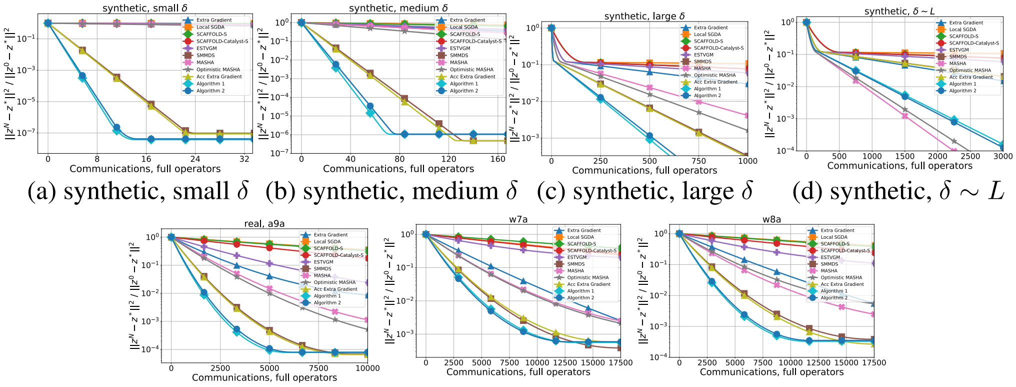
By substituting $H \sim K^2(\frac{L\sqrt{p}}{\delta} + 1)^2$, we can obtain that $\frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}[\|F(u_H^k)\|^2] \lesssim \frac{\delta^2\Omega^2}{pK}$. The discussion of the optimal choice of the parameter p, in this case, is the same as that following Theorem 5.2. Therefore, the following corollary is true.

Corollary 5.13. *Under the conditions of Theorem 5.12, $\mathcal{O}(\frac{n\delta^2\Omega^2}{\varepsilon^2})$ outer iterations is needed to achieve the accuracy ε (in terms of $\mathbb{E}\left[\|F(z)\|^2\right] \lesssim \varepsilon^2$) by Algorithm 1 with $p = \frac{1}{n}$.*

One can note that the number of information transmitted is n times less than the number of iterations in Corollary 5.13. Therefore, we have the communication complexity is equal to $\mathcal{O}\left(\delta^2\Omega^2/\varepsilon^2\right)$. This is better than the estimate $\mathcal{O}\left(L^2\Omega^2/\varepsilon^2\right)$ of a basic method that does not use local steps, similarity, and compression [33]. The problem is that our estimate of communication complexity does not benefit from large n. In the strongly monotone case, however, we observed a more pleasant situation (see Table 1). Unfortunately, this is a standard problem of the non-monotone case. For example, the methods from [15] with compression have the same complexity $\mathcal{O}\left(L^2\Omega^2/\varepsilon^2\right)$ as the basic method without compression. The methods from the paper [11] converge only in the special case when all local operators are equal to each other. Other papers from Table 1 do not consider the non-monotone case or consider it under other assumptions, then their complexity results are hard to compare with ours. Finally, it turns out that our work is the first to get an acceleration in terms of communication complexity in the non-monotone case. On the negative side, we can note that the estimate on the number of local iterations H in Theorem 5.12 is rather strict and grows with the number of iterations K. The results for Algorithm 2 in the non-monotone case can be obtained in a similar way.

