A Details on Introduction

A.1 Experimental motivation

1. Environment details of 2D goal reacher

- State space: $\mathcal{S} = \mathbb{R}^2$. For $(x, y) \in \mathcal{S}, |x| \leq 1, |y| \leq 1$.
- Action space: $\mathcal{A} = \{(\cos(\pi/4 \times k), \sin(\pi/4 \times k)) \mid k = 0, 1, ..., 7\}$ $(|\mathcal{A}| = 8)$
- Reward function: if the agent's state is in the Goal box, then it receives +6. Otherwise, it receives -0.5 rewards for every step.
- Transition probability: $s_{h+1} = s_h + a_h \cdot \epsilon$, where s_{h+1} is the next state, s_h is the current state, a_h is the current action, and $\epsilon \in \mathbb{R}^2$ with $||\epsilon||_2 = 1$ provides a stocasticity to the environment.
- Horizon length: $H = 13$
- Discounting factor: $\gamma = 0.99$
- Grid size: 10
- Goal box: The coordinates of the center of the time-varying goal box are (x_g, y_g)= $(0.9\cos(2\pi \times k/2500), 0.9\sin(2\pi \times k/2500))$, which changes for episode $k \in [5000]$. The width and height of the box are equal to 0.05.

2. Experiment details

To motivate our proposed meta-framework via a simple experiment, we used Q-learning as a component A of our meta-algorithm to update the policy. The three baselines (ProOLS, ONPG, FTML) of Figure 1(c) were trained with four learning rates $\eta \in \{0.001, 0.003, 0.005, 0.007\}$ and the entropy regularized parameter $\tau = 0.1$, where the shaded area of the three baselines is 95 % confidence area among 4 different learning rates. The `PTM-T` was trained with the model rollout length $\widehat{H} \in \{50, 60\}$, policy update iteration number $G \in \{10, 50\}$, entropy regularized parameter $\tau = 0.1$, Q-learning update parameter $\alpha_Q \in \{0.7, 0.9, 0.99\}$, and the learning rate $\eta = 0.001$. The shaded area of `PTM-T` is 95 % confidence area among the 12 different cases above. All four algorithms share the same agent's policy network structure.

B Related Works

Existing methods for non-stationary environments can be grouped into three categories: 1) shoehorning: directly using established frameworks for stationary MDPs by assuming no extra mechanisms are needed since non-stationarity already exists in standard RL due to policy updates; 2) model-based policy updates: updating models with new data, using short rollouts to prevent model exploitation [24, 29], online model updates, or latent factor identification [4, 13–16]; and 3) anticipating future changes by forecasting policy gradients or value functions [7, 30, 20, 10, 31].

The advantage of the model-free method is its computational efficiency, allowing for direct learning of complex policies from raw data [32, 33], while the advantage of the model-based method is its data efficiency, allowing one to learn fast by learning how the environment works [34, 35]. However, both advantages are weakened in non-stationary environments since the optimizing non-stationary loss function induced by time-varying data distribution makes the model-free method challenging to adaptively obtain the optimal policy [36, 37] and the model-based method challenging to estimate accurate non-stationary models [20, 10].

Model-free method in non-stationary RL. [8] uses meta-learning among the training tasks to find initial hyperparameters of the policy networks that can be quickly fine-tuned when facing testing tasks that have not been encountered before. However, access to a prior distribution of training tasks is not available in real-world problems. To mitigate this issue, [9] proposed the Follow-The-Meta-Leader (FTML) algorithm that continuously improves an initialization of parameters for non-stationary input data. However, it internally entails a lag when tracking optimal policy as it maximizes the current performance over all the past samples uniformly. To alleviate the lag problem, [7, 37] focused on directly forecasting the non-stationary performance gradient to adapt the time-varying optimal policies. However, it still has problems of showing empirical analysis on bandit settings or a low-dimensional environment and lack of theoretical analysis which provides a bound on the adapted policy's performance. [30] proposed adaptive Q-learning with a restart strategy and established its near-optimal dynamic regret bound. In addition, [36] proposed two model-free policy optimization algorithms based on the restart strategy and showed that dynamic regret satisfies polynomial space and time complexities. However, the provable model-free methods in [30, 36] still lack empirical evidence and adaptability in complex environments. Furthermore, since the agent can execute a policy in a fixed environment only once due to the non-stationarity of the environment, most existing model-free methods only update the policy once for each environment, which prevents the tracking of the time-varying optimal policies.

Model-based method in non-stationary RL. The work [14] learned the model change factors and their representation in heterogeneous domains with varying reward functions and dynamics. However, it has restrictions for use in non-stationary environments, meaning that it is applicable only for constant change factors or the domain adaptation setting. [4] proposed a Bayesian optimal learning policy algorithm by conditioning the action on both states and latent vectors that capture the agent's uncertainty in the environment. Also, [15] brought insights from recent causality research to model non-stationarity as latent change factors across different environments, and learn policy conditioning on latent factors of the causal graphs. However, learning an optimal policy conditioning on the latent states [4, 13–16] makes the theoretical analysis intractable. The recent works [20, 10, 31] proposed model-based algorithms with a provable guarantee, but their algorithms are not scalable for complex environments and lack empirical evaluation for complex environments.

C Details on Problem Statement and Notations

C.1 Details on Notations

Environment Interaction. First, we denote the state and action at the wall-clock time $\mathtt{t}_k$ of step h as $s_h^{\mathtt{t}_k}$ and $a_h^{\mathtt{t}_k}$, respectively. As mentioned in the main paper, we interchangeably use the symbols $s_h^{(k)}$ and $a_h^{(k)}$ for $s_h^{\mathtt{t}_k}$ and $a_h^{\mathtt{t}_k}$. At the wall-clock time $\mathtt{t}_k$, the agent starts from an initial state $s_0^{\mathtt{t}_k} \sim \rho$. At step $h \in [H]$ of the episode k, the agent takes the action $a_h^{\mathtt{t}_k} = \pi^{\mathtt{t}_k}(\cdot|s_h^{\mathtt{t}_k})$ from the current state $s_h^{\mathtt{t}_k}$. The agent then receives the reward $r_h^{\mathtt{t}_k} \sim R_{\mathtt{t}_k}(s_h^{\mathtt{t}_k}, a_h^{\mathtt{t}_k})$ and moves to the next state $s_{h+1}^{\mathtt{t}_k} \sim P_{\mathtt{t}_k}(s_{h+1}^{\mathtt{t}_k}|s_h^{\mathtt{t}_k}, a_h^{\mathtt{t}_k})$. The trajectory ends when the agent reaches $s_H^{\mathtt{t}_k}$.

Future MDP $\widehat{\mathcal{M}}_{t_{k+1}}$. Our work creates a one-episode-ahead MDP $\widehat{\mathcal{M}}_{t_{k+1}}$ based on the observed data from the p lastest MDPs $\{\mathcal{M}_{t_{k-p+1}}, ..., \mathcal{M}_{t_k}\}$ when the agent is stated in episode k. We define $\widehat{\mathcal{M}}_{t_{k+1}} \coloneqq \langle \mathcal{S}, \mathcal{A}, H, \widehat{P}_{\mathtt{t}_{k+1}}, \widehat{R}_{t_{k+1}}, \gamma \rangle$, where $\widehat{P}_{\mathtt{t}_{k+1}}$ and $\widehat{R}_{t_{k+1}}$ are the *forecasted* future transition probability and reward function, respectively. As mentioned in the main paper, the agent also interacts with the created future MDP $\widehat{\mathcal{M}}_{t_{k+1}}$ in the same way as it did with the original MDP $\mathcal{M}_{t_k}$. We denote the state, action, and policy in $\widehat{\mathcal{M}}_{t_{k+1}}$ as $\widehat{s}_h^{t_{k+1}}, \widehat{a}_h^{t_{k+1}}, \widehat{\pi}^{t_{k+1}}$, or equivalently $\widehat{s}_h^{(k+1)}, \widehat{a}_h^{(k+1)}, \widehat{\pi}^{(k+1)}$, respectively. We elaborate our main methodology in Section 3.

State value and state-action value functions. For any given policy π and the MDP $\mathcal{M}_{\mathtt{t}_k}$, we denote the state value function at the wall-clock time $\mathtt{t}_k$(episode k) as $V^{\pi,\mathtt{t}_k} : \mathcal{S} \to \mathbb{R}$ and the state-action value function k as $Q^{\pi,\mathtt{t}_k} : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$. We define

$$V^{\pi,\mathtt{t}_k}(s) \coloneqq \mathbb{E}_{\mathcal{M}_{\mathtt{t}_k},\pi}\left[\sum_{h=0}^{H-1} \gamma^h r_h^{\mathtt{t}_k} \;\middle|\; s_0^{\mathtt{t}_k} = s\right],$$

$$Q^{\pi,\mathtt{t}_k}(s,a) \coloneqq \mathbb{E}_{\mathcal{M}_{\mathtt{t}_k},\pi}\left[\sum_{h=0}^{H-1} \gamma^h r_h^{\mathtt{t}_k} \;\middle|\; s_0^{\mathtt{t}_k} = s,\ a_0^{\mathtt{t}_k} = a\right].$$

Also, given the future MDP $\widehat{\mathcal{M}}_{\mathtt{t}_{k+1}}$, we denote the *forecasted* state value as $\widehat{V}^{\pi,\mathtt{t}_{k+1}}(s) : \mathcal{S} \to \mathbb{R}$ and *forecasted* state-action value as $\widehat{Q}^{\pi,\mathtt{t}_{k+1}} : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$. We define

$$\widehat{V}^{\pi,\mathtt{t}_{k+1}}(s) \coloneqq \mathbb{E}_{\widehat{\mathcal{M}}_{\mathtt{t}_{k+1}},\pi}\left[\sum_{h=0}^{H-1} \gamma^h \widehat{r}_h^{\mathtt{t}_{k+1}} \;\middle|\; \widehat{s}_0^{\mathtt{t}_{k+1}} = s\right],$$

$$\widehat{Q}^{\pi,\mathtt{t}_{k+1}}(s,a) \coloneqq \mathbb{E}_{\widehat{\mathcal{M}}_{\mathtt{t}_{k+1}},\pi}\left[\sum_{h=0}^{H-1} \gamma^h \widehat{r}_h^{\mathtt{t}_{k+1}} \;\middle|\; \widehat{s}_0^{\mathtt{t}_{k+1}} = s,\ \widehat{a}_0^{\mathtt{t}_{k+1}} = a\right].$$

As mentioned in the main paper, we simplify the symbols $V^{\pi,\mathsf{t}_k}, Q^{\pi,\mathsf{t}_k}, \widehat{V}^{\pi,\mathsf{t}_{k+1}}, \widehat{Q}^{\pi,\mathsf{t}_{k+1}}$ as $V^{\pi,(k)}, Q^{\pi,(k)}, \widehat{V}^{\pi,(k+1)}, \widehat{Q}^{\pi,(k+1)}$.

Dynamic regret. Aside from stationary MDPs, the agent aims to maximize the cumulative expected reward throughout the K episodes by adopting a sequence of policies $\{\pi^{\mathsf{t}_k}\}_{1:K}$. In non-stationary MDPs, the optimality of the policies is evaluated in terms of the dynamic regret $\mathfrak{R}\left(\{\pi^{\mathsf{t}_k}\}_{1:K}, K\right)$ defined as

$$\mathfrak{R}\left(\{\pi^{\mathsf{t}_k}\}_{1:K}, K\right) \coloneqq \sum_{k=1}^{K}\left(V^{*,\mathsf{t}_k}(\rho) - V^{\pi^{\mathsf{t}_k},\mathsf{t}_k}(\rho)\right) \tag{C.1}$$

where $V^{*,\mathsf{t}_k}(= V^{\pi^{*,\mathsf{t}_k},\mathsf{t}_k})$ denotes the optimal state value function under the optimal policy π^{*,t_k} at the wall-clock time t_k (episode k) and $V^{\pi^{\mathsf{t}_k},\mathsf{t}_k}$ denotes the state value with agent's k^{th} episode's policy π^k. Dynamic regret is a stronger evaluation than the standard static regret that considers the optimality of a single policy over all episodes.

State value and state-action value functions at step h. We denote the state value function and the state-action value function for any policy π at *step* h of the wall-clock time t_k as V_h^{π,t_k} and Q_h^{π,t_k}, respectively. We define

$$V_h^{\pi,\mathsf{t}_k}(s) \coloneqq \mathbb{E}_{\mathcal{M}_{\mathsf{t}_k},\pi}\Big[\sum_{i=h}^{H-1}\gamma^{i-h} r_i^{\mathsf{t}_k} \mid s_h^{\mathsf{t}_k} = s\Big],$$

$$Q_h^{\pi,\mathsf{t}_k}(s,a) \coloneqq \mathbb{E}_{\mathcal{M}_{\mathsf{t}_k},\pi}\Big[\sum_{i=h}^{H-1}\gamma^{i-h} r_i^{\mathsf{t}_k} \mid s_h^{\mathsf{t}_k} = s, a_h^{\mathsf{t}_k} = a\Big].$$

Then, the corresponding Bellman equation is

$$Q_h^{\pi,\mathsf{t}_k}(s,a) = \big(R_{\mathsf{t}_k} + \gamma P_{\mathsf{t}_k} V_{h+1}^{\pi,\mathsf{t}_k}\big)(s,a), \;\; V_h^{\pi,\mathsf{t}_k}(s) = \big\langle Q_h^{\pi,\mathsf{t}_k}(s,\cdot), \pi(\cdot|s)\big\rangle_{\mathcal{A}}, \;\; V_H^{\mathsf{t}_k,\pi}(s) = 0 \; \forall s \in \mathcal{S} \tag{C.2}$$

where $(P_{\mathsf{t}_k} f)(s,a) \coloneqq \mathbb{E}_{s'\sim P^{\mathsf{t}_k}(\cdot|s,a))}[f(s')]$ for every function $f : \mathcal{S} \to \mathbb{R}$.

We denote $V_h^{*,\mathsf{t}_k}(s) = V_h^{\pi^{*,\mathsf{t}_k},\mathsf{t}_k}(s)$ as the optimal state value function at step h of episode k. We omit the subscript h when $h = 0$, that is, $V^{\pi,k} = V_0^{\pi,k}$, $Q^{\pi,k} = Q_0^{\pi,k}$. Then, the corresponding Bellman equation is

$$\begin{gathered} Q_h^{*,\mathsf{t}_k}(s,a) = \big(R_{\mathsf{t}_k} + \gamma P_{\mathsf{t}_k} V_{h+1}^{*,\mathsf{t}_k}\big)(s,a), \;\; V_h^{*,\mathsf{t}_k}(s) = \big\langle Q_h^{*,\mathsf{t}_k}(s,\cdot), \pi^{*,\mathsf{t}_k}(\cdot|s)\big\rangle_{\mathcal{A}}, \\ \pi^{*,\mathsf{t}_k}(s) = \max_a Q_h^{*,\mathsf{t}_k}(s,a). \end{gathered} \tag{C.3}$$

We also denote the *forecasted* state value at the wall-clock time t_{k+1} of step h when the agent is stated at time t_k as $\widehat{V}_h^{\pi,\mathsf{t}_{k+1}}$ and the *forecasted* state-action value as $\widehat{Q}_h^{\pi,\mathsf{t}_{k+1}}$ in a forecasted MDP $\widehat{\mathcal{M}}_{\mathsf{t}_{k+1}}$. We define

$$\widehat{V}_h^{\pi,\mathsf{t}_{k+1}}(s) \coloneqq \mathbb{E}_{\widehat{\mathcal{M}}_{\mathsf{t}_{k+1}},\pi}\Big[\sum_{i=h}^{H-1}\gamma^{i-h} \widehat{r}_i^{\mathsf{t}_{k+1}} \mid \widehat{s}_h^{\mathsf{t}_{k+1}} = s\Big], \tag{C.4}$$

$$\widehat{Q}_h^{\pi,\mathsf{t}_{k+1}}(s,a) \coloneqq \mathbb{E}_{\widehat{\mathcal{M}}_{\mathsf{t}_{k+1}},\pi}\Big[\sum_{i=h}^{H-1}\gamma^{i-h} \widehat{r}_i^{\mathsf{t}_{k+1}} \mid \widehat{s}_h^{\mathsf{t}_{k+1}} = s, \widehat{a}_h^{\mathsf{t}_{k+1}} = a\Big]. \tag{C.5}$$

Then, the Bellman equation is given by

$$\begin{gathered} \widehat{Q}_h^{\pi,\mathsf{t}_{k+1}}(s,a) = \big(\widehat{R}_{\mathsf{t}_{k+1}} + \gamma \widehat{P}_{\mathsf{t}_{k+1}} \widehat{V}_{h+1}^{\pi,\mathsf{t}_{k+1}}\big)(s,a), \;\; \widehat{V}_h^{\pi,\mathsf{t}_{k+1}}(s) = \big\langle \widehat{Q}_h^{\pi,\mathsf{t}_{k+1}}(s,\cdot), \pi(\cdot|s)\big\rangle_{\mathcal{A}}, \\ \widehat{V}_H^{\pi,\mathsf{t}_{k+1}}(s) = 0 \; \forall s \in \mathcal{S}. \end{gathered} \tag{C.6}$$

We denote the *future* optimal policy of the *future* value function $\widehat{V}^{\pi,\mathsf{t}_{k+1}}$ as $\widehat{\pi}^{*,\mathsf{t}_{k+1}}$. Then the Bellman equation also holds for $\widehat{Q}_h^{\pi,\mathsf{t}_{k+1}}(s)$ and $\widehat{V}_h^{\pi,\mathsf{t}_{k+1}}(s)$ as follows:

$$\begin{gathered} \widehat{Q}_h^{*,\mathsf{t}_{k+1}}(s,a) = \big(\widehat{R}_{\mathsf{t}_{k+1}} + \gamma \widehat{P}_{\mathsf{t}_{k+1}} \widehat{V}_{h+1}^{*,\mathsf{t}_{k+1}}\big)(s,a), \;\; \widehat{V}_h^{*,\mathsf{t}_{k+1}}(s) = \big\langle \widehat{Q}_h^{*,\mathsf{t}_{k+1}}(s,\cdot), \widehat{\pi}^{*,\mathsf{t}_{k+1}}(\cdot|s)\big\rangle_{\mathcal{A}}, \\ \widehat{\pi}^{*,\mathsf{t}_{k+1}}(s) = \max_a \widehat{Q}_h^{*,\mathsf{t}_{k+1}}(s,a). \end{gathered} \tag{C.7}$$

As mentioned in the main paper, we simplify the notations $V_h^{\pi,\mathsf{t}_k}, Q_h^{\pi,\mathsf{t}_k}, \widehat{V}_h^{\pi,\mathsf{t}_{k+1}}, \widehat{Q}_h^{\pi,\mathsf{t}_{k+1}}$ as $V_h^{\pi,(k)}, Q_h^{\pi,(k)}, \widehat{V}_h^{\pi,(k+1)}, \widehat{Q}_h^{\pi,(k+1)}$.

Unnormalized (discounted) occupancy measure. We define the unnormalized (discounted) occupancy measure $\nu_{s_0,a_0}^{\pi,\mathsf{t}_k} \in \Delta_{1/(1-\gamma)}(\mathcal{S}\times\mathcal{A})$ at wall-clock time t_k (episode k) for a given policy π together with an initial state s_0 and the action a_0 as

$$\nu_{s_0,a_0}^{\pi,\mathsf{t}_k}(s,a) \coloneqq \sum_{h=0}^{\infty}\gamma^h \mathbb{P}(s_h = s, a_h = a \mid s_0, a_0 \ ; \ \pi, P_{\mathsf{t}_k}) \ , \ \ \forall (s,a)\in\mathcal{S}\times\mathcal{A} \tag{C.8}$$

where $\mathbb{P}(s_h = s, a_h = a \mid s_0, a_0 \ ; \ \pi, P^{\mathsf{t}_k})$ is the probability of visiting (s,a) at step h when following policy π from (s_0,a_0) with the transition probability $P_{\mathsf{t}_{k+1}}$.

We also define the unnormalized non-stationary (discounted) *forecasted* occupancy measure $\widehat{\nu}_{s_0}^{\pi,\mathsf{t}_{k+1}} \in \Delta_{1/(1-\gamma)}(\mathcal{S}\times\mathcal{A})$ for a given policy π, an initial state s_0, an action a_0, and a forecasted future transition probability $\widehat{P}_{\mathsf{t}_{k+1}}$:

$$\widehat{\nu}_{s_0,a_0}^{\pi,\mathsf{t}_{k+1}}(s,a) \coloneqq \sum_{h=0}^{\infty}\gamma^h \mathbb{P}(s_h = s, a_h = a \mid s_0, a_0, \pi, \widehat{P}_{\mathsf{t}_{k+1}}) \ \ , \forall (s,a)\in\mathcal{S}\times\mathcal{A} \tag{C.9}$$

where the probability is defined in a forecasted environment with $\widehat{P}_{\mathsf{t}_{k+1}}$.

Model prediction error. To measure how well our meta-function predicts the future environment, we define two different *model prediction error*s $\iota_\infty^{\mathsf{t}_{k+1}}, \iota_h^{\mathsf{t}_{k+1}} : \mathcal{S}\times\mathcal{A}\to\mathbb{R}$, which denote the Bellman equation error when using $\widehat{V}$ and $\widehat{Q}$ estimated in the future MDP instead of the true V and Q functions:

$$\bar{\iota}_\infty^{\mathsf{t}_{k+1}}(s,a) \coloneqq \left(R_{\mathsf{t}_{k+1}} + \gamma P_{\mathsf{t}_{k+1}} \widehat{V}_\infty^{*,\mathsf{t}_{k+1}} - \widehat{Q}_\infty^{*,\mathsf{t}_{k+1}}\right)(s,a), \tag{C.10}$$

$$\iota_h^{\mathsf{t}_{k+1}}(s,a) \coloneqq \left(R_{\mathsf{t}_{k+1}} + \gamma P_{\mathsf{t}_{k+1}} \widehat{V}_{h+1}^{\overline{\pi}^{\mathsf{t}_{k+1}},\mathsf{t}_{k+1}} - \widehat{Q}_h^{\overline{\pi}^{\mathsf{t}_{k+1}},\mathsf{t}_{k+1}}\right)(s,a). \tag{C.11}$$

As mentioned in the main paper, we allow $\bar{\iota}_\infty^{\mathsf{t}_{k+1}}(s,a)$ and $\iota_h^{\mathsf{t}_{k+1}}(s,a)$ to be interchangeably expressed by the symbols $\bar{\iota}_\infty^{(k+1)}(s,a)$ and $\iota_h^{(k+1)}(s,a)$.