6 Experiments

We conduct experiments on the robust linear regression problem. This problem is defined as $\min_{w \in \mathbb{R}^d} \max_{\|r_i\| \leq D} \frac{1}{2N} \sum_{i=1}^{N} (w^T(x_i + r_i) - y_i)^2 + \frac{\lambda}{2}\|w\|^2 - \frac{\beta}{2}\|r\|^2$, where w are model weights, $\{x_i, y_i\}_{i=1}^N$ is the training dataset and r is artificially added noise. We use ℓ_2-regularization on both w and r. We consider a network with $n = 25$ devices and two types of datasets: synthetic and real. Synthetic data allows us to control the factor δ, which measures statistical similarity of functions over different nodes. We take local datasets of size $b = 100$ and generate the dataset $\{\hat{x}_i, \hat{y}_i\}_{i=1}^n$ at the server randomly, with each entry drawn from the Standard Gaussian distribution. The datasets at other devices ($i = 2, \ldots, n$) are obtained by perturbing $\{\hat{x}_i, \hat{y}_i\}_{i=1}^n$ with normal noise ξ_i, having zero mean and controlled variance. Real datasets are taken from LibSVM library [19]. We uniformly divide the whole dataset between devices. The solution z^* is found using a large number of iterations of `Extra Gradient` [50, 42] method. Algorithms 1 and 2 are compared with methods from Table 1 (`Extra Gradient` [50, 42], `Local SGDA` [25], `FedAvg-S` [36], `SCAFFOLD-S` [36], `SCAFFOLD-Catalyst-S` [36], `ESTVGM` [11], `SMMDS` [16], `MASHA` [15], `Optimistic MASHA` [12], `Accelerated Extra Gradient` [51]) implemented in Python 3.7. All the methods are tuned as in the theory of the corresponding papers. The number of full operator transmissions by one of the arbitrary devices is used as the x-asis. The experiment results are shown in Figure 1. It is evident that Algorithm 1 and 2 consistently outperform all other competing algorithms by a significant margin. Furthermore, as the value of δ decreases, the superiority over `Local SGDA`, `SCAFFOLD-S`, `SCAFFOLD-Catalyst-S`, `MASHA`, and `Optimistic MASHA` becomes even more pronounced.



(a) synthetic, small δ (b) synthetic, medium δ (c) synthetic, large δ (d) synthetic, $\delta \sim L$

(e) real, a9a (f) real, w7a (g) real, w8a

Figure 1: Comparison of state-of-the-art methods for distributed VI. The comparison is made on synthetic datasets with small, medium, and large δ, as well as the real datasets a9a, w7a, w8a from LibSVM. The x-axis denotes the number of full operators transmitted by one of the devices.

7 Conclusion and future works

We presented two new algorithms for solving distributed VI and SPP: Algorithm 1, which combines compression, similarity, and local steps techniques; and Algorithm 2, which uses partial participation instead of compression. Both algorithms have the best communication complexity, outperforming existing methods in theory and practice. For future work we plan to i) derive new methods to stochastic settings (finite-sum or bounded variance setups); ii) obtain lower bounds on communication complexity with compression.

Acknowledgments

This research of A. Beznosikov has been supported by The Analytical Center for the Government of the Russian Federation (Agreement No. 70-2021-00143 dd. 01.11.2021, IGK 000000D730321P5Q0002).

References

[1] Ahmet Alacaoglu and Yura Malitsky. Stochastic variance reduction for variational inequality methods. *arXiv preprint arXiv:2102.08352*, 2021.

[2] Dan Alistarh, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. QSGD: Communication-efficient SGD via gradient quantization and encoding. In *Advances in Neural Information Processing Systems*, pages 1709–1720, 2017.

[3] Dan Alistarh, Torsten Hoefler, Mikael Johansson, Sarit Khirirat, Nikola Konstantinov, and Cédric Renggli. The convergence of sparsified gradient methods. In *Advances in Neural Information Processing Systems*, 2018.

[4] Yossi Arjevani and Ohad Shamir. Communication complexity of distributed convex learning and optimization. *Advances in neural information processing systems*, 28, 2015.

[5] Sanjeev Arora, Nadav Cohen, and Elad Hazan. On the optimization of deep networks: Implicit acceleration by overparameterization. In *Proceedings of the 35th International Conference on Machine Learning (ICML)*, 2018.

[6] Francis Bach, Rodolphe Jenatton, Julien Mairal, and Guillaume Obozinski. Optimization with sparsity-inducing penalties. *arXiv preprint arXiv:1108.0775*, 2011.

[7] Francis Bach, Julien Mairal, and Jean Ponce. Convex sparse matrix factorizations. *arXiv preprint arXiv:0812.1869*, 2008.

[8] Babak Barazandeh, Tianjian Huang, and George Michailidis. A decentralized adaptive momentum method for solving a class of min-max optimization problems. *Signal Processing*, 189:108245, 2021.

[9] Debraj Basu, Deepesh Data, Can Karakus, and Suhas Diggavi. Qsparse-local-sgd: Distributed sgd with quantization, sparsification and local computations. *Advances in Neural Information Processing Systems*, 32, 2019.

[10] Aharon Ben-Tal, Laurent El Ghaoui, and Arkadi Nemirovski. *Robust Optimization*. Princeton University Press, 2009.

[11] Aleksandr Beznosikov, Pavel Dvurechensky, Anastasia Koloskova, Valentin Samokhin, Sebastian U Stich, and Alexander Gasnikov. Decentralized local stochastic extra-gradient for variational inequalities. *arXiv preprint arXiv:2106.08315*, 2021.

[12] Aleksandr Beznosikov and Alexander Gasnikov. Compression and data similarity: Combination of two techniques for communication-efficient solving of distributed variational inequalities. In *International Conference on Optimization and Applications*, pages 151–162. Springer, 2022.

[13] Aleksandr Beznosikov, Samuel Horváth, Peter Richtárik, and Mher Safaryan. On biased compression for distributed learning. *arXiv preprint arXiv:2002.12410*, 2020.

[14] Aleksandr Beznosikov, Boris Polyak, Eduard Gorbunov, Dmitry Kovalev, and Alexander Gasnikov. Smooth monotone stochastic variational inequalities and saddle point problems–survey. *arXiv preprint arXiv:2208.13592*, 2022.

[15] Aleksandr Beznosikov, Peter Richtárik, Michael Diskin, Max Ryabinin, and Alexander Gasnikov. Distributed methods with compressed communication for solving variational inequalities, with theoretical guarantees. *arXiv preprint arXiv:2110.03313*, 2021.

[16] Aleksandr Beznosikov, Gesualdo Scutari, Alexander Rogozin, and Alexander Gasnikov. Distributed saddle-point problems under data similarity. *Advances in Neural Information Processing Systems*, 34:8172–8184, 2021.

[17] Antonin Chambolle and Thomas Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *Journal of mathematical imaging and vision*, 40(1):120–145, 2011.

[18] Ernie Chan, Marcel Heimlich, Avi Purkayastha, and Robert van de Geijn. Collective communication: theory, practice, and experience. *Concurrency and Computation: Practice and Experience*, 19(13):1749–1783, 2007.

[19] Chih-Chung Chang and Chih-Jen Lin. Libsvm: a library for support vector machines. *ACM transactions on intelligent systems and technology (TIST)*, 2(3):1–27, 2011.

[20] Tatjana Chavdarova, Gauthier Gidel, François Fleuret, and Simon Lacoste-Julien. Reducing noise in gan training with variance reduced extragradient. *arXiv preprint arXiv:1904.08598*, 2019.

[21] Laurent Condat, Ivan Agarsky, and Peter Richtárik. Provably doubly accelerated federated learning: The first theoretically successful combination of local training and compressed communication. *arXiv preprint arXiv:2210.13277*, 2022.

[22] Cong D Dang and Guanghui Lan. On the convergence properties of non-Euclidean extragradient methods for variational inequalities with generalized monotone operators. *Computational Optimization and Applications*, 60(2):277–310, 2015.

[23] Constantinos Daskalakis, Andrew Ilyas, Vasilis Syrgkanis, and Haoyang Zeng. Training gans with optimism. *arXiv preprint arXiv:1711.00141*, 2017.

[24] Jia Deng, Wei Dong, Richard Socher, Li-Jia Li, Kai Li, and Li Fei-Fei. Imagenet: A large-scale hierarchical image database. In *2009 IEEE Conference on Computer Vision and Pattern Recognition*, pages 248–255, 2009.