Local time-elapsing variation budget. Aside from the time-elapsing variation budget, we define the *local* time-elapsing variation budgets $B_p^{(k-w:k)}$ and $B_r^{(k-w:k)}$ that quantifie how fast the environment changes over wall-clock times $\{\mathsf{t}_{k-w+1}, \mathsf{t}_{k+1}, ..., \mathsf{t}_k\}$ where $k-w, k\in[K]$:

$$B_p^{(k-w+1:k)}(\Delta_\pi) \coloneqq \sum_{\tau=k-w+1}^{k} \sup_{s,a} \|P_{\mathsf{t}_{\tau+1}}(\cdot\,|s,a) - P_{\mathsf{t}_\tau}(\cdot\,|s,a)\|_1,$$

$$B_r^{(k-w+1:k)}(\Delta_\pi) \coloneqq \sum_{\tau=k-w+1}^{k} \sup_{s,a} |R_{\mathsf{t}_{k+1}}(s,a) - R_{\mathsf{t}_k}(s,a)|.$$

D Proof of Theoretical Analysis

D.1 Preliminary for ProST-T and theoretical analysis

In this subsection, we elaborate on the `ProST-T`'s environment setting and its components f, g.

D.1.1 Environment setting

We consider the tabular environment have the following properties:

1. First, $P_{(k)}$ and $R_{(k)}$ are represented by the inner products of the feature functions $\psi : \mathcal{S}\times\mathcal{S}\times\mathcal{A}\to\mathbb{R}^{|\mathcal{S}|^2|\mathcal{A}|}, \varphi : \mathcal{S}\times\mathcal{A}\to\mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ and the non-stationary variables $o_{(k)}^p, o_{(k)}^r \in \mathcal{O}$, respectively, where $o_{(k)}^p : \mathcal{S}\times\mathcal{S}\times\mathcal{A}\to\mathbb{R}^{|\mathcal{S}|^2|\mathcal{A}|}$ and $o_{(k)}^r : \mathcal{S}\times\mathcal{A}\to\mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$. That is, $P_{(k)} =< \psi, o_{(k)}^p >$ and $R_{(k)} =< \varphi, o_{(k)}^r >$.

2. Second, the agent estimates $o^p_{(k)}$ and $o^r_{(k)}$ rather than observing them. More specifically, we consider the non-stationary variable set $\mathcal{O}$ to be the set $\{P_{(k)}\}_{1:K}, \{R_{(k)}\}_{1:K}$. The agent then attempts to *estimate* o_k (denote $P_{(k)}$ as $o^p_{(k)}$ and $R_{(k)}$ as $o^r_{(k)}$) through its w lastest trajectories, where Assumption 2 does not need to be satisfied in this setting. That is, the agent estimates $P_{(k)}$ by $\hat{o}^p_k$ and $R_{(k)}$ by $\hat{o}^r_k$ from observations of last w trajectories, i.e., $\tau_{k-(w-1):k}$.

We elaborate on the above two settings below:

1. $P_{(k)}, R_{(k)}$ are inner products of ψ, φ and $o^p_{(k)}, o^r_{(k)}$.

Let us define a set of one-hot reward vectors over all states and the action space, namely $\mathbb{1}_r \coloneqq \{\varphi^y \in \{0,1\}^{|\mathcal{S}||\mathcal{A}|} \mid \sum_{i=1}^{|\mathcal{S}||\mathcal{A}|} \varphi^y_i = 1\}$, and similarly define a set of one-hot transition probability vectors, namely $\mathbb{1}_p \coloneqq \{\psi^y \in \{0,1\}^{|\mathcal{S}|^2|\mathcal{A}|} \mid \sum_{i=1}^{|\mathcal{S}|^2|\mathcal{A}|} \psi^y_i = 1\}$. We then define one-to-one functions φ and ψ such that $\varphi : \mathcal{S} \times \mathcal{A} \to \mathbb{1}_r$ and $\psi : \mathcal{S} \times \mathcal{S} \times \mathcal{A} \to \mathbb{1}_p$. Namely, $\varphi(s,a)(\psi(s',s,a))$ is a one-hot vector such that the $(i)^{th}$ entry equals 1. We use the notation $\varphi^k_h = \varphi(s^{(k)}_h, a^{(k)}_h)$ for the observed $(s^{(k)}_h, a^{(k)}_h)$ on the trajectory τ_k, and similarly $\psi^k_h = \psi(s^{(k)}_{h+1}, s^{(k)}_h, a^{(k)}_h)$.

Then, we set $\mathcal{O} = \{P_{(k)}, R_{(k)}\}_{k=1}^{\infty}$ in `ProST-T`. Also, we set o_k to consist of two parameters as $o_k = (o^p_{(k)}, o^r_{(k)})$. We define a function $o^p_{(k)} \coloneqq \{o : \mathcal{S} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}^{|\mathcal{S}|^2|\mathcal{A}|} \mid o(s',s,a) = P_{(k)}(s'|s,a),\ \forall(s',s,a)\}$ and a function $o^r_{(k)} \coloneqq \{o : \mathcal{S}\times\mathcal{A} \to \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \mid o(s,a) = R_{(k)}(s,a),\ \forall(s,a)\}$. Then, the transition probability and reward value $P_{(k)}$ and $R_{(k)}$ can be constructed by the inner products of the stationary functions φ and ψ and the unknown non-stationary parameters $o^p_{(k)}$ and $o^r_{(k)}$ as follows,

$$P_{(k)}(s' \mid s,a) \coloneqq \ < \psi(s',s,a), o^p_{(k)}(s',s,a) > \text{ for } \forall(s',s,a), \tag{D.1}$$

$$R_{(k)}(s,a) \coloneqq \ < \varphi(s,a), o^r_{(k)}(s,a) > \text{ for } \forall(s,a). \tag{D.2}$$

For notational simplicity, we use $< \psi, o^p_{(k)} >$ and $< \varphi, o^r_{(k)} >$ to show the inner products of the functions $\psi, o^p_{(k)}$ and $\varphi, o^r_{(k)}$, respectively. Therefore, $P_{(k)} =< \psi, o^p_{(k)} >$ and $R_{(k)} =< \varphi, o^r_{(k)} >$.

To give an intuitive explanation, note that $o^p_{(k)}$ contains all transition probabilities for all (s',s,a) in a vector form with size $\mathbb{R}^{|\mathcal{S}|^2|\mathcal{A}|}$ and $o^r_{(k)}$ contains all rewards for all (s,a) in a vector form with size $\mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$.

2. The agent estimates $o^r_{(k)}$ and $o^p_{(k)}$ rather than observing them

We have defined the functions $o^p_{(k)}$ and $o^r_{(k)}$ as the transition probability and reward functions at episode k, respectively. Now, the agent strives to estimate $o^p_{(k)}$ and $o^r_{(k)}$, denoted as $\widehat{o}^p_{(k)}$ and $\widehat{o}^r_{(k)}$, from the current trajectory τ_k:

$$\widehat{o}^p_{(k)}(s',s,a) = \frac{n_{(k)}(s',s,a)}{\lambda + n_{(k)}(s,a)}, \quad \forall(s',s,a) \in \mathcal{S} \times \mathcal{S} \times \mathcal{A},$$

$$\widehat{o}^r_{(k)}(s,a) = \frac{\sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s^{(k)}_h, a^{(k)}_h)\right] \cdot r^{(k)}_h}{n_k(s,a)}, \quad \forall(s,a) \in \mathcal{S} \times \mathcal{A}$$

where $n_{(k)}(s,a)$ denotes visitation count of state s under action a through trajectory $\tau_{(k)}$ and $n_{(k)}(s,a,s')$ denotes visitation count of state s under action a and subsequent next state s' through trajectory. We denote $\widehat{o}^p_{k,h} = \widehat{o}^p_{(k)}(s^{(k)}_{h+1}, s^{(k)}_h, a^{(k)}_h)$ and $\widehat{o}^r_{k,h} = \widehat{r}^k_h(s^{(k)}_h, a^{(k)}_h)$.

It can be verified that the following relations hold at episode k for the state and action pairs from the k^{th} trajectory $\{s^{(k)}_0, a^{(k)}_0, s^{(k)}_1, a^{(k)}_1, .., s^{(k)}_{H-1}, a^{(k)}_{H-1}, s^{(k)}_H\}$:

$$P_{(k)}(s^{(k)}_{h+1} \mid s^{(k)}_h, a^{(k)}_h) = \ < \psi(s^{(k)}_{h+1}, s^{(k)}_h, a^{(k)}_h), o^p_{(k)}(s^{(k)}_{h+1}, s^{(k)}_h, a^{(k)}_h) > \quad, \forall h \in [H], \tag{D.3}$$

$$R_{(k)}(s^{(k)}_h, a^{(k)}_h) = \ < \varphi(s^{(k)}_h, a^{(k)}_h), o^r_{(k)}(s^{(k)}_h, a^{(k)}_h) > \quad, \forall h \in [H]. \tag{D.4}$$

Note that the observed non-stationary parameters $\widehat{o}^p_{(k)}$ and $\widehat{o}^r_{(k)}$ can be interpreted partially observed vectors.

D.1.2 Functions f and g

The function f estimates and the function g predicts as follows:

1. **Function** f: f forecasts one-episode-ahead non-stationary parameters $\hat{o}^p_{(k+1)}$ and $\hat{o}^r_{(k+1)}$ by minimizing the following loss function $\mathcal{L}_{f^\diamond}$ with the regularization parameter $\lambda \in \mathbb{R}_+$:

$$\mathcal{L}_{f^\diamond}(\phi\ ;\ \widehat{o}^\diamond_{(k-w+1:k)}) = \lambda\|\phi\|^2 + \sum_{s=k-w+1}^{k}\sum_{h=0}^{H-1}\left((\square^s_h)^\top\phi - \widehat{o}^\diamond_{s,h}\right)$$

   where $\diamond = r, p$ and $\square = \varphi$ if $\diamond = r$. We set $\square = \psi$ if $\diamond = p$. We let $\phi^k_{f^\diamond} = \mathrm{argmin}_\phi \mathcal{L}_{f^\diamond}(\widehat{o}^\diamond_{k-(w-1):k})$. We use $\phi^k_{f^\diamond}$ as $\widehat{o}^\diamond_{k+1}$.

2. **Function** g: Then g predicts the functions $\widehat{P}_{(k+1)}$ and $\widehat{R}_{(k+1)}$, denoted as $\widehat{g}^P_{(k+1)}$ and $\widehat{g}^R_{(k+1)}$, as $\widehat{P}_{(k+1)} = \widehat{g}^P_{(k+1)} :=< \varphi, \widehat{o}^p_{(k+1)} >$ and $\widehat{R}_{(k+1)} = \widehat{g}^R_{(k+1)} :=< \varphi, \widehat{o}^r_{k+1} > +2\Gamma^{(k)}_w$, where $\Gamma^{(k)}_w(s,a) : \mathcal{S}\times\mathcal{A} \to \mathbb{R}$ is the exploration bonus term that adapts the counter-based bonus terms in the literature.

We elaborate on above two procedures below:

1. The function f solves an optimization problem to obtain the future $\widehat{o}_{(k+1)}$.

The function $g \circ f$ forecasts the $k+1^{th}$ episode's non-stationary parameters as $(\widehat{o}^p_{(k+1)}, \widehat{o}^r_{(k+1)})$ from $\widehat{o}_{(k-w+1:k)}$, where w is the sliding window length (past reference length). The function f forecasts $o^p_{(k+1)}$ and $o^r_{(k+1)}$ by minimizing the following two regularized least-squares optimization problems [18].

$$\widehat{o}^p_{(k+1)} = \underset{o\in\mathbb{R}^{|\mathcal{S}|^2|\mathcal{A}|}}{\arg\min}\left(\lambda\|o\|^2 + \sum_{s=k-w+1,h=0}^{k,H}\left((\psi^s_h)^\top o - \widehat{o}^p_{s,h}\right)\right) \tag{D.5}$$

$$\widehat{o}^r_{(k+1)} = \underset{o\in\mathbb{R}^{|\mathcal{S}||\mathcal{A}|}}{\arg\min}\left(\lambda\|o\|^2 + \sum_{s=k-w+1,h=0}^{k,H-1}\left((\varphi^s_h)^\top o - \widehat{o}^r_{s,h}\right)\right) \tag{D.6}$$

2. The function g predicts $\widehat{P}_{(k+1)}$ and $\widehat{R}_{(k+1)}$ from $\widehat{o}_{k+1}$.

From the equations (17a) and (17b) of the paper [31], the explicit solutions of (D.5) and (D.6) are given as

$$\widehat{o}^p_{(k+1)}(s',s,a) = \frac{\sum_{t=k-w+1}^{k} n_t(s',s,a)}{\lambda + \sum_{t=k-w+1}^{k} n_t(s,a)},\ \widehat{o}^r_{(k+1)}(s,a) = \frac{\sum_{t=k-w+1}^{k}\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a) = (s^t_h, a^t_h)\right]\cdot r^t_h}{\lambda + \sum_{t=k-w+1}^{k} n_t(s,a)}. \tag{D.7}$$

Then, the `ProST-T` predicts the future model using the functions $\widehat{g}^P_{k+1}$ and $\widehat{g}^R_{k+1}$ as follows:

$$\begin{aligned}\widehat{g}^P_{k+1}(s',s,a) &:=< \varphi(s',s,a), \widehat{o}^p_{(k+1)}(s',s,a) >,\\ \widetilde{g}^R_{k+1}(s,a) &:=< \varphi(s,a), \widehat{o}^r_{(k+1)}(s,a) >,\\ \widehat{g}^R_{k+1}(s,a) &:= \widetilde{g}^R_{k+1}(s,a) + 2\Gamma^{(k)}_w(s,a).\end{aligned}$$

We utilize the exploration bonus $\Gamma^{(k)}_w(s,a) : \mathcal{S}\times\mathcal{A}\to\mathbb{R}$ to explore those state and action pairs that are less visited. We define it as $\Gamma^{(k)}_w(s,a) = \beta\left(\sum_{t=k-w+1}^{k} n_t(s,a) + \lambda\right)^{-1/2}$ with $\beta > 0$. Then, we use $\widehat{g}^P_{k+1}$ and $\widehat{g}^R_{k+1}$ to denote the future MDP's $\widehat{P}_{(k+1)}$ and $\widehat{R}_{(k+1)}$, respectively. From the following analysis, we write $\widehat{P}_{(k+1)} = \widehat{g}^P_{(k+1)}$, $\widetilde{R}_{(k+1)} = \widetilde{g}^R_{(k+1)}$, and $\widehat{R}_{(k+1)} = \widehat{g}^R_{(k+1)}$.

D.1.3 Baseline algorithms `Alg` and `Alg`$_\tau$

The `ProST-T` utilizes softmax parameterization that naturally ensures that the policy lies in the probability simplex. For any function that satisfies $\theta : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$, the policy $\pi^{(k)}$ is generated by the softmax transformation of $\theta^{(k)}$ at the wall-clock time $\mathtt{t}_k$. Furthermore, to promote exploration and discourage premature convergence to suboptimal policies in a non-stationary environment, we implement a widely used strategy known as entropy regularization. We augment the future state value function with an additional $\pi^{(k)}(s)$ entropy term, denoted by $\tau\mathcal{H}(s, \pi^{(k)})$, where $\tau > 0$. We perform a theoretical analysis with two baseline algorithms : Natural Policy Gradient (NPG) `Alg` and Natural Policy Gradient (NPG) with entropy regularization `Alg`$_\tau$

Softmax parameterization. For any function that satisfies $\theta : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$, the policy $\pi^{(k)}$ is generated by the softmax transformation of $\theta^{(k)}$ at the wall-clock time $\mathtt{t}_k$. Using the notation $\pi^{(k)} = \pi_{\theta^{(k)}}$, the soft parameterization is defined as

$$\pi_{\theta^{(k)}}(a|s) := \frac{\exp\left(\theta^{(k)}(s,a)\right)}{\sum_{a' \in \mathcal{A}} \exp\left(\theta^{(k)}(s,a')\right)} \quad , \forall (s,a) \in \mathcal{S} \times \mathcal{A}.$$

Under the softmax parameterization, the NPG update rule admits a simple form of update rule given in line 17 of Algorithm 2 in Appendix F.1. This is elaborated in [21].

Entropy regularized value maximization. For any policy π, we define the *forecasted* entropy-regularized state value function $\widehat{V}_\tau^{\pi,\mathtt{t}_{\mathtt{t}_k+1}}(s)$ as

$$\widehat{V}_\tau^{\pi,\mathtt{t}_{k+1}}(s) := \widehat{V}^{\pi,\mathtt{t}_k+1}(s) + \tau\mathcal{H}(s,\pi)$$

where $\tau \geq 0$ is a regularization parameter and $\mathcal{H}(s,\pi)$ is a discounted entropy defined as

$$\mathcal{H}(s,\pi) := \mathbb{E}_{\widehat{\mathcal{M}}_{(k+1)}}\left[\sum_{h=0}^{H-1} -\gamma^h \log \pi(\hat{a}_h^{(k+1)}|\hat{s}_h^{(k+1)})|\hat{s}_0^{(k+1)} = s\right].$$

Also, we define the *forecasted* regularized Q-function $\widehat{Q}_\tau^{\pi,(k+1)}$ as

$$\widehat{Q}_\tau^{\pi,\mathtt{t}_{k+1}}(s,a) = \hat{r}_h^{\mathtt{t}_{k+1}} + \gamma\mathbb{E}_{s' \sim \widehat{P}_{\mathtt{t}_{k+1}}(\cdot|s,a)}\left[\widehat{V}_\tau^{\pi,\mathtt{t}_{k+1}}(s')\right]$$
$$\text{where } (s', s, a) = (\hat{s}_{h+1}^{(k+1)}, \hat{s}_h^{(k+1)}, \hat{a}_h^{(k+1)}).$$

D.2 Notation for theoretical analysis

This subsection introduces some notations that we will use in the proofs.

At the wall-clock time $\mathtt{t}_k$, we define the *forecasting model error* $\Delta_{\mathtt{t}_k}^r(s,a)$ and *forecasting transition probability model error* $\Delta_{\mathtt{t}_k}^p(s,a)$ below:

$$\Delta_{\mathtt{t}_k}^r(s,a) := \left|\left(R_{(k+1)} - \widetilde{R}_{(k+1)}\right)(s,a)\right|, \tag{D.8}$$

$$\Delta_{\mathtt{t}_k}^p(s,a) := \left\|\left(P_{(k+1)} - \widehat{P}_{(k+1)}\right)(\cdot \mid s,a)\right\|_1. \tag{D.9}$$

Recall that $\widetilde{R}_{(k+1)}$ and $\widehat{P}_{(k+1)}$ estimate the future reward and transition probability by solving the optimization problems (D.5) and (D.6).

We define a model error that considers the bonus term as

$$\Delta_{\mathtt{t}_k}^{Bonus,r}(s,a) := \left|\left(R_{(k+1)} - \widehat{R}_{(k+1)}\right)(s,a)\right|$$

where $\widehat{R}_{(k+1)}(s,a) = \widetilde{R}_{(k+1)}(s,a) + 2\Gamma_w^{(k)}(s,a)$.

We also define the *empirical* forecasting reward model error $\bar{\Delta}_{\mathtt{t}_k,h}^r$ and the *empirical* forecasting transition probability model error $\bar{\Delta}_{\mathtt{t}_k,h}^p$:

$$\bar{\Delta}_{\mathtt{t}_k,h}^r := \left|\left(R_{(k+1)} - \widetilde{R}_{(k+1)}\right)(s_h^{(k+1)}, a_h^{(k+1)})\right|,$$
$$\bar{\Delta}_{\mathtt{t}_k,h}^p := \left\|\left(P_{(k+1)} - \widehat{P}_{(k+1)}\right)(\cdot \mid s_h^{(k+1)}, a_h^{(k+1)})\right\|_1$$

as well as the *empirical* bonus based on the reward model error:

$$\bar{\Delta}^{Bonus,r}_{\mathtt{t}_k,h} \coloneqq \left|\left(R_{(k+1)} - \widehat{R}_{(k+1)}\right)(s_h^{(k+1)}, a_h^{(k+1)})\right|.$$

Likewise, we define *total empirical* forecasting reward model error $\bar{\Delta}^r_K$ and the *total empirical* forecasting transition probability model error $\bar{\Delta}^p_K$:

$$\bar{\Delta}^r_K \coloneqq \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \bar{\Delta}^r_{\mathtt{t}_k,h}, \tag{D.10}$$

$$\bar{\Delta}^p_K \coloneqq \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \bar{\Delta}^p_{\mathtt{t}_k,h}. \tag{D.11}$$

We simplify the symbols $\Delta^r_{\mathtt{t}_k}(s,a), \Delta^p_{\mathtt{t}_k}(s,a), \Delta^{Bonus,r}_{\mathtt{t}_k}(s,a), \bar{\Delta}^r_{\mathtt{t}_k,h}, \bar{\Delta}^p_{\mathtt{t}_k,h}, \bar{\Delta}^{Bonus,r}_{\mathtt{t}_k,h}$ as $\Delta^r_{(k)}(s,a), \Delta^p_{(k)}(s,a), \Delta^{Bonus,r}_{(k)}(s,a), \bar{\Delta}^r_{(k),h}, \bar{\Delta}^p_{(k),h}, \bar{\Delta}^{Bonus,r}_{(k),h}$, respectively.

We also define a variable $\Lambda^{\mathtt{t}_k}_w(s,a)$ that quantifies the visitation:

$$\Lambda^{\mathtt{t}_k}_w(s,a) = \left[\lambda + \sum_{t=(1\wedge k-w+1)}^{k} n_t(s,a)\right]^{-1}. \tag{D.12}$$

It can be verified that

$$\Gamma^{\mathtt{t}_k}_w(s,a) = \beta\sqrt{\Lambda^{\mathtt{t}_k}_w(s,a)}. \tag{D.13}$$

As before, we simplify the notations $\Lambda^{\mathtt{t}_k}_w(s,a)$ and $\Gamma^{\mathtt{t}_k}_w(s,a)$ as $\Lambda^{(k)}_w(s,a)$ and $\Gamma^{(k)}_w(s,a)$. We define $r_{\max}, \widetilde{r}_{\max}, R^{\max}_{(k+1)}$, and $\widetilde{R}^{\max}_{(k+1)}$ as follows:

$$R^{\max}_{(k+1)} \coloneqq \max_{(s,a)} |R_{(k+1)}(s,a)|,$$

$$r_{\max} \coloneqq \max_{1\le k\le K-1} R^{\max}_{(k+1)},$$

$$\widetilde{R}^{\max}_{(k+1)} \coloneqq \max_{(s,a)} |\widetilde{R}_{(k+1)}(s,a)|,$$

$$\widetilde{r}_{\max} \coloneqq \max_{1\le k\le K-1} \widetilde{R}^{\max}_{(k+1)}$$

and since $\|\widehat{R}_{(k+1)}(s,a)\|_\infty \le \|\widetilde{R}_{(k+1)}(s,a)\|_\infty + \|2\Gamma^{(k)}_w(s,a)\|_\infty = \widetilde{R}^{\max}_{(k+1)} + \frac{2\beta}{\sqrt{\lambda}}$, we define $\hat{r}^{k+1}_{\max}$ as

$$\hat{r}^{\max}_{(k+1)} \coloneqq \widetilde{R}^{\max}_{(k+1)} + \frac{2\beta}{\sqrt{\lambda}}.$$

Also, since β and λ are hyperparameters independent of k, we have that

$$\hat{r}_{\max} = \widetilde{r}_{\max} + \frac{2\beta}{\sqrt{\lambda}}. \tag{D.14}$$

D.3 Proofs

Proof of Theorem 1. Following the definition of the dynamic regret (Definition C.1), it can be separated into three terms:

$$\begin{aligned}
&\mathfrak{R}\left(\{\widehat{\pi}^{(k+1)}\}_{1:K-1}, K\right)\\
&= \sum_{k=1}^{K-1}\left(V^{*,(k+1)}(s_0) - V^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right)\\
&= \underbrace{\sum_{k=1}^{K-1}\left(V^{*,(k+1)}(s_0) - \widehat{V}^{*,(k+1)}(s_0)\right)}_{①} + \underbrace{\sum_{k=1}^{K-1}\left(\widehat{V}^{*,(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right)}_{②}\\
&+ \underbrace{\sum_{k=1}^{K-1}\left(\widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) - V^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right)}_{③}
\end{aligned}$$

1. Upper bound on ①. The gap between $V^{\pi^{*,(k+1)},(k+1)}(s_0)$ and $\widehat{V}^{\widehat{\pi}^{*,(k+1)},(k+1)}(s_0)$ comes from the gap between two optimal value functions evaluated for two different MDPs: $\mathcal{M}_{(k+1)}$ and $\widehat{\mathcal{M}}_{(k+1)}$.

We will first come up with an upper bound on the difference between $Q_h^{*,(k+1)}(s,a)$ and $\widehat{Q}_h^{*,(k+1)}(s,a)$ for any $(s,a) \in \mathcal{S} \times \mathcal{A}$. The difference can be separated into three terms as follows:

$$\begin{aligned} Q_h^{*,(k+1)}(s,a) - \widehat{Q}_h^{*,(k+1)}(s,a) \le & \underbrace{\|Q_h^{*,(k+1)}(s,a) - Q_\infty^{*,(k+1)}(s,a)\|_\infty}_{\textcircled{1.1}} \\ & + \underbrace{\left(Q_\infty^{*,(k+1)}(s,a) - \widehat{Q}_\infty^{*,(k+1)}(s,a)\right)}_{\textcircled{1.2}} \\ & + \underbrace{\|\widehat{Q}_h^{*,(k+1)}(s,a) - \widehat{Q}_\infty^{*,(k+1)}(s,a)\|_\infty}_{\textcircled{1.3}} \end{aligned}$$

1.1. Terms $\textcircled{1.1}$ and $\textcircled{1.3}$.

First, the term $\textcircled{1.1}$ can be bounded as follows:

$$\begin{aligned} \textcircled{1.1} &= \left\| \mathbb{E}_{\mathcal{M}_{(k+1)},\pi^*}\Big[\sum_{i=0}^{H-h-1} \gamma^i r_{i+h}^{(k+1)} - \sum_{i=0}^{\infty} \gamma^i r_i^{(k+1)} \mid s_h^{(k+1)} = s, a_h^{(k+1)} = a \Big] \right\|_\infty \\ &\le \left| \sum_{i=H-h}^{\infty} \gamma^i r_{\max} \right| \\ &= \frac{\gamma^{H-h}}{1-\gamma} r_{\max} \end{aligned}$$

Through a similar process, we can also obtain the upper bound: $\textcircled{1.3} \le \gamma^{H-h}/(1-\gamma)\hat{r}_{\max}$.

1.2. Term $\textcircled{1.2}$.

An upper bound on the term $\textcircled{1.2}$ can be obtained by utilizing $\bar{\iota}_\infty^{(k+1)}(s,a)$ (Def (C.10)). Then, the Q-function gap between $Q_\infty^{*,(k+1)}$ and $\widehat{Q}_\infty^{*,(k+1)}$ can be represented using the Bellman equation as follows:

$$\begin{aligned} \textcircled{1.2} &= \left(Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}\right)(s,a) && \text{(D.15)} \\ &= \left(R_{(k+1)} + \gamma P_{(k+1)} V_\infty^{*,(k+1)}\right)(s,a) - \widehat{Q}_\infty^{*,(k+1)}(s,a) && \text{(D.16)} \\ &= \left(R_{(k+1)} + \gamma P_{(k+1)} \widehat{V}_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}\right)(s,a) + \gamma P_{(k+1)}\left(V_\infty^{*,(k+1)} - \widehat{V}_\infty^{*,(k+1)}\right)(s,a) \\ &\le \bar{\iota}_\infty^{k+1}(s,a) + \gamma P_{(k+1)}\left(V_\infty^{*,(k+1)} - \widehat{V}_\infty^{*,(k+1)}\right)(s,a) \\ &= \bar{\iota}_h^{k+1}(s,a) + \gamma P_{(k+1)}\left(\langle Q_\infty^{*,(k+1)}, \pi^{*,(k+1)}\rangle_{\mathcal{A}} - \langle \widehat{Q}_\infty^{*,(k+1)}, \widehat{\pi}^{*,(k+1)}\rangle_{\mathcal{A}}\right)(s,a) && \text{(D.17)} \\ &= \bar{\iota}_\infty^{k+1}(s,a) + \gamma P_{(k+1)}\Big(\langle Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}, \pi^{*,(k+1)}\rangle_{\mathcal{A}} \\ &+ \langle \widehat{Q}_\infty^{*,(k+1)}, \pi^{*,(k+1)} - \widehat{\pi}^{*,(k+1)}\rangle_{\mathcal{A}}\Big)(s,a) \\ &\le \bar{\iota}_\infty^{k+1}(s,a) + \gamma P_{(k+1)}\left(\langle Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}, \pi^{*,(k+1)}\rangle_{\mathcal{A}}\right)(s,a) && \text{(D.18)} \end{aligned}$$

where (D.16) and (D.17) hold by the definition of Bellman equation ((C.3) and (C.7)). Equation (D.18) holds by $\langle \widehat{Q}_\infty^{*,(k+1)}, \pi^{*,(k+1)} - \widehat{\pi}^{*,(k+1)}\rangle_{\mathcal{A}}(s,a) \le 0$ since $\widehat{\pi}^{*,(k+1)}$ is the optimal policy of $\widehat{Q}_\infty^{*,(k+1)}$. We now define the matrix operator $(\mathbb{P} \circ \pi)(s,a) : \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \to \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ as the transition matrix that captures how the state-action pair transitions from (s,a) to (s',a') when following the policy π in an environment with the transition probability $\mathbb{P}$. Also, define the one-vector $\mathbb{1}_{(s,a)} \in \mathbb{R}^{|\mathcal{S}|\mathcal{A}|}$ such that the $(s,a)^{\text{th}}$ entity is one and the remaining entries are zero. Then, the equation (D.15) becomes the same as the $(s,a)^{\text{th}}$ entity of the vector $\mathbb{1}_{(s,a)} \cdot (Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)})(s,a)$. Also, the right-hand side of equation (D.18) can be represented as

$$\begin{aligned} P_{(k+1)}\left(\langle Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}, \pi^{*,(k+1)}\rangle_{\mathcal{A}}\right)(s,a) &= (P_{(k+1)} \circ \pi^{*,(k+1)}) \\ &\quad \cdot \left(\mathbb{1}_{(s,a)} \cdot (Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)})\right)(s,a) \\ &= (\mathbb{P}_{\pi^*}^{k+1})\left(\mathbb{1}_{(s,a)} \cdot (Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)})\right)(s,a) \end{aligned}$$

where we denote $P_{(k+1)} \circ \pi^{*,(k+1)} \coloneqq \mathbb{P}_{\pi^*}^{(k+1)}$ for notational simplicity.

Then, we can reformulate the inequality (between (D.15) and (D.18)) into a vector form which holds element-wise for all s,a:

$$\begin{aligned} \left(\mathbb{1}_{(s,a)} \cdot \left(Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}\right)\right)(s,a) \leq& \mathbb{1}_{(s,a)} \cdot \bar{\iota}_\infty^{(k+1)}(s,a) \\ &+ \gamma(\mathbb{P}_{\pi^*}^{(k+1)})\left(\mathbb{1}_{(s,a)} \cdot (Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)})\right)(s,a) \end{aligned}$$

Then, rearranging the above inequality yields that

$$\begin{aligned} \mathbb{1}_{(s,a)} \cdot \left(Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}\right)(s,a) &\leq (\mathbb{I} - \gamma\mathbb{P}_{\pi^*}^{k+1})^{-1}\mathbb{1}_{(s,a)} \cdot \bar{\iota}_\infty^{k+1}(s,a) \qquad \text{(D.19)} \\ &= \frac{1}{1-\gamma}\bar{\iota}_\infty^{k+1}(s,a) \end{aligned}$$

Now, note that $(\mathbb{I} - \gamma\mathbb{P}_{\pi^*}^{(k+1)})^{-1}$ can be expanded with an infinite summation of the matrix operator $P_{(k+1)} \circ \pi^{*,(k+1)}$ as $(\mathbb{I} - \gamma\mathbb{P}_{\pi^*}^{(k+1)})^{-1} = \mathbb{I} + \gamma\mathbb{P}_{\pi^*}^{(k+1)} + (\gamma\mathbb{P}_{\pi^*}^{(k+1)})^2 + \ldots$s. Since, $\mathbb{1}_{(s,a)}$ can be viewed as the Dirac delta state-action distribution that always yields (s,a), it holds that $\nu_{(s,a)}^{\pi^{*,(k+1)},(k+1)} = (\mathbb{I} - \gamma\mathbb{P}_{\pi^*}^{(k+1)})^{-1}\mathbb{1}_{(s,a)}$, where ν is the unnormalized occupancy measure of (s,a) in light of Definition (C.8). Then taking the l_1 norm over the inequality (D.19) yields the that

$$\begin{aligned} \left\|\mathbb{1}_{(s,a)} \cdot \left(Q_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)}\right)(s,a)\right\|_1 &\leq \left\|(\mathbb{I} - \gamma\mathbb{P}_{\pi^*}^{k+1})^{-1}\mathbb{1}_{(s,a)} \cdot \bar{\iota}_\infty^{k+1}(s,a)\right\|_1 \\ &= \left\|(\mathbb{I} - \gamma\mathbb{P}_{\pi^*}^{k+1})^{-1}\mathbb{1}_{(s,a)}\right\|_1 \cdot \left|\bar{\iota}_\infty^{k+1}(s,a)\right| \\ &= \frac{1}{1-\gamma}\left|\bar{\iota}_\infty^{k+1}(s,a)\right| \qquad \text{(D.20)} \end{aligned}$$

Equation (D.20) holds since $\nu_{(s,a)}^{\pi^{*,(k+1)},(k+1)}$ is an unnormalized probability distribution.

Then, for every $(s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]$, it follows from combining the terms (1.1), (1.2) and (1.3) that

$$Q_h^{*,(k+1)}(s,a) - \widehat{Q}_h^{*,(k+1)}(s,a) \leq \frac{\gamma^{H-h}}{1-\gamma}(r_{\max} + \hat{r}_{\max}) + \frac{1}{1-\gamma}\left|\bar{\iota}_\infty^{(k+1)}(s,a)\right|$$

1.3. Combining the terms (1.1), (1.2) and (1.3).

Finally, an upper bound on ① is derived as

$$\begin{aligned} ① &= \sum_{k=1}^{K-1}\left(V^{\pi^{*,(k+1)},(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{*,(k+1)},(k+1)}(s_0)\right) \\ &\leq \sum_{k=1}^{K-1}\left\|Q^{*,(k+1)} - \widehat{Q}^{*,(k+1)}\right\|_\infty \\ &= \sum_{k=1}^{K-1} \cdot \frac{\gamma^H}{1-\gamma}(r_{\max} + \hat{r}_{\max}) + \frac{1}{1-\gamma}\sum_{k=1}^{K-1}\|\bar{\iota}_\infty^{k+1}\|_\infty \\ &= (K-1) \cdot \frac{\gamma^H}{1-\gamma}(r_{\max} + \hat{r}_{\max}) + \frac{1}{1-\gamma}\bar{\iota}_\infty^K \qquad \text{(D.21)} \end{aligned}$$

where we have defined $\bar{\iota}_{\infty}^{K} \coloneqq \sum_{k=1}^{K-1} \left\| \bar{\iota}_{\infty}^{(k+1)} \right\|_{\infty}$ in Theorem 1.

2. Upper bound on ②.

The gap between $\widehat{V}^{*,(k+1)}(s_0)$ and $\widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)$ comes from the optimization error between the optimal policy $\widehat{\pi}^{*,(k+1)}$ and the policy $\widehat{\pi}^{(k+1)}$, which are both driven from the same MDP $\widehat{\mathcal{M}}_{(k+1)}$. We also separate this gap into three terms:

$$
\begin{aligned}
\text{②'s } (k)^{\text{th}} \text{ term} &= \widehat{V}^{*,(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) \\
&= \left(\widehat{V}^{*,(k+1)}(s_0) - \widehat{V}_{\infty}^{*,(k+1)}(s_0)\right) + \left(\widehat{V}_{\infty}^{*,(k+1)}(s_0) - \widehat{V}_{\infty}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right) + \\
&+ \left(\widehat{V}_{\infty}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right) &\text{(D.22)} \\
&\leq \underbrace{\left(\widehat{V}_{\infty}^{*,(k+1)}(s_0) - \widehat{V}_{\infty}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right)}_{\textcircled{2.1}} + \frac{2\gamma^H \widehat{r}_{\max}}{1-\gamma} &\text{(D.23)}
\end{aligned}
$$

where the subscript ∞ in the notations $\widehat{V}_{\infty}^{\pi,(k+1)}(s_0)$ and $\widehat{V}_{\infty,\tau}^{\pi,(k+1)}(s_0)$ indicate the forecasted value function and the forecasted entropy-regularized value function when $H = \infty$ (infinite horizon MDPs). Equation (D.22) holds since $\widehat{V}^{\pi,(k+1)}(s) - \widehat{V}_{\infty}^{\pi,(k+1)}(s) = \mathbb{E}_{\widehat{\mathcal{M}}_{(k+1)},\pi}\left[\sum_{h=H}^{\infty} \gamma^h \widehat{r}_h^{(k+1)} \;\middle|\; s = \widehat{s}_0^{(k+1)}\right] \leq \frac{\gamma^H}{1-\gamma}\widehat{r}_{\max}$ holds for all $\pi \in \Pi$.

2.1. Upper bound on ② - NPG without entropy regularization (`Alg`). The term $\textcircled{2.1}$ in (D.23) can be bounded as

$$
\begin{aligned}
\textcircled{2.1} &= \widehat{V}_{\infty}^{*,(k+1)}(s_0) - \widehat{V}_{\infty}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) \\
&\leq \frac{\log|\mathcal{A}|}{\eta G} + \frac{1}{(1-\gamma)^2 G} &\text{(D.24)}
\end{aligned}
$$

due to Theorem 5.3 in [38]. Now, combining D.23 and D.24 offers an upper bound of the term ②'s $(k)^{\text{th}}$ term as follows:

$$
\begin{aligned}
\text{②'s } (k)^{\text{th}} \text{ term} &= \widehat{V}^{\widehat{\pi}^{*,(k+1)},(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) \\
&\leq \frac{1}{(1-\gamma)^2 G} + \frac{\log|\mathcal{A}|}{\eta G} + \frac{2\gamma^H \widehat{r}_{\max}}{1-\gamma}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
② &= \sum_{k=1}^{K-1}\left(\widehat{V}^{\widehat{\pi}^{*,(k+1)},(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right) \\
&\leq (K-1)\left(\frac{1}{(1-\gamma)^2 G} + \frac{\log|\mathcal{A}|}{\eta G} + \frac{2\gamma^H \widehat{r}_{\max}}{1-\gamma}\right) &\text{(D.25)}
\end{aligned}
$$

2.2. Upper bound on ② - NPG with entropy regularization ($\mathtt{Alg}_{\tau}$).

The term $\textcircled{2.1}$ in (D.23) can be further bounded as follows:

$$\begin{aligned}
\textcircled{2.1} =& \widehat{V}_{\infty}^{*,(k+1)}(s_0) - \widehat{V}_{\infty}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) \\
=& \left(\widehat{V}_{\infty}^{*,(k+1)}(s_0) - \widehat{V}_{\infty,\tau}^{*,(k+1)}(s_0)\right) + \left(\widehat{V}_{\infty,\tau}^{*,(k+1)}(s_0) - \widehat{V}_{\infty,\tau}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right) \\
&+ \left(\widehat{V}_{\infty,\tau}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) - \widehat{V}_{\infty}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right) \\
\le& \left\|\widehat{V}_{\infty}^{*,(k+1)}(s_0) - \widehat{V}_{\infty,\tau}^{*,(k+1)}(s_0)\right\|_{\infty} + \left\|\widehat{V}_{\infty,\tau}^{*,(k+1)}(s_0) - \widehat{V}_{\infty,\tau}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right\|_{\infty} \\
&+ \left\|\widehat{V}_{\infty,\tau}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) - \widehat{V}_{\infty}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right\|_{\infty} \\
\le& \underbrace{\left\|\widehat{V}_{\infty,\tau}^{*,(k+1)}(s_0) - \widehat{V}_{\infty,\tau}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right\|_{\infty}}_{\textcircled{2.2}} + \frac{2\tau\log|\mathcal{A}|}{1-\gamma} \qquad \text{(D.26)}
\end{aligned}$$

where (D.26) holds since $\left\|\widehat{V}_{\infty}^{\pi,(k+1)}(s_0) - \widehat{V}_{\infty,\tau}^{\pi,(k+1)}(s_0)\right\|_{\infty} = \tau\max_s|\mathcal{H}(s,\pi)| \le \tau\frac{\log|\mathcal{A}|}{1-\gamma}$ holds for all π.

We now bound the term $\textcircled{2.2}$ in (D.26). With the policy-update rule of `ProST-T` (Algorithm 2 in Appendix F.2), suppose that for a given $g \in [\Delta_\pi]$, we have obtained an *inexact* soft Q-function value of the policy $\widehat{\pi}_{(g)}$ as $\widetilde{Q}_{\tau}^{\widehat{\pi}_{(g)}}$, where $\widehat{Q}_{\tau}^{\widehat{\pi}_{(g)}}$ denotes an *exact* soft forecated Q-function value and g is the iteration index. The approximation gap $|\widetilde{Q}_{\tau}^{\widehat{\pi}_{(g)}} - \widehat{Q}_{\tau}^{\widehat{\pi}_{(g)}}|$ results from computing Q using a finite number of samples. For a hyperparameter δ, let the maximum of the approximation gap over (s,a) is smaller than δ, namely $\|\widetilde{Q}_{\tau}^{\widehat{\pi}_{(g)}} - \widehat{Q}_{\tau}^{\widehat{\pi}_{(g)}}\|_\infty \le \delta$ holds. Then, for iteration $g = 1, 2, .., \Delta_\pi$, the policy-update rule of `ProST-T` can be written as

$$\begin{aligned}
&\widehat{\pi}_{(g+1)}(\cdot|s) = \frac{1}{Z_{(g)}}\cdot\left(\widehat{\pi}_{(g)}(\cdot|s)\right)^{1-\frac{\eta\tau}{1-\gamma}}\exp\left(\frac{\eta\widetilde{Q}_{\tau}^{\widehat{\pi}_{(g)}}(s,a)}{1-\gamma}\right) \\
&\text{where } \ \|\widetilde{Q}_{\tau}^{\widehat{\pi}_{(g)}}(s,a) - \widehat{Q}_{\tau}^{\widehat{\pi}_{(g)}}(s,a)\|_\infty \le \delta \ \text{ for } \ \forall(s,a)\in\mathcal{S}\times\mathcal{A}
\end{aligned}$$

where $Z_{(g)}(s) = \sum_{a\in\mathcal{A}}\left(\widehat{\pi}_{(g)}(a|s)\right)^{1-\frac{\eta\tau}{1-\gamma}}\exp\left((\eta\widehat{Q}_{\tau}^{\widehat{\pi}_{(g)}}(s,a))/(1-\gamma)\right)$.

In light of Theorem 2 in [21], when the learning rate is such that $0 \le \eta \le (1-\gamma)/\tau$, then the approximate entropy-regularized NPG method satisfies the linear convergence theorem for every $g \in [\Delta_\pi]$:

$$\|\widehat{Q}_{\tau}^{*,(k+1)} - \widehat{Q}_{\tau}^{\widehat{\pi}_{(g)}}\|_\infty \le \gamma\left[(1-\eta\tau)^{g-1}C_1 + C_2\right] \tag{D.27}$$

$$\|\log\widehat{\pi}^{*,(k+1)} - \log\widehat{\pi}_{(g)}\|_\infty \le 2\tau^{-1}\left[(1-\eta\tau)^{g-1}C_1 + C_2\right] \tag{D.28}$$

where

$$\begin{aligned}
C_1 &:= \|\widehat{Q}_{\tau}^{*,(k+1)} - \widehat{Q}_{\tau}^{\widehat{\pi}_{(0)}}\|_\infty + 2\tau\left(1-\frac{\eta\tau}{1-\gamma}\right)\|\log\widehat{\pi}^{*,(k+1)} - \log\widehat{\pi}_{(0)}\|_\infty \\
&= \|\widehat{Q}_{\tau}^{*,(k+1)} - \widehat{Q}_{\tau}^{\pi^{(k)}}\|_\infty + 2\tau\left(1-\frac{\eta\tau}{1-\gamma}\right)\|\log\widehat{\pi}^{*,(k+1)} - \log\widehat{\pi}^{(k)}\|_\infty \qquad \text{(D.29)} \\
C_2 &:= \frac{2\delta}{1-\gamma}\left(1+\frac{\gamma}{\eta\tau}\right) \qquad \text{(D.30)}
\end{aligned}$$

The equation (D.29) holds since the policy that the agent executes at the wall-clock time $\mathfrak{t}_k$ (episode k), i.e., $\pi^{(k)}$, is same as the initial policy of the policy iteration, i.e., $\widehat{\pi}_{(0)}$, at the wall-clock time $\mathfrak{t}_k$. Also, the policy that the agent executes at the wall-clock time $\mathfrak{t}_{k+1}$, i.e., $\widehat{\pi}^{(k+1)}$, is same as the policy after Δ_π steps of the soft policy iteration, i.e., $\widehat{\pi}_{(\Delta_\pi)}$ at the wall-clock time $\mathfrak{t}_{k+1}$.