[25] Yuyang Deng and Mehrdad Mahdavi. Local stochastic gradient descent ascent: Convergence analysis and communication efficiency. In *International Conference on Artificial Intelligence and Statistics*, pages 1387–1395. PMLR, 2021.

[26] Jelena Diakonikolas, Constantinos Daskalakis, and Michael Jordan. Efficient methods for structured nonconvex-nonconcave min-max optimization. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, pages 2746–2754. PMLR, 2021.

[27] Zehao Dou and Yuanzhi Li. On the one-sided convergence of adam-type algorithms in non-convex non-concave min-max optimization. *arXiv preprint arXiv:2109.14213*, 2021.

[28] Ernie Esser, Xiaoqun Zhang, and Tony F Chan. A general framework for a class of first order primal-dual algorithms for convex optimization in imaging science. *SIAM Journal on Imaging Sciences*, 3(4):1015–1046, 2010.

[29] Francisco Facchinei and Jong-Shi Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Series in Operations Research. Springer, 2003.

[30] Gauthier Gidel, Hugo Berard, Gaëtan Vignoud, Pascal Vincent, and Simon Lacoste-Julien. A variational inequality perspective on generative adversarial networks. *arXiv preprint arXiv:1802.10551*, 2018.

[31] Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In *Advances in Neural Information Processing Systems (NIPS)*, 2014.

[32] Eduard Gorbunov, Konstantin P. Burlachenko, Zhize Li, and Peter Richtarik. Marina: Faster non-convex distributed learning with compression. In Marina Meila and Tong Zhang, editors, *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pages 3788–3798. PMLR, 18–24 Jul 2021.

[33] Eduard Gorbunov, Adrien Taylor, Samuel Horváth, and Gauthier Gidel. Convergence of proximal point and extragradient-based methods beyond monotonicity: the case of negative comonotonicity, 2022.

[34] P. T. Harker and J.-S. Pang. Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Mathematical programming*, 1990.

[35] Hadrien Hendrikx, Lin Xiao, Sebastien Bubeck, Francis Bach, and Laurent Massoulie. Statistically preconditioned accelerated gradient method for distributed optimization. In *International Conference on Machine Learning*, pages 4203–4227. PMLR, 2020.

[36] Charlie Hou, Kiran K Thekumparampil, Giulia Fanti, and Sewoong Oh. Efficient algorithms for federated saddle point optimization. *arXiv preprint arXiv:2102.06333*, 2021.

[37] Yu-Guan Hsieh, Franck Iutzeler, Jérôme Malick, and Panayotis Mertikopoulos. Explore aggressively, update conservatively: Stochastic extragradient methods with variable stepsize scaling. *Advances in Neural Information Processing Systems*, 33:16223–16234, 2020.

[38] Alfredo N Iusem, Alejandro Jofré, Roberto Imbuzeiro Oliveira, and Philip Thompson. Extragradient method with variance reduction for stochastic variational inequalities. *SIAM Journal on Optimization*, 27(2):686–724, 2017.

[39] Yujia Jin and Aaron Sidford. Efficiently solving MDPs with stochastic mirror descent. In *Proceedings of the 37th International Conference on Machine Learning (ICML)*, volume 119, pages 4890–4900. PMLR, 2020.

[40] Thorsten Joachims. A support vector method for multivariate performance measures. pages 377–384, 01 2005.

[41] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In C.J. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K.Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 26. Curran Associates, Inc., 2013.

[42] Anatoli Juditsky, Arkadii S. Nemirovskii, and Claire Tauvel. Solving variational inequalities with stochastic mirror-prox algorithm, 2008.

[43] Peter Kairouz, H Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Kallista Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, et al. Advances and open problems in federated learning. *Foundations and Trends® in Machine Learning*, 14(1–2):1–210, 2021.