Now, the term $\textcircled{2.2}$ can be bounded as follows:

$$
\begin{aligned}
\textcircled{2.2} &= \|\widehat{V}_\tau^{*,(k+1)} - \widehat{V}_\tau^{\widehat{\pi}^{(k+1)}}\|_\infty \\
&= \|\widehat{V}_\tau^{*,(k+1)} - \widehat{V}_\tau^{\widehat{\pi}^{(\Delta_\pi)}}\|_\infty \\
&\le \|\widehat{Q}_\tau^{*,(k+1)} - \widehat{Q}_\tau^{\widehat{\pi}^{(\Delta_\pi)}}\|_\infty + \tau\|\log\widehat{\pi}^{*,(k+1)} - \log\widehat{\pi}_{(g)}\|_\infty \\
&\le (\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}C_1 + C_2\right] \qquad \text{(D.31)}
\end{aligned}
$$

Combining (D.23,D.26 and D.31) offers an upper bound on the term ②'s $k^{(th)}$ term as follows,

$$
\begin{aligned}
\text{②'s } (k)^{th} \text{ term} &= \widehat{V}^{\widehat{\pi}^{*,(k+1)},(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0) \\
&\le (\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}C_1 + C_2\right] + \frac{2\gamma^H\widehat{r}_{\max}}{1-\gamma} + \frac{2\tau\log|\mathcal{A}|}{1-\gamma} \qquad \text{(D.32)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
② &= \sum_{k=1}^{K-1}\left(\widehat{V}^{\widehat{\pi}^{*,(k+1)},(k+1)}(s_0) - \widehat{V}^{\widehat{\pi}^{(k+1)},(k+1)}(s_0)\right) \\
&\le (K-1)\left((\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}C_1 + C_2\right] + \frac{2\gamma^H\widehat{r}_{\max}}{1-\gamma} + \frac{2\tau\log|\mathcal{A}|}{1-\gamma}\right) \qquad \text{(D.33)}
\end{aligned}
$$

where (D.32) and (D.33) hold when $0 \le \eta \le (1-\gamma)/\tau$

3. Upper bound on ③.

By recalling Definition (C.11), note that $\iota_h^{(k+1)}(\widehat{s}_h^{(k+1)}, \widehat{a}_h^{(k+1)})$ is an *empirical* estimated model prediction error, measuring the gap between $\mathcal{M}_{(k+1)}$ and $\widehat{\mathcal{M}}_{(k+1)}$. Specifically, at episode k, the `ProST` algorithm creates the future MDP $\widehat{\mathcal{M}}_{(k+1)}$ and evaluates $\widehat{V}$ and $\widehat{Q}$ using $\widehat{\pi}^{(k+1)}$. Subsequently at episode $k+1$, the agent uses $\widehat{\pi}^{(k+1)}$ to rollout a trajectory $\{s_0^{(k+1)}, a_0^{(k+1)}, s_1^{(k+1)}, a_1^{(k+1)}, ..., s_{H-1}^{(k+1)}, a_{H-1}^{(k+1)}, s_H^{(k+1)}\}$. Based on this observation, one can write

$$
\begin{aligned}
\iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) =& R_{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) + \gamma(P_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
&- \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \\
=& R_{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) + \gamma(P_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
&- Q_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) + Q_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \\
&- \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \\
=& \gamma P_{(k+1)}(\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)} - V_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
&+ Q_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) - \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \qquad \text{(D.34)}
\end{aligned}
$$

Equation (D.34) holds due to (C.6). Now, we define the operator $\widehat{\mathfrak{I}}^{(k+1)}$ for a function $f : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ as follows:

$$
(\widehat{\mathfrak{I}}^{(k+1)} f)(s) := \langle f(s,\cdot), \widehat{\pi}^{(k+1)}(\cdot|s)\rangle_{\mathcal{A}}
$$

Recall that $\widehat{V}_h^{\widehat{\pi}^{(k+1)},(k+1)}(s) = \left\langle \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}, \widehat{\pi}^{(k+1)}\right\rangle_{\mathcal{A}}$ and $V_h^{\widehat{\pi}^{(k+1)},(k+1)}(s) = \left\langle Q_h^{\widehat{\pi}^{(k+1)},(k+1)}, \widehat{\pi}^{(k+1)}\right\rangle_{\mathcal{A}}$ in light of (C.6) and (C.2). Then, the gap between $\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)})$

and $V_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)})$ can be expanded as

$$\begin{aligned}
&\widehat{V}_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}) - V_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}) \\
&= \left(\widehat{\mathfrak{I}}^{(k+1)}\left(\widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)} - Q_h^{\widehat{\pi}^{(k+1)},(k+1)}\right)\right)(s_h^{(k+1)}) \\
&= \left(\widehat{\mathfrak{I}}^{(k+1)}\left(\widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)} - Q_h^{\widehat{\pi}^{(k+1)},(k+1)}\right)\right)(s_h^{(k+1)}) - \iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \\
&+ \gamma P_{(k+1)}(\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)} - V_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
&+ \left(Q_h^{\widehat{\pi}^{(k+1)},(k+1)} - \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}\right)(s_h^{(k+1)}, a_h^{(k+1)})
\end{aligned}$$

Now, we define two sequences $\{D_{h,1}^{(k+1)}\}$ and $\{D_{h,1}^{(k+1)}\}$, where $(k,h) = (0,0),(0,1),...,(K-1,H)$. We define $D_{h,1}^{(k+1)}$ and $D_{h,2}^{(k+1)}$ as

$$\begin{aligned}
D_{h,1}^{(k+1)} :=&\gamma^h \left(\widehat{\mathfrak{I}}^{(k+1)}\left(\widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)} - Q_h^{\widehat{\pi}^{(k+1)},(k+1)}\right)\right)(s_h^{(k+1)}) \\
&- \gamma^h \left(\widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)} - Q_h^{\widehat{\pi}^{(k+1)},(k+1)}\right)(s_h^{(k+1)}, a_h^{(k+1)}) \\
D_{h,2}^{(k+1)} :=&\gamma^{h+1} P_{(k+1)}(\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)} - V_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
&- \gamma^{h+1}\left(\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)} - V_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}\right)(s_{h+1}^{(k+1)})
\end{aligned}$$

Therefore, we have the following recursive formula over h:

$$\begin{aligned}
&\gamma^h \left(\widehat{V}_h^{\widehat{\pi}^{(k+1)},(k+1)} - V_h^{\widehat{\pi}^{(k+1)},(k+1)}\right)(s_h^{(k+1)}) \\
&= D_{h,1}^{(k+1)} + D_{h,2}^{(k+1)} + \gamma^{h+1}\left(\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)} - V_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}\right)\left(s_{h+1}^{(k+1)}\right) - \gamma^h \iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)})
\end{aligned}$$

The summation over $h = 0,1,..,H-1$ yields that

$$\begin{aligned}
&\widehat{V}_0^{\widehat{\pi}^{(k+1)},(k+1)}(s_0^{(k+1)}) - V_0^{\widehat{\pi}^{(k+1)},(k+1)}(s_0^{(k+1)}) \\
&= \sum_{h=0}^{H-1}\left(D_{h,1}^{(k+1)} + D_{h,2}^{(k+1)}\right) - \sum_{h=0}^{H-1}\gamma^h \iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}).
\end{aligned}$$

Now, for every $(k,h) \in [K] \times [H]$, we define $\mathcal{F}_{h,1}^{(k)}$ as a $\sigma-$algebra generated by state-action sequences $\{(s_i^\tau, a_i^\tau)\}_{(\tau,i)\in[k-1]\times[H]} \cup \{(s_i^k, a_i^k)\}_{i\in[h]}$ and define $\mathcal{F}_{h,2}^{(k)}$ as a σ-algebra generated by $\{(s_i^\tau, a_i^\tau)\}_{(\tau,i)\in[k-1]\times[H]} \cup \{(s_i^k, a_i^k)\}_{i\in[h]} \cup \{s_{h+1}^{(k)}\}$. A filtration $\{\mathcal{F}_{h,m}^{(k)}\}_{(k,h,m)\in[K]\times[H]\times[2]}$ is a sequence of σ- algebras in terms of the time index $t(k,h,m) = 2(k-1)H + 2h + m$ such that $\mathcal{F}_{h,m}^{(k)} \subset \mathcal{F}_{h',m'}^{k'}$ for every $t(k,h,m) \le t((k'),h',m')$. The estimates $\widehat{V}_h^{\pi,(k+1)}$ and $\widehat{Q}_h^{\pi,(k+1)}$ are $\mathcal{F}_{1,1}^{(k+1)}$ measurable since they are forecasted from the past k historical trajectories. Now, since $D_{h,1}^{(k+1)} \in \mathcal{F}_{h,1}^{(k+1)}$ and $D_{h,2}^{(k+1)} \in \mathcal{F}_{h,2}^{(k+1)}$ hold, $\mathbb{E}[D_{h,1}^{(k+1)}|\mathcal{F}_{h-1,2}^{(k+1)}] = 0$ and $\mathbb{E}[D_{h,2}^{(k+1)}|\mathcal{F}_{h,1}^{(k+1)}] = 0$. Notice that $t(k,0,2) = t(k-1,H,2)$ and $\mathcal{F}_{0,2}^{(k)} = \mathcal{F}_{H,2}^{(k-1)}$ for $\forall k \ge 2$. Therefore, one can define a martingale sequence adapted to the filtration $\{\mathcal{F}_{h,m}^{(k)}\}_{(k,h,m)\in[K]\times[H]\times[2]}$:

$$s_{h,j}^{(k+1)} = \sum_{k'=1}^{k}\sum_{h'=0}^{H-1}\left(D_{h',1}^{k'} + D_{h',2}^{k'}\right) + \sum_{h'=0}^{h}\left(D_{h',1}^{(k+1)} + D_{h',2}^{(k+1)}\right) + \sum_{(k',h',j)\in[K]\times[H]\times[2]} D_{h',j}^{k'}$$

Let

$$\sum_{k=1}^{K-1}\sum_{h=0}^{H-1}\left(D_{h,1}^{(k+1)} + D_{h,2}^{(k+1)}\right) = S_{H,2}^{K-1}$$

Since $\gamma^h \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}, \gamma^{h+1}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)} \in [0, \hat{r}_{\max}/(1-\gamma)]$ and $\gamma^h Q_h^{\widehat{\pi}^{(k+1)},(k+1)}, \gamma^{h+1} V_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)} \in [0, r_{\max}/(1-\gamma)]$, it holds that $|D_{h,1}^{(k+1)}|$, $|D_{h,s}^{(k+1)}| \leq (r_{\max} \vee \hat{r}_{\max})/(1-\gamma)$ for $\forall (k,h) \in [K-1] \times [H]$. Then, by the Azuma-Hoeffding inequlaity, the following inequality holds:

$$\mathbb{P}\left(|S_{H,2}^{K-1}| \leq s\right) \geq 2\exp\left(\frac{-s^2}{16\left(\frac{r_{\max}\vee\hat{r}_{\max}}{1-\gamma}\right)^2 \cdot (K-1)H}\right)$$

For any $p \in (0,1)$, if we set $s = 4(r_{\max} \vee \hat{r}_{\max})(1-\gamma)^{-1}\sqrt{(K-1)H\log(4/p)}$, then the inequality holds with probability at least $1-p/2$. The term ③ can be bounded as

$$\begin{aligned} ③ &= \sum_{k=1}^{K-1}\sum_{h=0}^{H-1}\left(D_{h,1}^{(k+1)} + D_{h,2}^{(k+1)}\right) - \sum_{k=1}^{K-1}\sum_{h=0}^{H-1}\gamma^h \iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \\ &\leq \frac{4(r_{\max}\vee\hat{r}_{\max})}{1-\gamma}\sqrt{(K-1)H\log(4/p)} - \iota_H^K \end{aligned} \tag{D.35}$$

4. Upper bound on dynamic regret.

4.1. Upper bound on dynamic regret - without entropy regularization.

For without entropy-regularized case, combining the equations (D.21), (D.25) and (D.35) leads to the following upper bound on the dynamic regret for a future policy $\{\widehat{\pi}\}$ that holds with probability at least $1-p/2$:

$$\begin{aligned} &\mathfrak{R}\left(\{\widehat{\pi}^{(k+1)}\}_{1:K-1}, K)\right) \\ &= ① + ② + ③ \\ &\leq (K-1)\cdot\frac{\gamma^H}{1-\gamma}(r_{\max}+\hat{r}_{\max}) + \frac{1}{1-\gamma}\bar{\iota}_\infty^K \\ &+ (K-1)\left(\frac{1}{(1-\gamma)^2\Delta_\pi} + \frac{\log|\mathcal{A}|}{\eta\Delta_\pi} + \frac{2\gamma^H\widehat{r}_{\max}}{1-\gamma}\right) \\ &+ \frac{4(r_{\max}\vee\hat{r}_{\max})}{1-\gamma}\sqrt{(K-1)H\log(4/p)} - \iota_H^K \end{aligned}$$

Taking an upper bound on $r_{\max}$ and $\hat{r}_{\max}$ using $(r_{\max}\vee\hat{r}_{\max})$ yields the following upper bound that holds with probability at least $1-p/2$:

$$\begin{aligned} &\mathfrak{R}\left(\{\widehat{\pi}^{(k+1)}\}_{1:K-1}, K)\right) \\ &\leq (K-1)\left(\frac{1}{(1-\gamma)^2\Delta_\pi} + \frac{\log|\mathcal{A}|}{\eta\Delta_\pi} + \frac{4\gamma^H(\widehat{r}_{\max}\vee r_{\max})}{1-\gamma}\right. \\ &\left.+ \frac{4(r_{\max}\vee\hat{r}_{\max})}{1-\gamma}\sqrt{\frac{H\log(4/p)}{K-1}}\right) + \frac{1}{1-\gamma}\bar{\iota}_\infty^K - \iota_H^K \end{aligned}$$

4.2. Upper bound on dynamic regret - with entropy regularization.

For the entropy-regularized case, combining the equations (D.21), (D.33), (D.35) leads to the following upper bound on the dynamic regret for a future policy $\{\widehat{\pi}\}$ that holds with probability at least $1-p/2$:

$$
\begin{aligned}
&\mathfrak{R}\left(\{\widehat{\pi}^{(k+1)}\}_{1:K-1}, K)\right) \\
&= \text{①} + \text{②} + \text{③} \\
&\leq (K-1)\cdot\frac{\gamma^H}{1-\gamma}(r_{\max}+\hat{r}_{\max}) + \frac{1}{1-\gamma}\bar{\iota}_\infty^K \\
&+ (K-1)\left((\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}C_1 + C_2\right] + \frac{2\gamma^H\widehat{r}_{\max}}{1-\gamma} + \frac{2\tau\log|\mathcal{A}|}{1-\gamma}\right) \\
&+ \frac{4(r_{\max}\vee\hat{r}_{\max})}{1-\gamma}\sqrt{(K-1)H\log(4/p)} - \iota_H^K
\end{aligned}
$$

Then, the following holds with probability at least $1-p/2$:

$$
\begin{aligned}
&\mathfrak{R}\left(\{\widehat{\pi}^{(k+1)}\}_{1:K-1}, K)\right) \\
&\leq (K-1)\Bigg((\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}C_1 + C_2\right] + \frac{4\gamma^H(\widehat{r}_{\max}\vee r_{\max})}{1-\gamma} + \frac{2\tau\log|\mathcal{A}|}{1-\gamma} \\
&+ \frac{4(r_{\max}\vee\hat{r}_{\max})}{1-\gamma}\sqrt{\frac{H\log(4/p)}{K-1}}\Bigg) + \frac{1}{1-\gamma}\bar{\iota}_\infty^K - \iota_H^K
\end{aligned}
$$

4.3. Upper bound of Theorem 1.

Then, combining **4.1**, **4.2** provides the expression,

$$\mathfrak{R}\left(\{\widehat{\pi}^{(k+1)}\}_{1:K-1}, K)\right) \leq \mathfrak{R}_I + \mathfrak{R}_{II}$$

where $\mathfrak{R}_{II} = \mathfrak{R}_{\texttt{Alg}}$ if we use `Alg` as the baseline algorithm and $\mathfrak{R}_{II} = \texttt{Alg}_\tau$ if we use $\mathfrak{R}_{\texttt{Alg}_\tau}$ as the baseline algorithm:

$$
\begin{aligned}
\mathfrak{R}_I &= \frac{1}{1-\gamma}\bar{\iota}_\infty^K - \iota_H^{(k)} + C_p\sqrt{K-1} \\
\mathfrak{R}_{\texttt{Alg}} &= C_{\texttt{Alg}}(\Delta_\pi)\cdot(K-1) \\
\mathfrak{R}_{\texttt{Alg}_\tau} &= C_{\texttt{Alg}_\tau}(\Delta_\pi)\cdot(K-1)
\end{aligned}
$$

where the corresponding constants are

$$C_p = \frac{4(r_{\max}\vee\hat{r}_{\max})}{1-\gamma}\sqrt{H\log(4/p)},\quad C_{\texttt{Alg}}(\Delta_\pi) = \left(\frac{1}{(1-\gamma)^2} + \frac{\log|\mathcal{A}|}{\eta}\right)\cdot\frac{1}{\Delta_\pi} + \frac{4\gamma^H(\widehat{r}_{\max}\vee r_{\max})}{1-\gamma}$$

$$C_{\texttt{Alg}_\tau}(\Delta_\pi) = (\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}C_1 + C_2\right] + \frac{4\gamma^H(\widehat{r}_{\max}\vee r_{\max})}{1-\gamma} + \frac{2\tau\log|\mathcal{A}|}{1-\gamma}$$

□

Lemma 1 (Conditions on Δ_π and H to guarantee the optimal threshold 2ϵ of ② without entropy regularization). *We decompose the term ② as*

$$\text{②'s } (k)^{th} \textit{ term} = \underbrace{\frac{1}{(1-\gamma)^2\Delta_\pi} + \frac{\log|\mathcal{A}|}{\eta\Delta_\pi}}_{\text{②-ⓐ} \leq \epsilon} + \underbrace{\frac{2\gamma^H\widehat{r}_{max}}{1-\gamma}}_{\text{②-ⓑ} \leq \epsilon}$$

To guarantee that the terms ② − ⓐ and ② − ⓑ are each less than or equal to ϵ, it suffices to satisfy the following conditions for τ, η, Δ_π and H:

$$
\begin{aligned}
\text{②} - \text{ⓐ} &: \Delta_\pi \geq \left(\frac{1}{(1-\gamma)^2} + \frac{\log|\mathcal{A}|}{\eta}\right)\cdot\frac{1}{\epsilon} \\
\text{②} - \text{ⓑ} &: H \geq \frac{\log(\frac{1-\gamma}{2\widehat{r}_{max}}\epsilon)}{\log(\gamma)} \quad \textit{or} \quad H \geq \frac{1}{1-\gamma}\log\left(\frac{2\widehat{r}_{max}}{(1-\gamma)\epsilon}\right)
\end{aligned}
$$

Lemma 2 (**Conditions on τ, Δ_π, H to guarantee the optimal threshold 4ϵ of ② with entropy regularization**)**.** *We decompose the term ② as*

$$\text{②'s } (k)^{th} \text{ term} = \underbrace{(\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}C_1\right]}_{\text{②-ⓐ} \le \epsilon} + \underbrace{(\gamma+2)\,C_2}_{\text{②-ⓑ} \le \epsilon} + \underbrace{\frac{2\gamma^H \widehat{r}_{max}}{1-\gamma}}_{\text{②-ⓒ} \le \epsilon} + \underbrace{\frac{2\tau \log|\mathcal{A}|}{1-\gamma}}_{\text{②-ⓓ} \le \epsilon}$$

To guarantee that the terms ② − ⓑ, ② − ⓒ and ② − ⓓ are each less than or equal to ϵ, it suffices to satisfy the following conditions for τ, η, Δ_π and H:

$$\text{②} - \text{ⓑ} : \delta \le \frac{\epsilon}{(\gamma+2)\cdot\frac{2}{1-\gamma}\cdot(1+\frac{\gamma}{\eta\tau})} \tag{D.36}$$

$$\text{②} - \text{ⓒ} : H \ge \frac{\log(\frac{1-\gamma}{2\widehat{r}_{max}}\epsilon)}{\log(\gamma)} \quad or \quad H \ge \frac{1}{1-\gamma}\log\left(\frac{2\widehat{r}_{max}}{(1-\gamma)\epsilon}\right) \tag{D.37}$$

$$\text{②} - \text{ⓓ} : \tau \le \frac{1-\gamma}{2\log|\mathcal{A}|}\epsilon \tag{D.38}$$

and the term ② − ⓐ offers the lower bound of iteration Δ_π as follows.

$$\text{②} - \text{ⓐ} : \Delta_\pi \ge \frac{\log\left(\frac{\epsilon}{C_1(\gamma+2)}\right)}{\log(1-\eta\tau)} + 1 \quad or \quad \Delta_\pi \ge \frac{1}{\eta\tau}\log\left(\frac{C_1(\gamma+2)}{\epsilon}\right) + 1 \tag{D.39}$$

The inequalities (D.37) and (D.39) results from applying the first-order Taylor series on $\log(\gamma)$ and $\log(1-\eta\tau)$ since $\gamma \in (0,1]$ and $\eta \in (0,(1-\gamma)/\tau]$. The inequalities (D.36) and (D.39) implies that if the learning rate η is fixed in the admissible range, then the iteration complexity scales inversely proportional to τ, and the upper bound on δ, which we will denote it as δ_{max}, also scales proportional to τ.

Now, the best guaranteed convergence can be achieved when $\eta^* = (1-\gamma)/\tau$ (associated with the value of η that minimizes the equation (D.29)), for which conditions of hyperparameters Δ_{π,η^*} and δ_{η^*} are

$$\text{②} - \text{ⓐ} : \Delta_{\pi,\eta^*} \ge \frac{1}{1-\gamma}\log\left(\frac{\|\widehat{Q}_\tau^{*,(k+1)} - \widehat{Q}_\tau^{\widehat{\pi}^{(0)}}\|_\infty(\gamma+2)}{\epsilon}\right) + 1$$

$$\text{②} - \text{ⓑ} : \delta_{\eta^*} \le \frac{\epsilon(1-\gamma)^2}{2(\gamma+2)}.$$

When $\eta^* = (1-\gamma)/\tau$, the iteration complexity is now proportional to the effective horizon $1/(1-\gamma)$ modulo some log factor, where the iteration complexity and δ_{max} are now independent of the choice of the regularization parameter τ.

Lemma 3 (**Sample complexity to guarantee the optimal threshold 4ϵ of ②**)**.** *We define δ_{max} as right-hand side of the equation (D.36). If we have the number of samples per state-action pairs is at least the order of*

$$\frac{1}{(1-\gamma)^3\delta_{max}^2}$$

up to some logarithmic factor, then $\delta \le \delta_{max}$ holds with high probability and we can guarantee the optimal threshold 4ϵ with high probability for the upper bound of ②, provided (D.37), (D.38) and (D.39) hold.

Proof of Theorem 2. **1. ProST-T $\iota_H^{(k)}$:**

The *empirical* estimated model prediction error $\iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)})$ is represented as follows (Definition (C.11)):

$$
\begin{aligned}
-\iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) =& - R_{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) - \gamma(P_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
&+ \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) && \text{(D.40)} \\
=& - R_{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) - \gamma(P_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
&+ \widehat{R}_{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) + \gamma(\widehat{P}_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s_h^{(k+1)}, a_h^{(k+1)}) \\
& && \text{(D.41)} \\
=& \left(\widehat{R}_{(k+1)} - R_{(k+1)}\right)(s_h^{(k+1)}, a_h^{(k+1)}) \\
&+ \gamma\left(\left(\widehat{P}_{(k+1)} - P_{(k+1)}\right)\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}\right)(s_h^{(k+1)}, a_h^{(k+1)}) \\
\leq& \bar{\Delta}_{(k),h}^{Bonus,r} + \gamma \left\|\left(\widehat{P}_{(k+1)} - P_{(k+1)}\right)(\cdot \mid s_h^{(k+1)}, a_h^{(k+1)})\right\|_1 \left\|\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}(\cdot)\right\|_\infty \\
\leq& \bar{\Delta}_{(k),h}^{Bonus,r} + \gamma\bar{\Delta}_{k,h}^{p}\frac{\gamma^{H-h}\hat{r}_{\max}}{1-\gamma} && \text{(D.42)} \\
\leq& \bar{\Delta}_{(k),h}^{r} + 2\Gamma_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)}) + \gamma\bar{\Delta}_{(k),h}^{p}\frac{\gamma^{H-h}\hat{r}_{\max}}{1-\gamma} && \text{(D.43)}
\end{aligned}
$$

The equation (D.41) holds due to the future Bellman equation (C.6), the equation (D.42) holds since $\left\|\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}(\cdot)\right\|_\infty \leq \sum_{h'=h+1}^{H} \gamma^{h'-(h+1)}\hat{r}_{\max} \leq \gamma^{H-h}\hat{r}_{\max}/(1-\gamma)$, and the equation (D.43) holds since $\Delta_{(k)}^{Bonus,r}(s,a) \leq \left|\left(R_{(k+1)} - \widetilde{R}_{(k+1)}\right)(s,a)\right| + |2\Gamma_w^{(k)}(s,a)| = \Delta_{(k)}^{r}(s,a) + 2\Gamma_w^{(k)}(s,a)$ for all (s,a). The summation of the empirical model prediction error over all episodes and all steps can be bounded as

$$
-\iota_H^K = \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} -\gamma^h \iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \leq \underbrace{\bar{\Delta}_K^r}_{①} + \underbrace{\sum_{k=1}^{K-1}\sum_{h=0}^{H-1} 2\Gamma_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)})}_{②} + \frac{\gamma\hat{r}_{\max}}{1-\gamma}\underbrace{\bar{\Delta}_K^p}_{③} \tag{D.44}
$$

We use Lemma 8 to bound the term ①, Lemma 9 and (D.13) to bound the term ②, and Lemma 11 (or Lemma 10) to bound the term ③:

$$
\begin{aligned}
① &\leq wHB_r(\Delta_\pi) + \lambda r_{\max} \cdot (K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} && \text{(D.45)} \\
② &\leq 2\beta(K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} && \text{(D.46)} \\
③ &\leq \left(|\mathcal{S}|\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda\right)(K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} + wHB_p(\Delta_\pi) && \text{(D.47)}
\end{aligned}
$$

where the inequality (D.47) holds with probability at least $1-\delta$, where $\delta \in (0,1)$. Now, combining (D.45), (D.46) and (D.47) that

$$
\begin{aligned}
-\iota_H^K =& -\sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \\
\le& \underbrace{\bar{\Delta}_K^r}_{\text{①}} + \underbrace{\sum_{k=1}^{K-1}\sum_{h=0}^{H-1} 2\Gamma_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)})}_{\text{②}} + \frac{\gamma \hat{r}_{\max}}{1-\gamma}\underbrace{\bar{\Delta}_K^p}_{\text{③}} \\
\le& wHB_r(\Delta_\pi) + \lambda r_{\max}\cdot(K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda+wH}{\lambda}\right)} + 2\beta(K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda+wH}{\lambda}\right)} \\
&+ \frac{\gamma\hat{r}_{\max}}{1-\gamma}\left(\left(|\mathcal{S}|\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)}+\lambda\right)(K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda+wH}{\lambda}\right)} + wHB_p(\Delta_\pi)\right) \\
\le& wH\left(B_r(\Delta_\pi) + \frac{\gamma\hat{r}_{\max}}{1-\gamma}B_p(\Delta_\pi)\right) \\
&+ (K-1)\sqrt{H}\left(\lambda r_{\max} + 2\beta + \frac{\gamma\hat{r}_{\max}}{1-\gamma}\left(|\mathcal{S}|\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)}+\lambda\right)\right)\sqrt{\frac{1}{w}}\sqrt{\log\left(\frac{\lambda+wH}{\lambda}\right)}
\end{aligned}
\tag{D.48}
$$

2. ProST-T $\bar{\iota}_\infty^K$:

Recall that $\bar{\iota}_\infty^K = \sum_{k=1}^{K-1}\bar{\iota}_\infty^{(k+1)}$. For the same δ that we used in the previous proof of [1.`ProST-T` $\iota_H^{(k)}$] (see equation (D.48)), $\bar{\iota}_\infty^k$ can be bounded as follows with probability at least $1-\delta$:

$$
\begin{aligned}
\bar{\iota}_\infty^{(k+1)} =& R_{(k+1)} + \gamma P_{(k+1)}\widehat{V}_\infty^{*,(k+1)} - \widehat{Q}_\infty^{*,(k+1)} \\
=& R_{(k+1)} + \gamma P_{(k+1)}\widehat{V}_\infty^{*,(k+1)} - \left(\widehat{R}_{(k+1)} + \gamma\widehat{P}_{(k+1)}\widehat{V}_\infty^{*,(k+1)}\right) \qquad \text{(D.49)} \\
=& R_{(k+1)} + \gamma P_{(k+1)}\widehat{V}_\infty^{*,(k+1)} - \left(\widetilde{R}_{(k+1)} + 2\Gamma_w^{(k)}(s,a) + \gamma\widehat{P}_{(k+1)}\widehat{V}_\infty^{*,(k+1)}\right) \qquad \text{(D.50)} \\
=& R_{(k+1)} + \gamma P_{(k+1)}\widehat{V}_\infty^{*,(k+1)} - \left(\widetilde{R}_{(k+1)} + 2\beta(\Lambda_w^{(k)}(s,a))^{1/2} + \gamma\widehat{P}_{(k+1)}\widehat{V}_\infty^{*,(k+1)}\right) \qquad \text{(D.51)} \\
=& \left(R_{(k+1)} - \widetilde{R}_{(k+1)}\right) - \beta(\Lambda_w^{(k)}(s,a))^{1/2} + \gamma\left(P_{(k+1)} - \widehat{P}_{(k+1)}\right)\widehat{V}_\infty^{*,(k+1)} \\
&- \beta(\Lambda_w^{(k)}(s,a))^{1/2} \qquad \text{(D.52)} \\
\le& |R_{(k+1)} - \widetilde{R}_{(k+1)}| - \beta(\Lambda_w^{(k)}(s,a))^{1/2} + \gamma\|P_{(k+1)} - \widehat{P}_{(k+1)}\|_1\|\widehat{V}_\infty^{*,(k+1)}\|_\infty - \beta(\Lambda_w^{(k)}(s,a))^{1/2} \\
\le& \left(B_r^{(k-w+1:k)}(\Delta_\pi) + \lambda\Lambda_w^{(k)}(s,a)r_{\max}\right) - \beta(\Lambda_w^{(k)}(s,a))^{1/2} \qquad \text{(D.53)} \\
&+ \gamma\cdot\left(B_p^{(k-w+1:k)}(\Delta_\pi) + (\Lambda_w^{(k)}(s,a))^{1/2}\cdot|\mathcal{S}|\cdot\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda\Lambda_w^{(k)}(s,a)\right)\cdot\frac{\hat{r}_{\max}}{1-\gamma} \\
&- \beta(\Lambda_w^{(k)}(s,a))^{1/2} \qquad \text{(D.54)} \\
\le& \left(B_r^{(k-w+1:k)}(\Delta_\pi) + \lambda(\Lambda_w^{(k)}(s,a))^{1/2}r_{\max}\right) - \beta(\Lambda_w^{(k)}(s,a))^{1/2} \qquad \text{(D.55)} \\
&+ \gamma\cdot\left(B_p^{(k-w+1:k)}(\Delta_\pi) + (\Lambda_w^{(k)}(s,a))^{1/2}\cdot|\mathcal{S}|\cdot\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda(\Lambda_w^{(k)}(s,a))^{1/2}\right)\cdot\frac{\hat{r}_{\max}}{1-\gamma} \\
&- \beta(\Lambda_w^{(k)}(s,a))^{1/2} \qquad \text{(D.56)} \\
\le& B_r^{(k-w+1:k)}(\Delta_\pi) + \gamma B_p^{(k-w+1:k)}(\Delta_\pi) \\
&+ \underbrace{\left(\lambda r_{\max} - \beta + \gamma|\mathcal{S}|\cdot\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \frac{\lambda\hat{r}_{\max}}{1-\gamma} - \beta\right)}_{\le 0}(\Lambda_w^{(k)}(s,a))^{1/2} \qquad \text{(D.57)} \\
\le& B_r^{(k-w+1:k)}(\Delta_\pi) + \gamma B_p^{(k-w+1:k)}(\Delta_\pi) \qquad \text{(D.58)}
\end{aligned}
$$

The equation (D.49) holds by the future Bellman equation (C.7) when $H = \infty$, the equations (D.50) and (D.51) hold by the definition of $\widehat{R}_{(k+1)}$ together with (D.13). The inequalities (D.53) and (D.54) hold by Lemma 7, Lemma 10, (D.8) and (D.9). The inequalities (D.55) and (D.56) hold since $0 \leq \Lambda_w^{(k)}(s,a) < 1$. Now, the inequality (D.58) holds *if* the under-brace term of equation (D.57) is equal or smaller than zero. That gives us an additional condition on β to obtain the final inequality (D.58). Since $\hat{r}_{\max}$ is defined as $\tilde{r}_{\max} + \frac{2\beta}{\sqrt{\lambda}}$ where $\widetilde{r}_{\max}$ is a constant and $\hat{r}_{\max}$ is still function of β, λ (equation (D.14)), the condition is

$$\lambda r_{\max} - \beta + \gamma|\mathcal{S}| \cdot \sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \frac{\lambda}{1-\gamma} \cdot \left(\widetilde{r}_{\max} + \frac{2\beta}{\sqrt{\lambda}}\right) - \beta \leq 0$$

or equivalently,

$$\beta \geq \left(2 + \frac{2\sqrt{\lambda}}{1-\gamma}\right)^{-1} \left(\lambda r_{\max} + \gamma|\mathcal{S}| \cdot \sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)}\right) \tag{D.59}$$

Since (D.58) holds for all (s,a) if β satisfies (D.59), $\sum_{k=1}^{K-1} \bar{\iota}_\infty^K = \|\bar{\iota}_\infty^k\|_\infty$ is bounded as

$$\bar{\iota}_\infty^K \leq \sum_{k=1}^{K-1} \left(B_r^{(k-w+1:k)}(\Delta_\pi) + \gamma B_p^{(k-w+1:k)}(\Delta_\pi)\right) \leq w(B_r(\Delta_\pi) + \gamma B_p(\Delta_\pi))$$

because $\sum_{k=1}^{K-1} B_p^{(k-w+1:k)}(\Delta_\pi) = \sum_{\mathcal{E}=1}^{\lfloor \frac{K-1}{w} \rfloor} \sum_{k=(\mathcal{E}-1)w}^{\mathcal{E}w} B_p^{(k-w+1:k)}(\Delta_\pi) \leq wB_p(\Delta_\pi)$ holds and in the same way $\sum_{k=1}^{K-1} B_p^{(k-w+1:k)}(\Delta_\pi) \leq wB_r(\Delta_\pi)$ holds.

Then, the model prediction errors $-\iota_H^K, \bar{\iota}_\infty^K$ when utilizing the forecaster f as `SW-LSE` are

$$\begin{aligned}
-\iota_H^K \leq & wH\left(B_r(\Delta_\pi) + \frac{\gamma\hat{r}_{\max}}{1-\gamma}B_p(\Delta_\pi)\right) \\
& + (K-1)\sqrt{H}\left(\lambda r_{\max} + 2\beta + \frac{\gamma\hat{r}_{\max}}{1-\gamma}\left(|\mathcal{S}|\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda\right)\right)\sqrt{\frac{1}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)}, \\
\bar{\iota}_\infty^K \leq & w(B_r(\Delta_\pi) + \gamma B_p(\Delta_\pi))
\end{aligned}$$

Finally, the term $\mathfrak{R}_I$ can be bounded as

$$\begin{aligned}
\mathfrak{R}_I = & \frac{1}{1-\gamma}\bar{\iota}_\infty^K - \iota_H^K + C_p\sqrt{K-1} \\
\leq & \frac{1}{1-\gamma}\left(w(B_r(\Delta_\pi) + \gamma B_p(\Delta_\pi))\right) + wH\left(B_r(\Delta_\pi) + \frac{\gamma\hat{r}_{\max}}{1-\gamma}B_p(\Delta_\pi)\right) \\
& + (K-1)\sqrt{H}\left(\lambda r_{\max} + 2\beta + \frac{\gamma\hat{r}_{\max}}{1-\gamma}\left(|\mathcal{S}|\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda\right)\right)\sqrt{\frac{1}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} \\
& + C_p\sqrt{K-1} \\
\leq & \left(\left(\frac{1}{1-\gamma} + H\right)B_r(\Delta_\pi) + \frac{(1 + H\hat{r}_{\max})\gamma}{1-\gamma}B_p(\Delta_\pi)\right)w \\
& + (K-1)\sqrt{H}\left(\lambda r_{\max} + 2\beta + \frac{\gamma\hat{r}_{\max}}{1-\gamma}\left(|\mathcal{S}|\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda\right)\right)\sqrt{\frac{1}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} \\
& + C_p\sqrt{K-1}
\end{aligned}$$

Now, let $B(\Delta_\pi)$ be a conic combination of $B_r(\Delta_\pi)$ and $B_p(\Delta_\pi)$ as

$$\begin{aligned}
B(\Delta_\pi) = & \left(\frac{1}{1-\gamma} + H\right)B_r(\Delta_\pi) + \frac{(1 + H\hat{r}_{\max})\gamma}{1-\gamma}B_p(\Delta_\pi) \\
\leq & \left(\frac{1}{1-\gamma} + H\right)\Delta_\pi^{\alpha_r}B_r(1) + \frac{(1 + H\hat{r}_{\max})\gamma}{1-\gamma}\Delta_\pi^{\alpha_p}B_p(1) \\
= & C_{B_r}\Delta_\pi^{\alpha_r} + C_{B_p}\Delta_\pi^{\alpha_p}
\end{aligned} \tag{D.60}$$

where $C_{B_r} = \left(\frac{1}{1-\gamma} + H\right) B_r(1)$ and $C_{B_p} = \frac{(1+H\hat{r}_{\max})\gamma}{1-\gamma} B_p(1)$ are constants related to the total variation budget with reward and transition probability.

Recall the definitions of $B_r(\Delta_\pi)$ and $B_p(\Delta_\pi)$, as well as the inequalities $B_r(\Delta_\pi) \le \Delta_\pi^{\alpha_r} B_r(1)$ and $B_p(\Delta_\pi) \le \Delta_\pi^{\alpha_p} B_p(1)$. We denote $B_p(1)$ and $B_r(1)$ as time-elapsing variation budgets for one policy iteration. We also let the constant C_k be defined as

$$C_k = (K-1)\sqrt{H}\left(\lambda r_{\max} + 2\beta + \frac{\gamma \hat{r}_{\max}}{1-\gamma}\left(|\mathcal{S}|\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda\right)\right).$$

Then, an upper bound on $\mathfrak{R}_I$ can be obtained as

$$\mathfrak{R}_I \le B(\Delta_\pi)w + C_k\sqrt{\frac{1}{w}\log\left(\frac{\lambda + wH}{\lambda}\right)} + C_p\sqrt{K-1}.$$

□

Proof of Proposition 2. Now, we set the sliding window length w that is adaptive to Δ_π as follows:

$$\widetilde{w}(\Delta_\pi) = \left(\frac{C_k}{B(\Delta_\pi)}\right)^{2/3}.$$

Then,

$$\begin{aligned} &B(\Delta_\pi)\widetilde{w}(\Delta_\pi) + C_k\sqrt{\frac{1}{\widetilde{w}(\Delta_\pi)}}\sqrt{\log\left(\frac{\lambda + \widetilde{w}(\Delta_\pi)H}{\lambda}\right)} \\ =&C_k^{2/3}B(\Delta_\pi)^{1/3} + C_k^{2/3}B(\Delta_\pi)^{1/3}\sqrt{\log\left(1 + \frac{H}{\lambda}\left(\frac{C_k}{B(\Delta_\pi)}\right)^{2/3}\right)}. \end{aligned}$$

Since C_k is linear to $K-1$, the function $\mathfrak{R}_I$ satisfies that

$$\mathfrak{R}_I = \mathcal{O}\left(B(\Delta_\pi)^{1/3}(K-1)^{2/3} \cdot \sqrt{\log\left(\frac{K-1}{B(\Delta_\pi)}\right)}\right). \tag{D.61}$$

Now, by utilizing (D.60), if $B(\Delta_\pi) \le C_{B_r}\Delta_\pi^{\alpha_r} + C_{B_p}\Delta_\pi^{\alpha_p} = o(K)$ holds, then $\mathfrak{R}_I$ is sublinear to K. The corresponding condition is $B_r(1) + \frac{\hat{r}_{\max}}{1-\gamma}B_p(1) = o(K)$ with $\Delta_\pi < K$ since

$$\begin{aligned} &C_{B_r}\Delta_\pi^{\alpha_r} + C_{B_p}\Delta_\pi^{\alpha_p} = o(K) \\ &\left(C_{B_r} + C_{B_p}\right) \cdot \Delta_\pi^{\max(\alpha_r,\alpha_p)} = o(K) \\ &\left(\left(\frac{1}{1-\gamma} + H\right)B_r(1) + \left(\frac{1+H\hat{r}_{\max}}{1-\gamma} +\right)B_p(1)\right) \cdot \Delta_\pi^{\max(\alpha_r,\alpha_p)} = o(K) \\ &\left(\frac{1}{1-\gamma}\left(B_r(1) + B_p(1)\right) + H\left(B_r(1) + \frac{\hat{r}_{\max}}{1-\gamma}B_p(1)\right)\right) \cdot \Delta_\pi^{\max(\alpha_r,\alpha_p)} = o(K). \end{aligned}$$

This completes the proof. □

Proof of Theorem 3. We first prove multiple statements below. We denote the upper bound on $\mathfrak{R}_I$ as $\mathfrak{R}_I^{\max}$, and that of $\mathfrak{R}_{II}$ as $\mathfrak{R}_{II}^{\max}$.

1. The upper bound on $\mathfrak{R}_{II}(\Delta_\pi)$ (i.e., $\mathfrak{R}_{II}^{\max}$) is a non-increasing function, the upper bound on $\mathfrak{R}_I(\Delta_\pi)$ (i.e., $\mathfrak{R}_I^{\max}$) is a non-decreasing function , and both are convex in the region $\Delta_\pi \in$

$\mathbb{N}_I \cap \mathbb{N}_{II}$

$$\begin{aligned}
\frac{\partial \mathfrak{R}_{II}^{\max}(\Delta_\pi)}{\partial \Delta_\pi} &= \frac{\partial}{\partial \Delta_\pi}\left(C_1(K-1)(\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}\right]\right) \\
&= \log(1-\eta\tau)C_1(K-1)(\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}\right] \le 0 \\
\frac{\partial^2 \mathfrak{R}_{II}^{\max}(\Delta_\pi)}{\partial^2 \Delta_\pi} &= \frac{\partial^2}{\partial^2 \Delta_\pi}\left(C_1(K-1)(\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}\right]\right) \\
&= \left(\log(1-\eta\tau)\right)^2 C_1(K-1)(\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}\right] \ge 0
\end{aligned}$$

since $\Delta_\pi \in \mathbb{N}_I \cap \mathbb{N}_{II}$ satisfies $\Delta_\pi > 1$ and $\log(1-\eta\tau) \le 0$ holds under the hyperparameter assumption $0 \le \eta \le (1-\gamma)/\tau$, it follows from the Proposition 1 that

$$\begin{aligned}
\frac{\partial \mathfrak{R}_{I}^{\max}(\Delta_\pi)}{\partial \Delta_\pi} &= \frac{\partial}{\partial \Delta_\pi}\left(C_{B_r}\Delta_\pi^{\alpha_r} + C_{B_p}\Delta_\pi^{\alpha_p}\right) \\
&= \alpha_r C_{B_r}\Delta_\pi^{\alpha_r-1} + \alpha_p C_{B_p}\Delta_\pi^{\alpha_p-1} \ge 0 \\
\frac{\partial^2 \mathfrak{R}_{I}^{\max}(\Delta_\pi)}{\partial^2 \Delta_\pi} &= \frac{\partial^2}{\partial^2 \Delta_\pi}\left(C_{B_r}\Delta_\pi^{\alpha_r} + C_{B_p}\Delta_\pi^{\alpha_p}\right) \\
&= \alpha_r(\alpha_r-1)C_{B_r}\Delta_\pi^{\alpha_r-2} + \alpha_p(\alpha_p-1)C_{B_p}\Delta_\pi^{\alpha_p-2} \ge 0
\end{aligned}$$

when $\alpha_r, \alpha_p \ge 1$.

2. Suboptimal Δ_π^*

We slightly relax the upper bound $\mathfrak{R}_I(\Delta_\pi) \le C_{B_r}\Delta_\pi^{\alpha_r} + C_{B_p}\Delta_\pi^{\alpha_p}$ to $\mathfrak{R}_I(\Delta_\pi) = \left(C_{B_r} + C_{B_p}\right)\Delta_\pi^{\max(\alpha_r,\alpha_p)}$ and obtain Δ_π^* in the worst case by optimizing $\mathfrak{R}_I^{\max}(\Delta_\pi) + \mathfrak{R}_{II}^{\max}(\Delta_\pi)$.

1. $\boldsymbol{\max(\alpha_r, \alpha_p)} = 0$: this means that $\mathfrak{R}_I^{\max}(\Delta_\pi) = C_{B_r} + C_{B_p}$, where $\mathfrak{R}_I^{\max}$ is now independent of Δ_π. Then, an infinite number Δ_π guarantees a small dynamic regret $\mathfrak{R}_I$, which also leads to a small $\mathfrak{R}$. It can be checked that $\mathfrak{R}_{II}$ without entropy regularization decreases with the scale of $1/\Delta_\pi$, and $\mathfrak{R}_{II}$ with entropy regularization decreases with the scale of $\exp(\Delta_\pi)$. This also matches with the existing results on achieving a faster convergence with an entropy regularization.