[44] Aswin Kannan and Uday V Shanbhag. Optimal stochastic extragradient schemes for pseudomonotone stochastic variational inequality problems and their variants. *Computational Optimization and Applications*, 74(3):779–820, 2019.

[45] Sai Praneeth Karimireddy, Satyen Kale, Mehryar Mohri, Sashank Reddi, Sebastian Stich, and Ananda Theertha Suresh. SCAFFOLD: Stochastic controlled averaging for federated learning. In Hal Daumé III and Aarti Singh, editors, *Proceedings of the 37th International Conference on Machine Learning*, volume 119 of *Proceedings of Machine Learning Research*, pages 5132–5143. PMLR, 13–18 Jul 2020.

[46] Ahmed Khaled, Konstantin Mishchenko, and Peter Richtárik. Tighter theory for local SGD on identical and heterogeneous data. In *International Conference on Artificial Intelligence and Statistics*, pages 4519–4529, 2020.

[47] Ahmed Khaled, Konstantin Mishchenko, and Peter Richtarik. Tighter theory for local sgd on identical and heterogeneous data. In Silvia Chiappa and Roberto Calandra, editors, *Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics*, volume 108 of *Proceedings of Machine Learning Research*, pages 4519–4529. PMLR, 26–28 Aug 2020.

[48] Anastasia Koloskova, Nicolas Loizou, Sadra Boreiri, Martin Jaggi, and Sebastian U Stich. A unified theory of decentralized sgd with changing topology and local updates. *arXiv preprint arXiv:2003.10422*, 2020.

[49] Jakub Konečný, H. Brendan McMahan, Felix Yu, Peter Richtárik, Ananda Theertha Suresh, and Dave Bacon. Federated learning: strategies for improving communication efficiency. In *NIPS Private Multi-Party Machine Learning Workshop*, 2016.

[50] G. M. Korpelevich. The extragradient method for finding saddle points and other problems. 1976.

[51] Dmitry Kovalev, Aleksandr Beznosikov, Ekaterina Borodich, Alexander Gasnikov, and Gesualdo Scutari. Optimal gradient sliding and its application to distributed optimization under similarity. *arXiv preprint arXiv:2205.15136*, 2022.

[52] Dmitry Kovalev, Aleksandr Beznosikov, Abdurakhmon Sadiev, Michael Persiianov, Peter Richtarik, and Alexander Gasnikov. Optimal algorithms for decentralized stochastic variational inequalities. In S. Koyejo, S. Mohamed, A. Agarwal, D. Belgrave, K. Cho, and A. Oh, editors, *Advances in Neural Information Processing Systems*, volume 35, pages 31073–31088. Curran Associates, Inc., 2022.

[53] Alex Krizhevsky, Vinod Nair, and Geoffrey Hinton. Cifar-10 (canadian institute for advanced research). `http://www.cs.toronto.edu/~kriz/cifar.html`, 2009.

[54] Yann LeCun and Corinna Cortes. MNIST handwritten digit database. 2010.

[55] Shigang Li and Torsten Hoefler. Near-optimal sparse allreduce for distributed deep learning. In *Proceedings of the 27th ACM SIGPLAN Symposium on Principles and Practice of Parallel Programming*, pages 135–149, 2022.

[56] Tian Li, Anit Kumar Sahu, Ameet Talwalkar, and Virginia Smith. Federated learning: challenges, methods, and future directions. *IEEE Signal Processing Magazine*, 37(3):50–60, 2020.

[57] Tengyuan Liang and James Stokes. Interaction matters: A note on non-asymptotic local convergence of generative adversarial networks. In Kamalika Chaudhuri and Masashi Sugiyama, editors, *Proceedings of the Twenty-Second International Conference on Artificial Intelligence and Statistics*, volume 89 of *Proceedings of Machine Learning Research*, pages 907–915. PMLR, 16–18 Apr 2019.

[58] Tao Lin, Sebastian U Stich, Kumar Kshitij Patel, and Martin Jaggi. Don't use large mini-batches, use local sgd. *arXiv preprint arXiv:1808.07217*, 2018.