For the remaining case, we first compute the gradient of the term $\mathfrak{R}_I^{\max}(\Delta_\pi) + \mathfrak{R}_{II}^{\max}(\Delta_\pi)$ when $\mathfrak{R}_{II}^{\max}(\Delta_\pi)$ comes from entropy-regularized case:

$$\begin{aligned}
&\frac{\partial\left(\mathfrak{R}_I^{\max}(\Delta_\pi) + \mathfrak{R}_{II}^{\max}(\Delta_\pi)\right)}{\partial \Delta_\pi} \\
&= \max(\alpha_r, \alpha_p)\left(\alpha_r C_{B_r} + \alpha_p C_{B_p}\right)\Delta_\pi^{\max(\alpha_r,\alpha_p)-1} - \log\left(\frac{1}{1-\eta\tau}\right)C_1(K-1)(\gamma+2)\left[(1-\eta\tau)^{\Delta_\pi-1}\right] \\
&= k_I \Delta_\pi^{\max(\alpha_r,\alpha_p)-1} - k_{II}\left[(1-\eta\tau)^{\Delta_\pi-1}\right]
\end{aligned}$$

when $\mathfrak{R}_{II}^{\max}(\Delta_\pi)$ is for the case without entropy regularization, the gradient of the dynamic regret upper bound is given as

$$\begin{aligned}
&\frac{\partial\left(\mathfrak{R}_I^{\max}(\Delta_\pi) + \mathfrak{R}_{II}^{\max}(\Delta_\pi)\right)}{\partial \Delta_\pi} \\
&= \max(\alpha_r, \alpha_p)\left(\alpha_r C_{B_r} + \alpha_p C_{B_p}\right)\Delta_\pi^{\max(\alpha_r,\alpha_p)-1} - \left(\frac{1}{(1-\gamma)^2} + \frac{\log|\mathcal{A}|}{\eta}\right)\cdot\frac{1}{\Delta_\pi^2} \\
&= k_I \Delta_\pi^{\max(\alpha_r,\alpha_p)-1} - k_{II}\frac{1}{\Delta_\pi^2}
\end{aligned}$$

2. $\boldsymbol{\max(\alpha_r, \alpha_p) = 1}$: The relation $(1-\eta\tau)^{\Delta_\pi-1} = k_I/k_{II}$ should be satisfied for the entropy regularized case and $\Delta_\pi^{-2} = k_I/k_{II}$ should be satisfied in the case without entropy regularization, respectively. Then, it holds that $\Delta_\pi^* = \log_{1-\eta\tau}(k_I/k_{II}) + 1$ for the entropy regularized case and $\Delta_\pi^* = \sqrt{k_{II}/k_I}$ without regularization.

Now, for the case of the entropy regularized case, if $k_{II} = (1 - \eta\tau)k_I$ is satisfied, $\partial\left(\mathfrak{R}_I^{\max}(\Delta_\pi) + \mathfrak{R}_{II}^{\max}(\Delta_\pi)\right)/\partial\Delta_\pi = 0$ is equal to solving $\Delta_\pi^{\max(\alpha_r,\alpha_p)-1} = (1-\eta\tau)^{\Delta_\pi}$. Now, we use the Lambert W function to find Δ_π as follows:

$$
\begin{aligned}
&\Delta_\pi^{\max(\alpha_r,\alpha_p)-1} = (1-\eta\tau)^{\Delta_\pi}\\
&(\max(\alpha_r,\alpha_p)-1)\log\Delta_\pi = \Delta_\pi\log(1-\eta\tau)\\
&\Delta_\pi^{-1}\cdot\log\Delta_\pi = \frac{\log(1-\eta\tau)}{\max(\alpha_r,\alpha_p)-1}\\
&-\log\Delta_\pi\cdot e^{-\log\Delta_\pi} = -\frac{\log(1-\eta\tau)}{\max(\alpha_r,\alpha_p)-1}\\
&W\left[-\log\Delta_\pi\cdot e^{-\log\Delta_\pi}\right] = W\left[-\frac{\log(1-\eta\tau)}{\max(\alpha_r,\alpha_p)-1}\right]\\
&W\left[-\log\Delta_\pi\cdot e^{-\log G}\right] = W\left[-\frac{\log(1-\eta\tau)}{\max(\alpha_r,\alpha_p)-1}\right]\\
&-\log\Delta_\pi = W\left[-\frac{\log(1-\eta\tau)}{\max(\alpha_r,\alpha_p)-1}\right]\\
&\Delta_\pi^* = \exp\left(-W\left[-\frac{\log(1-\eta\tau)}{\max(\alpha_r,\alpha_p)-1}\right]\right) = \exp(-W[x])
\end{aligned}
$$

3. $\mathbf{0 < \max(\alpha_r,\alpha_p) < 1}$:
    - Without Entropy-regularization: $\Delta_\pi^* = (k_I/k_{II})^{1/(\max(\alpha_r,\alpha_p)+1)}$
    - With Entropy-regularization: Since $x = -\frac{\log(1-\eta\tau)}{\max(\alpha_r,\alpha_p)-1} < 0$, a small $|x|$ will have a large $-W(x) > 0$ value, which leads to a large Δ_π^*.
4. $\mathbf{\max(\alpha_r,\alpha_p) > 1}$:
    - Without Entropy-regularization: $\Delta_\pi^* = (k_I/k_{II})^{1/(\max(\alpha_r,\alpha_p)+1)}$
    - With Entropy-regularization: It holds that $x > 0$ and $-W(x) < 0$. Then $\Delta_\pi^* < 1$, which means that one iteration is enough.

□

From the proof of Theorem 2, we will develop Lemma 4, Lemma 5 and Lemma 6 to upper-bound two model prediction errors $-\iota_h^{(k)}$ and $\bar{\iota}_\infty^k$.

Lemma 4 (Upper bound on $-\iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)})$ by $\bar{\Delta}_{k,h}^r$, $\bar{\Delta}_{k,h}^p$)**.** *It holds that*

$$
-\iota_h^{(k+1)}(s_h^{(k+1)}, a_h^{(k+1)}) \le \bar{\Delta}_{k,h}^r + 2\Gamma_w^{(k)}(s,a) + \gamma\bar{\Delta}_{k,h}^p\frac{\gamma^{H-h}\hat{r}_{max}}{1-\gamma}
$$

Proof of Lemma 4. It follows from (D.40), (D.41), (D.42) and (D.43). □

Lemma 5 (Upper bound on $-\iota_h^{(k+1)}(s,a)$ by $\Delta_{(k)}^r$, $\Delta_{(k)}^p$)**.** *For every $(s,a) \in \mathcal{S}\times\mathcal{A}$, it holds that*

$$
-\iota_h^{(k+1)}(s,a) \le \Delta_{(k)}^r(s,a) + \gamma\Delta_{(k)}^p(s,a)\frac{\gamma^{H-h}\hat{r}_{max}}{1-\gamma} + 2\Gamma_w^{(k)}(s,a)
$$

Proof of Lemma 5.

$$
\begin{aligned}
-\iota_h^{(k+1)}(s,a) &= -R_{(k+1)}(s,a) - \gamma(P_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s,a) + \widehat{Q}_h^{\widehat{\pi}^{(k+1)},(k+1)}(s,a) \\
&= -R_{(k+1)}(s,a) - \gamma(P_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s,a) \\
&\quad + \widehat{R}_{(k+1)}(s,a) + \gamma(\widehat{P}_{(k+1)}\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)})(s,a) \\
&= \left(\widehat{R}_{(k+1)} - R_{(k+1)}\right)(s,a) + \gamma\left(\left(\widehat{P}_{(k+1)} - P_{(k+1)}\right)\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}\right)(s,a) \\
&\le \Delta_{(k)}^r(s,a) + 2\Gamma_w^{(k)}(s,a) + \gamma\left\|\left(\widehat{P}_{(k+1)} - P_{(k+1)}\right)(\cdot \mid s,a)\right\|_1 \left\|\widehat{V}_{h+1}^{\widehat{\pi}^{(k+1)},(k+1)}(\cdot)\right\|_\infty \\
&\le \Delta_{(k)}^r(s,a) + 2\Gamma_w^{(k)}(s,a) + \gamma\Delta_{(k)}^p(s,a)\frac{\gamma^{H-h}\hat{r}_{\max}}{1-\gamma}
\end{aligned}
$$

□

Lemma 6 (Upper bound on $\overline{\iota}_\infty^k$ by $\Delta_{(k)}^r$, $\Delta_{(k)}^p$)**.** *For every $(s,a) \in \mathcal{S} \times \mathcal{A}$, it holds that*

$$
\overline{\iota}_\infty^{k+1}(s,a) \le \Delta_{(k)}^r(s,a) + \Delta_{(k)}^p(s,a)\frac{\gamma\hat{r}_{max}}{1-\gamma} - 2\Gamma_w^{(k)}(s,a)
$$

Proof of Lemma 6. It results from (D.52),

$$
\begin{aligned}
\overline{\iota}_\infty^{k+1} &= \left(R_{(k+1)} - \widetilde{R}_{(k+1)}\right) - \beta(\Lambda_w^{(k)}(s,a))^{1/2} + \gamma\left(P_{(k+1)} - \widehat{P}_{(k+1)}\right)\widehat{V}_\infty^{*,(k+1)} - \beta(\Lambda_w^{(k)}(s,a))^{1/2} \\
&\le \left|R_{(k+1)} - \widetilde{R}_{(k+1)}\right| - \beta(\Lambda_w^{(k)}(s,a))^{1/2} + \gamma\left\|P_{(k+1)} - \widehat{P}_{(k+1)}\right\|_1 \left\|\widehat{V}_\infty^{*,(k+1)}\right\|_\infty - \beta(\Lambda_w^{(k)}(s,a))^{1/2} \\
&\le \Delta_{(k)}^r(s,a) - \beta(\Lambda_w^{(k)}(s,a))^{1/2} + \gamma\Delta_{(k)}^p(s,a)\frac{\hat{r}_{max}}{1-\gamma} - \beta(\Lambda_w^{(k)}(s,a))^{1/2} \\
&= \Delta_{(k)}^r(s,a) + \Delta_{(k)}^p(s,a)\frac{\gamma\hat{r}_{max}}{1-\gamma} - 2\Gamma_w^{(k)}(s,a)
\end{aligned}
$$

□

Lemma 7 (Upper bound on $\Delta_{(k)}^r(s,a)$)**.** *For every $(s,a) \in \mathcal{S} \times \mathcal{A}$, it holds that*

$$
\Delta_{(k)}^r(s,a) \le B_r^{(k-w:k)}(\Delta_\pi) + \lambda\Lambda_w^{(k)}(s,a)r_{max}
$$

Proof of Lemma 7. We directly utilize the proof of Lemma 35 in [31]. For every $(s,a) \in \mathcal{S} \times \mathcal{A}$, $\Delta^r_{(k)}(s,a)$ can be represented as

$$\Delta^r_{(k)}(s,a) \tag{D.62}$$

$$= |R_{(k+1)}(s,a) - \widetilde{R}_{(k+1)}(s,a)| \tag{D.63}$$

$$= |o^r_{(k+1)}(s,a) - \widetilde{o}^r_{(k+1)}(s,a)| \tag{D.64}$$

$$= \left| \frac{\sum_{t=(1\wedge k-w+1)}^{k} \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \cdot r_h^t}{\lambda + \sum_{t=(1\wedge k-w+1)}^{k} n_t(s,a)} - o^r_{(k+1)}(s,a) \right| \tag{D.65}$$

$$= \Lambda_w^{(k)}(s,a) \left| \sum_{t=(1\wedge k-w+1)}^{k} \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \cdot r_h^t - \left(\lambda + \sum_{t=(1\wedge k-w+1)}^{k} n_t(s,a) \right) o^r_{(k+1)}(s,a) \right| \tag{D.66}$$

$$= \Lambda_w^{(k)}(s,a) \left| \sum_{t=(1\wedge k-w+1)}^{k} \sum_{h=0}^{H-1} \left(\mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \left(r_h^t - o^r_{(k+1)}(s,a) \right) \right) - \lambda \cdot o^r_{(k+1)}(s,a) \right| \tag{D.67}$$

$$\leq \Lambda_w^{(k)}(s,a) \left(\sum_{t=(1\wedge k-w+1)}^{k} \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \cdot \left| r_h^t - o^r_{(k+1)}(s,a) \right| \right) + \lambda \Lambda_w^{(k)}(s,a) \left| o^r_{(k+1)}(s,a) \right| \tag{D.68}$$

$$\leq \Lambda_w^{(k)}(s,a) \left(\sum_{t=(1\wedge k-w+1)}^{k} n_t(s,a) \left(\left| r^t(s,a) - o^r_{(k+1)}(s,a) \right| \right) \right) + \lambda \Lambda_w^{(k)}(s,a) r_{\max} \tag{D.69}$$

$$\leq \max_{(1\wedge k-w+1) \leq t \leq k} \left(\left| r^t(s,a) - o^r_{(k+1)}(s,a) \right| \right) \Lambda_w^{(k)}(s,a) \left(\sum_{t=(1\wedge k-w+1)}^{k} n_t(s,a) \right) + \lambda \Lambda_w^{(k)}(s,a) r_{\max}$$

$$\leq \max_{(1\wedge k-w+1) \leq t \leq k} \left(\left| r^t(s,a) - o^r_{(k+1)}(s,a) \right| \right) + \lambda \Lambda_w^{(k)}(s,a) r_{\max}$$

$$\leq B_r^{(k-w:k)}(\Delta_\pi) + \lambda \Lambda_w^{(k)}(s,a) r_{\max} \tag{D.70}$$

Equations (D.64) and (D.65) hold by the definition of $o^r_{k+1}, \widetilde{o}^r_{k+1}$ (definition (D.7)), equation (D.66) holds by the definition (D.12), equation (D.67) holds since $n_t(s,a) \coloneqq \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right]$, and inequality (D.70) holds since $\max_{(1\wedge k-w+1) \leq t \leq k} \left(\left| r^t(s,a) - o^r_{(k+1)}(s,a) \right| \right) \leq |r^{(1\wedge k-w+1)}(s,a) - r^{(1\wedge k-w+1)+1}(s,a)| + \cdots + |r^k(s,a) - r^{k+1}(s,a)| = B_r^{(k-w:k)}(\Delta_\pi)$. □

Lemma 8 (Upper bound on $\bar{\Delta}^r_K$)**.** *For every $(s,a) \in \mathcal{S} \times \mathcal{A}$, it holds that*

$$\bar{\Delta}^r_K \leq wHB_r(\Delta_\pi) + \lambda r_{max} \cdot (K-1) \sqrt{\frac{H}{w}} \sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)}$$

Proof of Lemma 8. The total empirical forecasting model error up to $K-1$ is given as

$$\begin{aligned}
\bar{\Delta}^r_K &= \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \bar{\Delta}^r_{k,h} \\
&= \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \Delta^r_{(k)}(s_h^{(k+1)}, a_h^{(k+1)}) \\
&\le \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left(B_r^{(k-w:k)}(\Delta_\pi) + \lambda \Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)}) r_{\max}\right) && \text{(D.71)} \\
&= wHB_r(\Delta_\pi) + \lambda r_{\max} \cdot \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left(\Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)})\right) && \text{(D.72)} \\
&\le wHB_r(\Delta_\pi) + \lambda r_{\max} \cdot \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left(\sqrt{\Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)})}\right) \\
&\le wHB_r(\Delta_\pi) + \lambda r_{\max} \cdot (K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} && \text{(D.73)}
\end{aligned}$$

The inequality (D.71) holds by Lemma 7, the equation (D.72) holds since $\sum_{k=1}^{K-1} B_r^{(k-w:k)}(\Delta_\pi) = \sum_{\mathcal{E}=1}^{\lfloor \frac{K-1}{w} \rfloor} \sum_{k=(\mathcal{E}-1)w}^{\mathcal{E}w} B_r^{(k-w:k)}(\Delta_\pi) \le wB_r(\Delta_\pi)$, and the inequality (D.73) holds by Lemma 9. ☐

Lemma 9 (Upper bound on the term $\sum_{k=1}^{K-1}\sum_{h=0}^{H-1}\sqrt{\Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)})}$)**.** *It holds that*

$$\sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left(\sqrt{\Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)})}\right) \le (K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)}$$

Proof of lemma 9. We denote $\bar{\Lambda}_w^k = \lambda\mathbb{I} + \sum_{t=(1\wedge k-w+1)}^{k}\sum_{h=0}^{H-1}\varphi(s_h^t, a_h^t)\varphi(s_h^t, a_h^t)^\top$. Also, we denote $(\bar{\Lambda}_w^k)^{(1)} = \lambda\mathbb{I} + \varphi(s_h^{(1\wedge k-w+1)}, a_h^{(1\wedge k-w+1)})\varphi(s_h^{(1\wedge k-w+1)}, a_h^{(1\wedge k-w+1)})^\top$ Then, for every $(s,a) \in \mathcal{S}\times\mathcal{A}$, $\Lambda_w^{(k)}(s,a) = \varphi(s,a)(\bar{\Lambda}_w^k)^{-1}\varphi(s,a)^\top$ holds. Now, the following term can be bounded as

$$\begin{aligned}
&\sum_{k=1}^{K-1}\sum_{h=0}^{H-1}\sqrt{\Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)})} \\
&= \sum_{k=1}^{K-1}\sum_{h=0}^{H-1}\sqrt{\varphi(s_h^{(k+1)}, a_h^{(k+1)})(\bar{\Lambda}_w^k)^{-1}\varphi(s_h^{(k+1)}, a_h^{(k+1)})^\top} \\
&= \sum_{\mathcal{E}=1}^{\lfloor \frac{K-1}{w} \rfloor}\sum_{k=(\mathcal{E}-1)w+1}^{\mathcal{E}w}\sum_{h=0}^{H-1}\sqrt{\varphi(s_h^{(k+1)}, a_h^{(k+1)})(\bar{\Lambda}_w^k)^{-1}\varphi(s_h^{(k+1)}, a_h^{(k+1)})^\top} \\
&\le \sum_{\mathcal{E}=1}^{\lfloor \frac{K-1}{w} \rfloor}\sqrt{Hw}\sqrt{\sum_{k=(\mathcal{E}-1)w+1}^{\mathcal{E}w}\sum_{h=0}^{H-1}\varphi(s_h^{(k+1)}, a_h^{(k+1)})(\bar{\Lambda}_w^k)^{-1}\varphi(s_h^{(k+1)}, a_h^{(k+1)})^\top} && \text{(D.74)} \\
&\le \sum_{\mathcal{E}=1}^{\lfloor \frac{K-1}{w} \rfloor}\sqrt{Hw}\sqrt{\log\left(\frac{\det(\Lambda_w^{\mathcal{E}w+1})}{\det\left((\Lambda_w^{(\mathcal{E}-1)w+2})^{(1)}\right)}\right)} && \text{(D.75)} \\
&\le \left\lfloor \frac{K-1}{w} \right\rfloor \sqrt{Hw}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} && \text{(D.76)} \\
&\le (K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)}
\end{aligned}$$

The inequality (D.74) holds by the Cauchy–Schwarz inequality, (D.75) holds by Lemmas (D.1) and (D.2) in [39], and (D.76) holds since $(\Lambda_w^{(\mathcal{E}-1)w+2})^{(1)} \ge \lambda$ and $\Lambda_w^{\mathcal{E}w+1} \le \lambda + wH$. ☐

Lemma 10 (Upper bound on $\Delta^{p}_{(k)}(s,a)$)**.** *For every $(s,a)\in\mathcal{S}\times\mathcal{A}$ and given $\delta\in(0,1)$, the following holds with probability at least $1-\delta$:*

$$\Delta^{p}_{(k)}(s,a)\le B_p^{(k-w+1:k)}+(\Lambda_w^{(k)}(s,a))^{1/2}\cdot|\mathcal{S}|\cdot\sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)}+\lambda\Lambda_w^{(k)}(s,a)$$

***Proof of lemma 10*.** For every $(s,a)\in\mathcal{S}\times\mathcal{A}$, one can write:

$$\begin{aligned}
&\Delta^{p}_{(k)}(s,a)\\
&=\|P_{(k+1)}(\cdot|s,a)-\widehat{P}_{(k+1)}(\cdot|s,a)\|_1\\
&=\|o^{p}_{(k+1)}(\cdot,s,a)-\widehat{o}^{p}_{(k+1)}(\cdot,s,a)\|_1\\
&=\sum_{s'\in\mathcal{S}}\left|\frac{\sum_{t=k-w+1}^{k}n_t(s',s,a)}{\lambda+\sum_{t=k-w+1}^{k}n_t(s,a)}-o^{p}_{(k+1)}(s',s,a)\right|\\
&=\Lambda_w^{(k)}(s,a)\sum_{s'\in\mathcal{S}}\left|\sum_{t=k-w+1}^{k}n_t(s',s,a)-\left(\lambda+\sum_{t=k-w+1}^{k}n_t(s,a)\right)o^{p}_{(k+1)}(s',s,a)\right|\\
&\le\Lambda_w^{(k)}(s,a)\sum_{s'\in\mathcal{S}}\left(\left|\sum_{t=k-w+1}^{k}\left(n_t(s',s,a)-n_t(s,a)o^{p}_{(k+1)}(s',s,a)\right)\right|+\left|\lambda o^{p}_{(k+1)}(s',s,a)\right|\right)\\
&\le\Lambda_w^{(k)}(s,a)\sum_{s'\in\mathcal{S}}\left|\sum_{t=k-w+1}^{k}\left(n_t(s',s,a)-n_t(s,a)o^{p}_{(k+1)}(s',s,a)\right)\right|+\lambda\Lambda_w^{(k)}(s,a)
\end{aligned}\tag{D.77}$$

Recall that $n_t(s',s,a),\ n_t(s,a)$ is defined as

$$\begin{aligned}
n_t(s',s,a)&=\sum_{h=0}^{H-1}\mathbb{1}\left[(s',s,a)=(s^t_{h+1},s^t_h,a^t_h)\right]\\
&=\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a)=(s^t_h,a^t_h)\right]\cdot\mathbb{1}\left[s'=s^t_{h+1}\right]
\end{aligned}\tag{D.78}$$

$$n_t(s,a)=\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a)=(s^t_h,a^t_h)\right]\tag{D.79}$$

where $\mathbb{1}[\cdot]$ is an indicator function. Substituting (D.78) and (D.79) into (D.77) yields that

$$\begin{aligned}
&\Lambda_w^{(k)}(s,a)\sum_{s'\in\mathcal{S}}\left|\sum_{t=k-w+1}^{k}\left(n_t(s',s,a)-n_t(s,a)o^{p}_{(k+1)}(s',s,a)\right)\right|\\
&=\Lambda_w^{(k)}(s,a)\sum_{s'\in\mathcal{S}}\left|\sum_{t=k-w+1}^{k}\left(\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a)=(s^t_h,a^t_h)\right]\cdot\mathbb{1}\left[s'=s^t_{h+1}\right]\right.\right.\\
&\left.\left.-\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a)=(s^t_h,a^t_h)\right]\cdot o^{p}_{(k+1)}(s',s,a)\right)\right|\\
&=\Lambda_w^{(k)}(s,a)\sum_{s'\in\mathcal{S}}\left|\sum_{t=k-w+1}^{k}\left(\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a)=(s^t_h,a^t_h)\right]\left(\mathbb{1}\left[s'=s^t_{h+1}\right]-o^{p}_{(k+1)}(s',s,a)\right)\right)\right|\\
&\le\underbrace{\Lambda_w^{(k)}(s,a)\sum_{s'\in\$}\left|\sum_{t=k-w+1}^{k}\left(\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a)=(s^t_h,a^t_h)\right]\left(\mathbb{1}\left[s'=s^t_{h+1}\right]-o^{p}_{t}(s',s,a)\right)\right)\right|}_{\textcircled{\scriptsize 2.1}}\\
&+\underbrace{\Lambda_w^{(k)}(s,a)\sum_{s'\in\$}\left|\sum_{t=k-w+1}^{k}\left(\sum_{h=0}^{H-1}\mathbb{1}\left[(s,a)=(s^t_h,a^t_h)\right]\left(o^{p}_{t}(s',s,a)-o^{p}_{(k+1)}(s',s,a)\right)\right)\right|}_{\textcircled{\scriptsize 2.2}}
\end{aligned}$$

The term $\textcircled{2.1}$ can be upperbounded by utilizing the Lemmas (34) and (43) in [31]. For every $t \in [K]$ and $s' \in \mathcal{S}$, we define the random variable $\eta^t(s') \coloneqq \sum_{h=0}^{H-1} \left(\mathbb{1}\left[s' = s_{h+1}^t\right] - o_t^p(s', s_h^t, a_h^t)\right)$. Given $s' \in \mathcal{S}$, the sequence $\{\eta^\tau(s')\}_{\tau=1}^{\infty}$ is a zero-mean and $H/2$-sub Gaussian random variable. From the Lemma 43 in [31], we set $Y = \lambda\mathbb{I}$ and $X_t = \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right]$. Then, for a given $\delta \in (0,1)$, the following holds with probability at least $1-\delta$ for all $(s,a) \in \mathcal{S} \times \mathcal{A}$:

$$\begin{aligned}
&\left|(\Lambda_w^{(k)}(s,a))^{1/2} \sum_{t=k-w+1}^{k} \left(\sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \cdot \sum_{h=0}^{H-1} \mathbb{1}[s' = s_{h+1}^t] - o_t^p(s', s, a)\right)\right| \\
&\leq \sqrt{\frac{H^2}{2} \log\left(\frac{(\Lambda_w^{(k)}(s,a))^{-1/2} \cdot \lambda^{-1/2}}{\delta/H}\right)} \\
&= \sqrt{\frac{H^2}{2} \log\left(\frac{H}{\delta} \cdot \frac{1}{(\Lambda_w^{(k)}(s,a))^{1/2} \cdot \lambda^{1/2}}\right)} \\
&\leq \sqrt{\frac{H^2}{2} \log\left(\frac{H}{\delta} \cdot \frac{1}{\lambda}\right)} \qquad \text{(D.80)}
\end{aligned}$$

As a result, the following inequality holds with probability at least $1-\delta$:

$$\begin{aligned}
&\textcircled{2.1} \\
&= (\Lambda_w^{(k)}(s,a))^{1/2} \sum_{s' \in \mathbb{S}} \left|(\Lambda_w^{(k)}(s,a))^{1/2} \sum_{t=k-w+1}^{k} \left(\sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \cdot \sum_{h=0}^{H-1} \mathbb{1}[s' = s_{h+1}^t] \right.\right. \\
&\qquad \left.\left. - o_t^p(s', s, a)\right)\right| \\
&\leq (\Lambda_w^{(k)}(s,a))^{1/2} \cdot |\mathcal{S}| \cdot \sqrt{\frac{H^2}{2} \log\left(\frac{H}{\delta\lambda}\right)}
\end{aligned}$$

The term $\textcircled{2.2}$ can be bounded as

$$\begin{aligned}
\textcircled{2.2} &\leq \Lambda_w^{(k)}(s,a) \sum_{s' \in \mathcal{S}} \sum_{t=k-w+1}^{k} \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \left|o_t^p(s', s, a) - o_{(k+1)}^p(s', s, a)\right| \\
&= \Lambda_w^{(k)}(s,a) \sum_{t=k-w+1}^{k} \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \sum_{s' \in \mathcal{S}} \left|o_t^p(s', s, a) - o_{(k+1)}^p(s', s, a)\right| \\
&= \Lambda_w^{(k)}(s,a) \sum_{t=k-w+1}^{k} \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right] \left\|o_t^p(\cdot, s, a) - o_{(k+1)}^p(\cdot, s, a)\right\|_1 \\
&\leq \max_{t \in [k-w+1,k]} \left(\left\|o_t^p(\cdot, s, a) - o_{(k+1)}^p(\cdot, s, a)\right\|_1\right) \cdot \left(\Lambda_w^{(k)}(s,a) \sum_{t=k-w+1}^{k} \sum_{h=0}^{H-1} \mathbb{1}\left[(s,a) = (s_h^t, a_h^t)\right]\right) \\
&\leq \max_{t \in [k-w+1,k]} \left(\left\|o_t^p(\cdot, s, a) - o_{(k+1)}^p(\cdot, s, a)\right\|_1\right) \cdot 1 \\
&\leq B_p^{(k-w+1:k)}(\Delta_\pi) \qquad \text{(D.81)}
\end{aligned}$$

Then, by combining (D.77), (D.80) and (D.81), the term $\Delta_{(k)}^p(s,a)$ can be expressed as

$$\Delta_{(k)}^p(s,a) \leq B_p^{(k-w+1:k)}(\Delta_\pi) + (\Lambda_w^{(k)}(s,a))^{1/2} \cdot |\mathcal{S}| \cdot \sqrt{\frac{H^2}{2} \log\left(\frac{H}{\delta\lambda}\right)} + \lambda\Lambda_w^{(k)}(s,a).$$

□

Lemma 11 (Upper bound on $\bar{\Delta}_K^p$). *Given $\delta \in (0,1)$, the following inequality holds with probability at least $1-\delta$:*

$$\bar{\Delta}_K^p \leq \left(|\mathcal{S}|\sqrt{\frac{H^2}{2} \log\left(\frac{H}{\delta\lambda}\right)} + \lambda\right)(K-1)\sqrt{\frac{H}{w}}\sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} + wHB_p(\Delta_\pi)$$

Proof of lemma 11. The total empirical forecasting transition probability model error $\bar{\Delta}_K^p$ can be represented as follows,

$$\begin{aligned}
\bar{\Delta}_K^p &= \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \bar{\Delta}_{k,h}^p \\
&= \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \Delta_{(k)}^p(s_h^{(k+1)}, a_h^{(k+1)}) \\
&\leq \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left((\Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)}))^{1/2} |\mathcal{S}| \sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} \right) \\
&\quad + \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left(\max_{t\in[k-w+1,k]} \left\| o_t^p(\cdot, s_h^{(k+1)}, a_h^{(k+1)}) - o_{(k+1)}^p(\cdot, s_h^{(k+1)}, a_h^{(k+1)}) \right\|_1 \right) \\
&\quad + \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left(\lambda \Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)}) \right) \\
&\leq \left(|\mathcal{S}| \sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda \right) \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left((\Lambda_w^{(k)}(s_h^{(k+1)}, a_h^{(k+1)}))^{1/2} \right) \\
&\quad + \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} \left(\max_{t\in[k-w+1,k]} \left\| o_t^p(\cdot, s_h^{(k+1)}, a_h^{(k+1)}) - o_{(k+1)}^p(\cdot, s_h^{(k+1)}, a_h^{(k+1)}) \right\|_1 \right) \\
&\leq \left(|\mathcal{S}| \sqrt{\frac{H^2}{2}\log\left(\frac{H}{\delta\lambda}\right)} + \lambda \right) (K-1) \sqrt{\frac{H}{w}} \sqrt{\log\left(\frac{\lambda + wH}{\lambda}\right)} + wHB_p(\Delta_\pi)
\end{aligned}$$

□

Proof of Theorem 4 . Before introduing the proof, we first go over some details about Theorem 4 in the following paragraph.

The `W-LSE` involves solving the following joint optimization problem over $\phi_f^r \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$, $\phi_f^p \in \mathbb{R}^{|\mathcal{S}|^2|\mathcal{A}|}$ and $q \in \mathbb{R}^N$ to obtain a minimum upper bound on the dynamic regret:

$$\min_{\phi_f^\diamond, q} \mathcal{L}\left(\phi_f^\diamond, q \,;\, \square_{1:N}\right) \text{ where } \mathcal{L}\left(\phi_f^\diamond, q \,;\, \square_{1:N}\right) = \sum_{t=1}^{N} q_t \left(\widehat{\square}_{\phi_f^\diamond}^{k+1} - \square_t\right)^2 + \operatorname{disc}(q) + \frac{1}{wH} \cdot \lambda \|\phi_f^\diamond\|_2 \tag{D.82}$$

where $\diamond = r$ or p. If $\diamond = r$, then $\square = R(s,a)$ and if $\diamond = p$, then $\square = P(s', s, a)$. Moreover, $\square_{\phi_f^\diamond}$ means that $\square$ is parameterized by $\phi_f^\diamond$, and $\square_{1:N}$ are observed data of $\square$, and the $\operatorname{disc}(q) \coloneqq \sup_{f\in\mathcal{F}} \left(\mathbb{E}[f(\widehat{\square}^{k+1} \mid \square_{1:N}] - \sum_{t=1}^N q_t \mathbb{E}[\widehat{\square}^t | \square_{1:t-1}]\right)$ measures the non-stationarity of the environment. $\operatorname{disc}(q)$ could be measured and upper-bounded by the observed data. For example, if $\diamond = r$ and $\square = R$, then ϕ_f^r parameterizes the future reward function $\widehat{R}_{\phi_f^r}^{k+1}$, N is the total number of visits of (s,a) up to episode k, $R_{1:N}(s,a)$ is the set of reward values $\{R_1(s,a), R_1(s,a), \ldots, R_N(s,a)\}$ that the agent has received when visiting (s,a). We demonstrate a modified upper bound on $\mathfrak{R}_I$ when utilizing `W-LSE`. To do so, we define the forecasting reward model error $\Delta_{r,k}^1(s,a) = \left|\left(R_{(k+1)} - \widetilde{R}_{(k+1)}\right)(s,a)\right|$ and the forecasting transition probability model error as $\Delta_{(k)}^p(s,a) = \left\|\left(P_{(k+1)} - \widehat{P}_{(k+1)}\right)(\cdot \mid s,a)\right\|_1$ where $\widetilde{R}_{(k+1)}$ and $\widehat{P}_{(k+1)}$ are predicted reward,transition probability from function $g \circ f$ (Appendix D.2).

We now brought the Theorem 7 of [22] to offer an upper bound on the l_2-norm of the reward gap between $R_{(k+1)}(s,a)$ and $\widetilde{R}_{(k+1)}(s,a)$ as follows. To this end, we denote $X_{k,h} = (s_h^{(k)}, a_h^{(k)}) \in \mathcal{S} \times \mathcal{A}$, $Y_{k,h} = R_{(k)}(s_h^{(k)}, a_h^{(k)}) \in \mathbb{R}$ and assume that the environment provides the agent with a noisy reward $\widehat{Y}_{k,h} = Y_{k,h} + \eta$, where η is sampled from a zero-mean Gaussian. Define the kernel $\Psi(x) = \varphi(x) \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$, where $\varphi(x)$ is the one-hot vector that we have defined in Section D.1.1. Now, we set $r(x) = c^\top \varphi(x)$ where the vector $c \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ is the same as the estimated future reward vector $\widetilde{R}_{(k+1)} \in$

$\mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ and $r(x)$ is the same as the estimated future reward when $x = (s,a)$, namely $\widetilde{R}_{(k+1)}(s,a)$. Then, for data until episode k, i.e., $\mathcal{D}_{data} = \{(X_{1,0}, \widehat{Y}_{1,0}), (X_{1,1}, \widehat{Y}_{1,1}), .., (X_{k,H-1}, \widehat{Y}_{k,H-1})\}$, we denote $\mathcal{D}_{data}^{(s,a)} \coloneqq \{(X_{k,h}, \widehat{Y}_{k,h}) \mid X_{k,h} = (s,a)$ such that $(X_{k,h}, \widehat{Y}_{k,h}) \in \mathcal{D}_{data}\}$. We relabel $\mathcal{D}_{data}^{(s,a)}$ as $\{\left((s,a), \widehat{Y}_1\right), \left((s,a), \widehat{Y}_2\right), ..., \left((s,a), \widehat{Y}_N\right)\}$ such that $N(s,a) = \sum_{t=1}^{k} n_t(s,a)$ is the total number of visitations of (s,a) until episode k (Definition (D.79)). We use the shorthand notation N as $N(s,a)$, and $\sum_{t=1}^{N} q_t = 1$. For every $(s,a) \in \mathcal{S} \times \mathcal{A}$, the following inequalities hold with probability at least $1 - \delta$ for all functions $r \in \{x \to c^\top \Psi(x) : \|c\|_2 \le \Lambda\}$:

$$\mathbb{E}[\left(r(s,a) - \widehat{Y}_{N+1}\right)^2 | \mathcal{D}_{data}^{(s,a)}] \le \sum_{t=1}^{N} q_t \left(r(s,a) - \widehat{Y}_t\right)^2 + \text{disc}(q) + \frac{1}{wH} \cdot \lambda \|\bar{r}\|_2 \tag{D.83}$$

Take the expectation over η on both inequailty.

$$\mathbb{E}_\eta\left[\mathbb{E}\left[\left(r(s,a) - \widehat{Y}_{N+1}\right)^2 | \mathcal{D}_{data}^{(s,a)}\right]\right] \le \mathbb{E}_\eta\left[\sum_{t=1}^{N} q_t \left(r(s,a) - \widehat{Y}_t\right)^2 + \text{disc}(q) + \frac{1}{wH} \cdot \lambda \|\bar{r}\|_2\right],$$

$$\mathbb{E}\left[\left(r(s,a) - \widehat{Y}_{N+1}\right)^2 | \mathcal{D}_{data}^{(s,a)}\right] \le \sum_{t=1}^{N} \mathbb{E}_\eta\left[q_t \left(r(s,a) - \widehat{Y}_t\right)^2\right] + \text{disc}(q) + \frac{1}{wH} \cdot \lambda \|\bar{r}\|_2.$$

The left-hand side of (D.83) can be expressed as

$$\begin{aligned}\mathbb{E}\left[(r(s,a) - \widehat{Y}_{N+1} - \eta)^2\right] &= \mathbb{E}_\eta\left[(r(s,a) - Y_{N+1})^2\right] + \mathbb{E}_\eta\left[\eta^2\right] \\ &= (r(s,a) - Y_{N+1})^2 + \mathbb{E}\left[\eta^2\right]\end{aligned} \tag{D.84}$$

Also, the term $\sum_{t=1}^{N} \mathbb{E}_\eta\left[q_t \left(r(s,a) - \widehat{Y}_t\right)^2\right]$ of the right-hand side of equation (D.83) can be written as

$$\begin{aligned}\sum_{t=1}^{N} \mathbb{E}_\eta[q_t \left(r(s,a) - \widehat{Y}_t\right)^2] &= \sum_{t=1}^{N} \mathbb{E}_\eta[q_t \left(\left(r(s,a) - Y_t\right)^2 + \eta^2\right)] \\ &= \sum_{t=1}^{N} \mathbb{E}_\eta\left[q_t \left(\left(r(s,a) - Y_t\right)^2\right)\right] + \sum_{t=1}^{N} \mathbb{E}_\eta\left[q_t \eta^2\right] \\ &= \sum_{t=1}^{N} q_t \left(\left(r(s,a) - Y_t\right)^2\right) + \mathbb{E}_\eta\left[\eta^2\right]\end{aligned}$$

By eliminating $\mathbb{E}_\eta[\eta^2]$ from both sides, we obtain that

$$(r(s,a) - Y_{N+1})^2 \le \sum_{t=1}^{N} q_t \left(\left(r(s,a) - Y_t\right)^2\right) + \text{disc}(q) + \frac{1}{wH} \cdot \lambda \|\bar{r}\|_2 \tag{D.85}$$

Recall the definition of $r(s,a) = \widetilde{R}_{(k+1)}(s,a)$, $Y_t = R_t(s,a)$. Since t matches one of $(k,h) \in [K] \times [H]$ pairs, we can rewrite

$$\begin{aligned}\sum_{t=1}^{N} q_t \left(r(s,a) - \widehat{Y}_t\right)^2 &= \sum_{k'=1}^{k-1} \sum_{h=0}^{H-1} q_{(k',h)} \left(r(s,a) - Y_{(k,h)}\right)^2 \\ &= \sum_{k'=1}^{k-1} \sum_{h=0}^{H-1} q_{(k',h)} \left(\widetilde{R}_{(k+1)}(s,a) - R_h^{k'}(s,a)\right)^2 \\ &= \sum_{k'=1}^{k-1} \sum_{h=0}^{H-1} q_{(k',h)} \left(\widetilde{R}_{(k+1)}(s,a) - R^{k'}(s,a)\right)^2\end{aligned}$$

where if (s,a) is not visited at step h of episode k, then the corresponding $q_{(k',h)}$ is zero. As a result,

$$\begin{aligned}\Delta_{(k)}^{r}(s,a) \le &\sqrt{\min_{q,\bar{r}} \left(\sum_{k'=1}^{k-1} \sum_{h=0}^{H-1} q_{(k',h)} \left(\widetilde{R}_{(k+1)}(s,a) - R^{k'}(s,a)\right)^2 + \text{disc}(q) + \frac{1}{wH} \cdot \lambda \|\bar{r}\|_2\right)} \\ \le &\sqrt{\min_{q,\bar{r}} \left(\left(\max_{1 \le k' \le k} \left(\widetilde{R}_{(k+1)}(s,a) - R^{k'}(s,a)\right)\right)^2 \left(\sum_{k'=1}^{k-1} \sum_{h=0}^{H-1} q_{(k',h)}\right) + \text{disc}(q) + \frac{1}{wH} \cdot \lambda \|\bar{r}\|_2\right)}\end{aligned}$$

A similar analysis for $\Delta^p_{(k)}$ leads to the following inequality for all $(s', s, a) \in \mathcal{S} \times \mathcal{S} \times \mathcal{A}$:

$$\begin{aligned}
&\left|P_{(k+1)}(s' \mid s,a) - \widehat{P}_{(k+1)}(s' \mid s,a)\right| \\
&\leq \sqrt{\min_{q,\bar{p}}\left(\sum_{k'=1}^{k-1}\sum_{h=0}^{H-1} q_{(k',h)}\left(\widehat{P}_{(k+1)}(s'|s,a) - P^{k'}(s'|s,a)\right)^2 + \mathrm{disc}(q) + \frac{1}{wH}\cdot\lambda\|\bar{p}\|_2\right)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Delta^p_{(k)}(s,a) &\leq \sum_{s'\in\mathcal{S}} \sqrt{\min_{q,\bar{p}}\left(\sum_{k'=1}^{k-1}\sum_{h=0}^{H-1} q_{(k',h)}\left(\widehat{P}_{(k+1)}(s'|s,a) - P^{k'}(s'|s,a)\right)^2 + \mathrm{disc}(q) + \frac{1}{wH}\cdot\lambda\|\bar{p}\|_2\right)} \\
&\leq |\mathcal{S}|\sqrt{\min_{q,\bar{p}}\left(\sum_{k'=1}^{k-1}\sum_{h=0}^{H-1} q_{(k',h)}\left\|\widehat{P}_{(k+1)}(\cdot|s,a) - P^{k'}(\cdot|s,a)\right\|_\infty^2 + \mathrm{disc}(q) + \frac{1}{wH}\cdot\lambda\|\bar{p}\|_2\right)}
\end{aligned}$$

Recall the Corollary 5, Corollary6 and $\mathfrak{R}_I$ definition. Aftering fixing (s,a), the term $\mathfrak{R}_I(s,a)$ can be expressed as

$$\begin{aligned}
\mathfrak{R}_I =& \frac{1}{1-\gamma}\sum_{k=1}^{K-1}\bar{\iota}_\infty^{k+1} + \sum_{k=1}^{K-1}\sum_{h=0}^{H-1} -\iota_h^{(k+1)} + C_p\sqrt{K-1} \\
\leq& \frac{1}{1-\gamma}\sum_{k=1}^{K-1}\left(\Delta^r_{(k)}(s,a) + \Delta^p_{(k)}(s,a)\frac{\gamma\hat{r}_{max}}{1-\gamma} - 2\Gamma_w^{(k)}(s,a)\right) \\
&+ \sum_{k=1}^{K-1}\sum_{h=0}^{H-1}\left(\Delta^r_{(k)}(s,a) + \Delta^p_{(k)}(s,a)\frac{\gamma\hat{r}_{max}}{1-\gamma} + 2\Gamma_w^{(k)}(s,a)\right) \\
&+ C_p\sqrt{K-1} \\
\leq& \frac{1}{1-\gamma}\sum_{k=1}^{K-1}\left(\Delta^r_{(k)}(s,a) + \Delta^p_{(k)}(s,a)\frac{\gamma}{1-\gamma}\left(\tilde{r}_{\max} + \max(2\Gamma_w^{(k)}(s,a))\right) - 2\Gamma_w^{(k)}(s,a)\right) \\
&+ H\sum_{k=1}^{K-1}\left(\Delta^r_{(k)}(s,a) + \Delta^p_{(k)}(s,a)\frac{\gamma}{1-\gamma}\left(\tilde{r}_{\max} + \max(2\Gamma_w^{(k)}(s,a))\right) + 2\Gamma_w^{(k)}(s,a)\right) \\
&+ C_p\sqrt{K-1} \\
\leq& \sum_{k=1}^{K-1}\Bigg(\underbrace{\left(\frac{1}{1-\gamma} + H\right)\Delta^r_{(k)}(s,a) + \frac{\gamma\tilde{r}_{\max}}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\Delta^p_{(k)}(s,a) +}_{①} \\
&+ \frac{\gamma}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\max(2\Gamma_w^{(k)}(s,a))\Delta^p_{(k)}(s,a)\Bigg) \\
&+ \sum_{k=1}^{K-1} 2\left(-\frac{1}{1-\gamma} + H\right)\Gamma_w^{(k)}(s,a) \\
&+ C_p\sqrt{K-1}
\end{aligned}$$