[59] Mingrui Liu, Youssef Mroueh, Jerret Ross, Wei Zhang, Xiaodong Cui, Payel Das, and Tianbao Yang. Towards better understanding of adaptive gradient algorithms in generative adversarial nets. *arXiv preprint arXiv:1912.11940*, 2019.

[60] Mingrui Liu, Wei Zhang, Youssef Mroueh, Xiaodong Cui, Jerret Ross, Tianbao Yang, and Payel Das. A decentralized parallel algorithm for training generative adversarial nets. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2020.

[61] Weijie Liu, Aryan Mokhtari, Asuman Ozdaglar, Sarath Pattathil, Zebang Shen, and Nenggan Zheng. A decentralized proximal point-type method for saddle point problems. *arXiv preprint arXiv:1910.14380*, 2019.

[62] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards deep learning models resistant to adversarial attacks. In *International Conference on Learning Representations (ICLR)*, 2018.

[63] Grigory Malinovsky, Kai Yi, and Peter Richtárik. Variance reduced proxskip: Algorithm, theory and application to federated learning. *arXiv preprint arXiv:2207.04338*, 2022.

[64] L. O. Mangasarian. Parallel gradient distribution in unconstrained optimization. *SIAM Journal on Control and Optimization*, 33(6):1916–1925, 1995.

[65] Ryan Mcdonald, Mehryar Mohri, Nathan Silberman, Dan Walker, and Gideon S. Mann. Efficient large-scale distributed training of conditional maximum entropy models. In Y. Bengio, D. Schuurmans, J. D. Lafferty, C. K. I. Williams, and A. Culotta, editors, *Advances in Neural Information Processing Systems 22*, pages 1231–1239. Curran Associates, Inc., 2009.

[66] Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial intelligence and statistics*, pages 1273–1282. PMLR, 2017.

[67] Panayotis Mertikopoulos, Bruno Lecouat, Houssam Zenati, Chuan-Sheng Foo, Vijay Chandrasekhar, and Georgios Piliouras. Optimistic mirror descent in saddle-point problems: Going the extra (gradient) mile. *arXiv preprint arXiv:1807.02629*, 2018.

[68] Konstantin Mishchenko, Eduard Gorbunov, Martin Takáč, and Peter Richtárik. Distributed learning with compressed gradient differences. *arXiv preprint arXiv:1901.09269*, 2019.

[69] Konstantin Mishchenko, Grigory Malinovsky, Sebastian Stich, and Peter Richtárik. Proxskip: Yes! local gradient steps provably lead to communication acceleration! finally! In *International Conference on Machine Learning*, pages 15750–15769. PMLR, 2022.

[70] Giorgi Nadiradze, Amirmojtaba Sabour, Peter Davies, Ilia Markov, Shigang Li, and Dan Alistarh. Decentralized sgd with asynchronous, local and quantized updates. *arXiv preprint arXiv:1910.12308*, 2019.

[71] Arkadi Nemirovski. Prox-method with rate of convergence $o(1/t)$ for variational inequalities with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems. *SIAM Journal on Optimization*, 15:229–251, 01 2004.

[72] J. Von Neumann and O. Morgenstern. *Theory of games and economic behavior*. Princeton University Press, 1944.

[73] Shayegan Omidshafiei, Jason Pazis, Christopher Amato, Jonathan P. How, and John Vian. Deep decentralized multi-task multi-agent reinforcement learning under partial observability. In *Proceedings of the 34th International Conference on Machine Learning (ICML)*, volume 70, pages 2681–2690. PMLR, 2017.

[74] Wei Peng, Yu-Hong Dai, Hui Zhang, and Lizhi Cheng. Training gans with centripetal acceleration. *Optimization Methods and Software*, 35(5):955–973, 2020.

[75] Xun Qian, Peter Richtarik, and Tong Zhang. Error compensated distributed sgd can be accelerated. In M. Ranzato, A. Beygelzimer, Y. Dauphin, P.S. Liang, and J. Wortman Vaughan, editors, *Advances in Neural Information Processing Systems*, volume 34, pages 30401–30413. Curran Associates, Inc., 2021.

[76] Peter Richtárik, Igor Sokolov, and Ilyas Fatkhullin. EF21: A new, simpler, theoretically better, and practically faster error feedback. *arXiv preprint arXiv:2106.05203*, 2021.

[77] Gesualdo Scutari, Daniel P Palomar, Francisco Facchinei, and Jong-Shi Pang. Convex optimization, game theory, and variational inequality theory. *IEEE Signal Processing Magazine*, 27(3):35–49, 2010.

[78] Shai Shalev-Shwartz and Shai Ben-David. *Understanding machine learning: From theory to algorithms*. Cambridge university press, 2014.

[79] Ohad Shamir, Nati Srebro, and Tong Zhang. Communication-efficient distributed optimization using an approximate newton-type method. In Eric P. Xing and Tony Jebara, editors, *Proceedings of the 31st International Conference on Machine Learning*, volume 32 of *Proceedings of Machine Learning Research*, pages 1000–1008, Bejing, China, 22–24 Jun 2014. PMLR.

[80] Sebastian U Stich. Local sgd converges fast and communicates little. *arXiv preprint arXiv:1805.09767*, 2018.

[81] Sebastian U Stich, Jean-Baptiste Cordonnier, and Martin Jaggi. Sparsified sgd with memory. *arXiv preprint arXiv:1809.07599*, 2018.

[82] Rafał Szlendak, Alexander Tyurin, and Peter Richtárik. Permutation compressors for provably faster distributed nonconvex optimization. *arXiv preprint arXiv:2110.03300*, 2021.

[83] Joel A Tropp. User-friendly tail bounds for sums of random matrices. *Foundations of computational mathematics*, 12:389–434, 2012.

[84] Paul Tseng. A modified forward-backward splitting method for maximal monotone mappings. *SIAM Journal on Control and Optimization*, 38(2):431–446, 2000.

[85] Joost Verbraeken, Matthijs Wolting, Jonathan Katzy, Jeroen Kloppenburg, Tim Verbelen, and Jan S Rellermeyer. A survey on distributed machine learning. *ACM Computing Surveys*, 2019.

[86] Blake Woodworth, Kumar Kshitij Patel, and Nathan Srebro. Minibatch vs local sgd for heterogeneous distributed learning. *arXiv preprint arXiv:2006.04735*, 2020.

[87] Blake Woodworth, Kumar Kshitij Patel, Sebastian U Stich, Zhen Dai, Brian Bullins, H Brendan McMahan, Ohad Shamir, and Nathan Srebro. Is local sgd better than minibatch sgd? *arXiv preprint arXiv:2002.07839*, 2020.

[88] Blake E Woodworth, Brian Bullins, Ohad Shamir, and Nathan Srebro. The min-max complexity of distributed stochastic convex optimization with intermittent communication. In *Conference on Learning Theory*, pages 4386–4437. PMLR, 2021.

[89] Linli Xu, James Neufeld, Bryce Larson, and Dale Schuurmans. Maximum margin clustering. In L. Saul, Y. Weiss, and L. Bottou, editors, *Advances in Neural Information Processing Systems*, volume 17. MIT Press, 2005.

[90] Xiao-Tong Yuan and Ping Li. On convergence of distributed approximate newton methods: Globalization, sharper bounds and beyond. *arXiv preprint arXiv:1908.02246*, 2019.

[91] Martin Zinkevich, Markus Weimer, Lihong Li, and Alex J. Smola. Parallelized stochastic gradient descent. In J. D. Lafferty, C. K. I. Williams, J. Shawe-Taylor, R. S. Zemel, and A. Culotta, editors, *Advances in Neural Information Processing Systems 23*, pages 2595–2603. Curran Associates, Inc., 2010.