We set the term ① to be $2(\frac{1}{1-\gamma} + H)\Gamma_w^{(k)}(s,a)$, which requires redefining the exploration bonus term as

$$\Gamma_w^{(k)}(s,a) = \frac{1}{2}\Delta^r_{(k)}(s,a) + \frac{\gamma\tilde{r}_{\max}}{2(1-\gamma)}\Delta^p_{(k)}(s,a).$$

Also, note that $\Delta^p_{(k)}(s,a) = \sum_{s'\in\mathcal{S}} \left|\left(\widehat{P}_{(k+1)} - P_{(k+1)}\right)(s'|s,a)\right| \le |\mathcal{S}|$. Therefore,

$$\begin{aligned}
\mathfrak{R}_I &\le \sum_{k=1}^{K-1}\left(4H\Gamma_w^{(k)}(s,a) + \frac{2\gamma}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\max(\Gamma_w^{(k)}(s,a))\,|\mathcal{S}|\right) \\
&\le \sum_{k=1}^{K-1}\left(4H + \frac{2\gamma}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\right)\max(\Gamma_w^{(k)}(s,a)) \\
&= \left(4H + \frac{2\gamma\,|\mathcal{S}|}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\right)\sum_{k=1}^{K-1}\max(\Gamma_w^{(k)}(s,a)) \\
&\le \left(4H + \frac{2\gamma\,|\mathcal{S}|}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\right)\sum_{k=1}^{K-1}\left(\frac{1}{2}\max(\Delta^r_{(k)}(s,a)) + \frac{\gamma\tilde{r}_{\max}}{2(1-\gamma)}\max(\Delta^p_{(k)}(s,a))\right) \\
&= \left(4H + \frac{2\gamma\,|\mathcal{S}|}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\right)\sum_{k=1}^{K-1}\left(\frac{1}{2}\Delta^r_{(k)}(s,a) + \frac{\gamma\tilde{r}_{\max}}{2(1-\gamma)}\Delta^p_{(k)}(s,a)\right) \\
&= \left(4H + \frac{2\gamma\,|\mathcal{S}|}{1-\gamma}\left(\frac{1}{1-\gamma} + H\right)\right)\left(\frac{1}{2}\sum_{k=1}^{K-1}\Delta^r_{(k)}(s,a) + \frac{\gamma\tilde{r}_{\max}}{2(1-\gamma)}\sum_{k=1}^{K-1}\Delta^p_{(k)}(s,a)\right)
\end{aligned}$$

Note that above upper bound on $\mathfrak{R}_I$ holds under the following conditions for $\Delta^r_{(k)}(s,a)$ and $\Delta^p_{(k)}(s,a)$:

$$\Delta^r_{(k)}(s,a) \le \sqrt{\min_{q,\bar{r}}\left(\sum_{k'=1}^{k-1}\sum_{h=0}^{H-1} q_{(k',h)}\left(\widetilde{R}_{(k+1)}(s,a) - R^{k'}(s,a)\right)^2 + \mathrm{disc}(q) + \frac{1}{wH}\cdot\lambda\|\bar{r}\|_2\right)},$$

$$\Delta^p_{(k)}(s,a) \le \sum_{s'\in\mathcal{S}}\sqrt{\min_{q,\bar{p}}\left(\sum_{k'=1}^{k-1}\sum_{h=0}^{H-1} q_{(k',h)}\left(\widehat{P}_{(k+1)}(s'|s,a) - P^{k'}(s'|s,a)\right)^2 + \mathrm{disc}(q) + \frac{1}{wH}\cdot\lambda\|\bar{p}\|_2\right)}.$$

□

Proof of Remark 1. The proof starts with (D.85). Define

$$\begin{aligned}
q_t^{sw} &= \begin{cases}\frac{1}{wH} & \text{if } t\in(k-w,k] \\ 0 & \text{otherwise}\end{cases}, \\
r^{sw} &= \arg\min_{\bar{r}}\left(\lambda\|\bar{r}\|^2 + \sum_{t=1}^{N}\left(r(s,a) - \widehat{Y}_t\right)^2\right).
\end{aligned} \tag{D.86}$$

where r_{sw} is the same reward estimation as in (D.7). Then the minimum of (D.83) yields that

$$\begin{aligned}
&\min_{\bar{r},q}\left(\sum_{t=1}^{N} q_t\left(r(s,a) - \widehat{Y}_t\right)^2 + \mathrm{disc}(q) + \frac{1}{wH}\cdot\lambda\|\bar{r}\|_2\right) && \text{(D.87)} \\
&\le \min_{\bar{r}}\left(\sum_{t=1}^{N} q_t^{sw}\left(r(s,a) - \widehat{Y}_t\right)^2 + \mathrm{disc}(q^{sw}) + \frac{1}{Hw}\cdot\lambda\|\bar{r}\|_2\right) \\
&\le \frac{1}{Hw}\underbrace{\min_{\bar{r}}\left(\sum_{t=1}^{N}(Hw)\cdot q_t^{sw}\left(r(s,a) - \widehat{Y}_t\right)^2 + \lambda\|\bar{r}\|_2\right)}_{\text{①}} + \mathrm{disc}(q_{sw}). && \text{(D.88)}
\end{aligned}$$

The term ① is the optimization problem in (D.86) whose minimizer is r^{sw}. An inspection of (D.87) and (D.88) concludes that the optimal solution $(q^*,\bar{r}^*)$, namely the minimizer of (D.87) provides a smaller value than (q^{sw}, r^{sw}). Since the right-hand side (D.85) is same as (D.87), $(q^*,\bar{r}^*)$ provides a tighter upper bound on the left-hand side term of equation (D.83) than q^{sw}, r^{sw}. Therefore, (D.84) implies that the optimal solution $(q^*,\bar{r}^*)$ gives a tighter upper bound on $\Delta^r_{(k)}$ than using $(q^{sw},\bar{r}^{sw})$. One can repeat the above argument for the upper bound on $\Delta^p_{(k)}$. Then, by Corollary 5 and 6, the tighter upper bounds on $\Delta^r_{(k)}(s,a)$ and $\Delta^p_{(k)}(s,a)$ provide smaller upper bounds on $-\iota_H^{(k)}, \bar{\iota}_\infty^K$ and lead to a tighter upper bound on $\mathfrak{R}_I$. □

E Experimental design and results

E.1 Environment setting details

Reward function design.

All three environments share the same reward function structure and have an identical goal. The reward function R consists of three parts $R = R_h + R_f - R_c$, where R_h is the healthy reward, $R_f = k_f(x_{t+1} - x_t)/\Delta t, k_f > 0$ is the forward reward, and R_c is the control cost. The agents have a goal to run faster in the $+x$ direction, and therefore the faster they run, the higher the forward reward R_f is. We modify the environment to make the agent's desired directions change as the episode goes by. To be specific, we design the forward reward R_f to change as episodes progress in the form of $R_f^k = o_k \cdot k_f(x_{t+1} - x_t)/\Delta t$ where $o_k = a\sin(wbk)$ and k is a episode where $a, b > 0$ are constants. A positive o_k causes the agent to desire a forward $+x$ direction as an optimal policy, and a negative o_k causes it to desire a backward $-x$ direction. We generate different speeds of non-stationarity by changing the frequency variable $w \in \{1, 2, 3, 4, 5\}$.

Non-stationary variable o_k generator.

1. Sine function: The non-stationary parameter o_k is designed as $o_k = \sin(2\pi wk/37)$, where w is the integer speed of the environment change and k is the episode number. We change w in the set $[1, 2, 3, 4, 5]$. We divide $2\pi wk$ by 37, a prime number, to ensure that the environment has various non-stationary modes and to avoid certain non-stationary parameters appearing frequently.
2. Real data: we bring the stock price data to model a non-stationary real dataset.

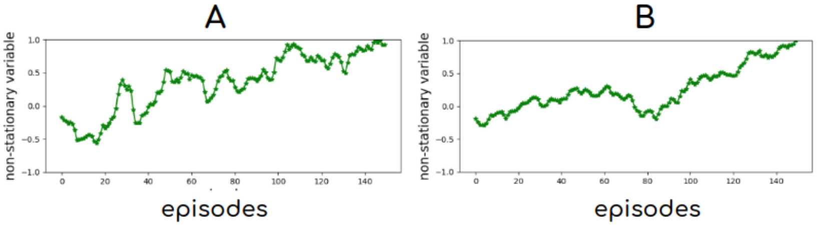


Figure 4: Nonstationary parameter from real data A,B

Non-stationary parameter o_k generator (ablation study). $B(G)$ satisfies the property of the time-elapsing variation budget that $B(G)$ increases as G increases. For the ablation study, we generate $o_k = \sin(2\pi \cdot G \cdot k/37)$, where $G \in \{38, 76, 114, 152, 190\}$. We estimated $B(G)$ as $\sum_{k=1}^{150} |o_{k+1} - o_k|$:

	$G = 38$	$G = 76$	$G = 114$	$G = 152$	$G = 190$
$B(G)$	15.98	31.85	47.49	62.79	77.64

E.2 Hyperparameters and implementation details

Training Details.

For the ARIMA model that serves as a forecatser f, we use the auto_arima function of pmdarima python package to find the optimal p, q, d. To compare the results between `ProST-G` and MBPO, we train the MBPO and `ProST-G` with the initial learning rate $lr = 0.0003$ with the decaying parameter 0.999. For `ProST-G`, We add the uniform noise $\eta \sim \text{Unif}([-b, b])$ to the non-stationary parameter o^k to generate the noisy non-stationary parameter $\hat{o}_k = o_k + \eta$ with different noise bounds $b \in \{0.01, 0.03, 0.05\}$. We denote $\text{Unif}([-b, b])$ as continuous uniform distributions over the interval $[-b, b]$.

To compare the results between `ProST-G` and ProOLS, ONPG, FTML, we train these three baselines with eight different initial learning rates $lr \in \{0.001, 0.003, 0.005, 0.007, 0.01, 0.03, 0.05, 0.07\}$.

Hyper parameters.

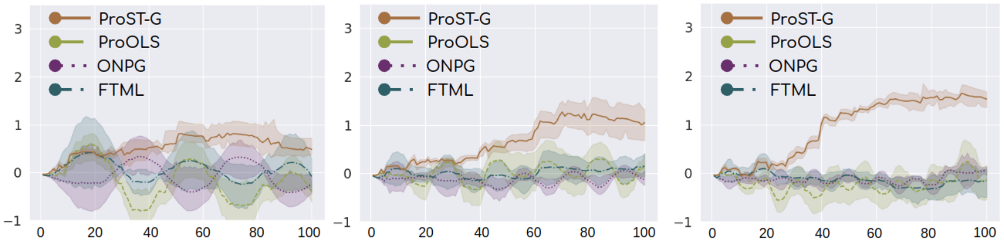
Letter	hyper parameters	Swimmer-v2	Half cheetah-v2	Hopper-v2
K	episodes	100	150	150
H	environment steps per episodes	100		
G	policy updates per epochs	50		
$\widehat{H}$	model rollout length	1 → 15 over episodes 20 →150		
N	iteration of policy update and policy evaluation	1		
M	model rollout batch size (D_{syn})	1e5		
τ	entropy regularization parameter	0.2		
γ	reward discounting factor	0.99		

Note that $\widehat{H}$ increases linearly within a certain range as episode goes by. We denote $h_{min} \rightarrow h_{max}$ over episodes $k_{min} \rightarrow k_{max}$ as $\widehat{H}(k) = \min(\max(h_{min} + (k - k_{min})/(k_{max} - k_{min}) \cdot (h_{max} - h_{min}), h_{min}), h_{max})$.

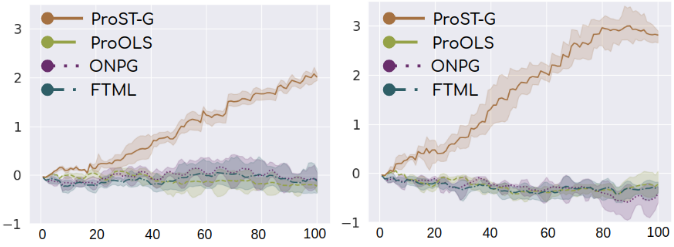
E.3 Full results

E.3.1 Non-stationarity: sine wave

(1) Swimmer-v2

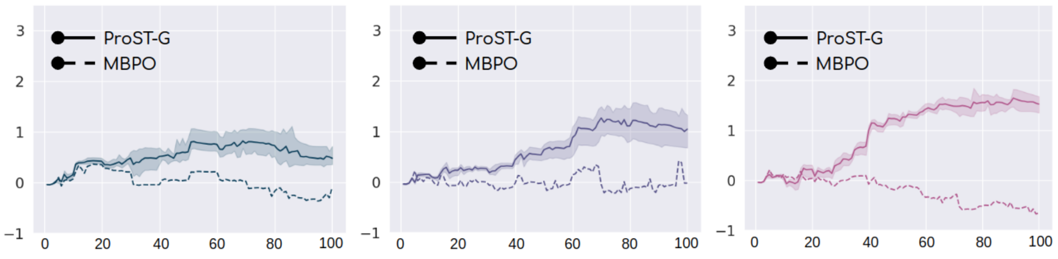


(a) Speed = 1 (b) Speed = 2 (c) Speed = 3

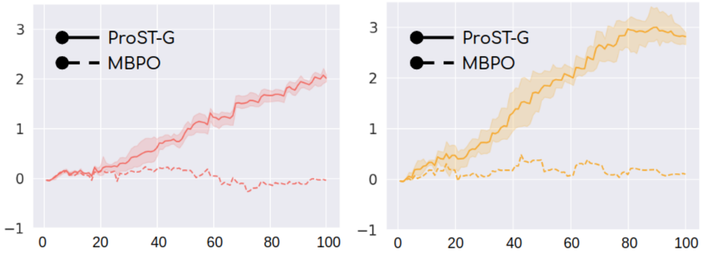


(d) Speed = 4 (e) Speed = 5

Figure 5: (a) ~ (e) the average rewards of `ProST-G`, and the three baselines: ProOLS, ONPG, FTML for 5 different speeds (x-axis indicates the episode). The shaded area of `ProST-G` is 95% confidence area among 3 different noise bounds, and the shaded areas of three baselines are the 95 % confidence area among 8 different learning rates.



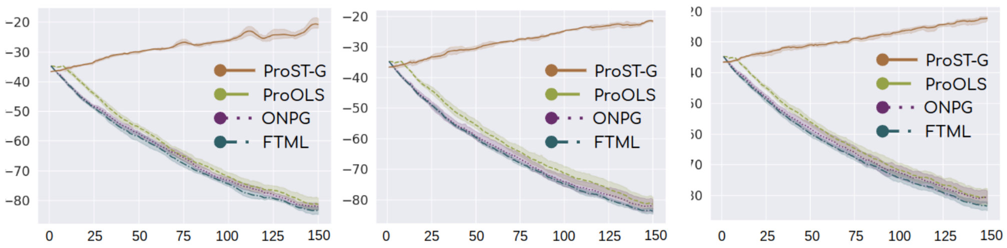
(a) Speed = 1 (b) Speed = 2 (c) Speed = 3



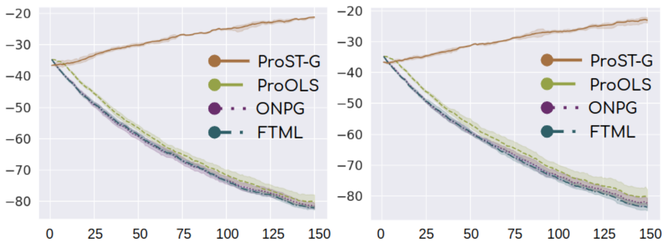
(d) Speed = 4 (e) Speed = 5

Figure 6: (a) ~ (e) the average rewards of `ProST-G` and MBPO. The shaded area of `ProST-G` is 95% confidence area among 3 different noise bounds.

(2) Halfcheetah-v2

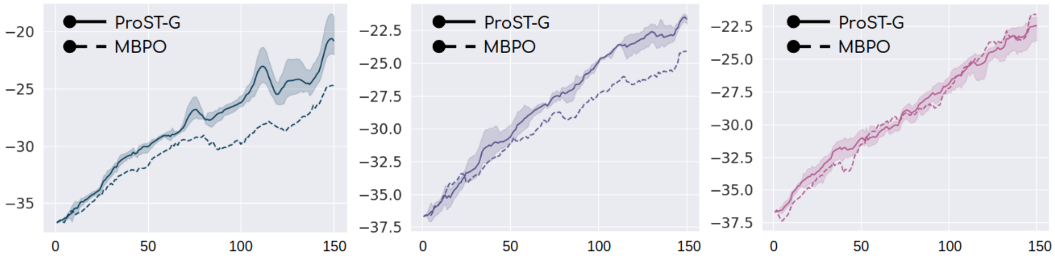


(a) Speed = 1 (b) Speed = 2 (c) Speed = 3

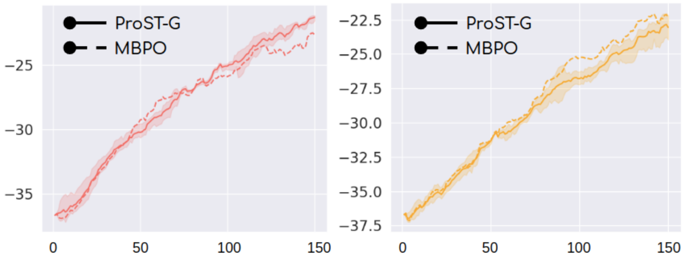


(d) Speed = 4 (e) Speed = 5

Figure 7: (a) ~ (e) the average rewards of `ProST-G`, and the three baselines: ProOLS, ONPG, FTML for 5 different speeds (x-axis indicates the episode). The shaded area of `ProST-G` is 95% confidence area among 3 different noise bounds, and the shaded ares of three baselines are the 95% confidence areas among 8 different learning rates.



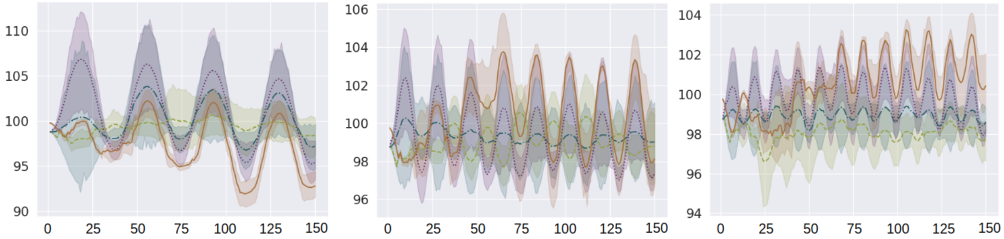
(a) Speed = 1 (b) Speed = 2 (c) Speed = 3



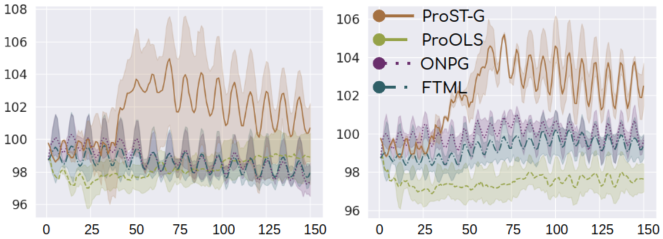
(d) Speed = 4 (e) Speed = 5

Figure 8: (a) ~ (e) the average rewards of `ProST-G` and MBPO (x-axis indicates the episode). The shaded area of `ProST-G` is 95% confidence area among 3 different noise bounds.

(3) Hopper-v2

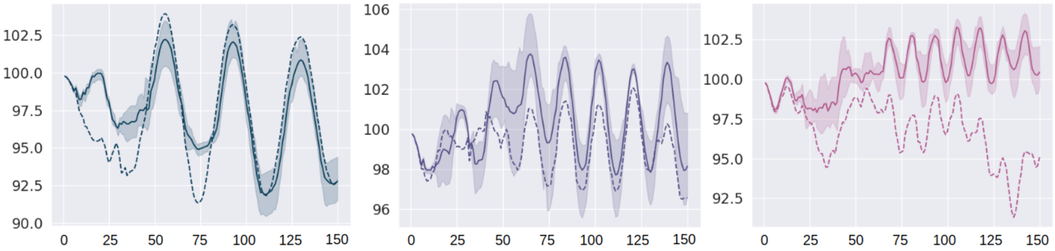


(a) Speed = 1 (b) Speed = 2 (c) Speed = 3

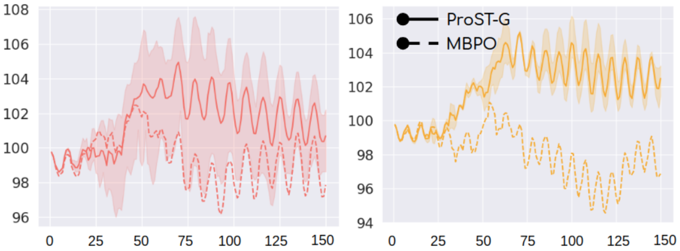


(d) Speed = 4 (e) Speed = 5

Figure 9: (a) ~ (e) the average rewards of `ProST-G`, and the three baselines : ProOLS, ONPG, FTML for 5 different speeds (x-axis indicates the episode). The shaded area of `ProST-G` is 95% confidence area among 3 different noise bounds, and the shaded areas of three baselines are the 95% confidence areas among 8 different learning rates.



(a) Speed = 1 (b) Speed = 2 (c) Speed = 3



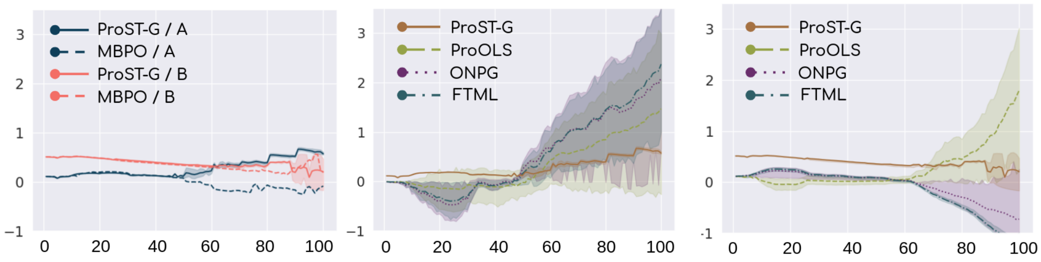
(d) Speed = 4 (e) Speed = 5

Figure 10: (a) ~ (e) the average rewards of `ProST-G` and MBPO (x-axis indicates the episode). The shaded area of `ProST-G` is 95% confidence area among 3 different noise bounds.

E.3.2 Non-stationarity : real data

The shaded area of `ProST-G` is 95% confidence area among 3 different noise bounds, and the shaded ares of three baselines are the 95% confidence area among 8 different learning rates.

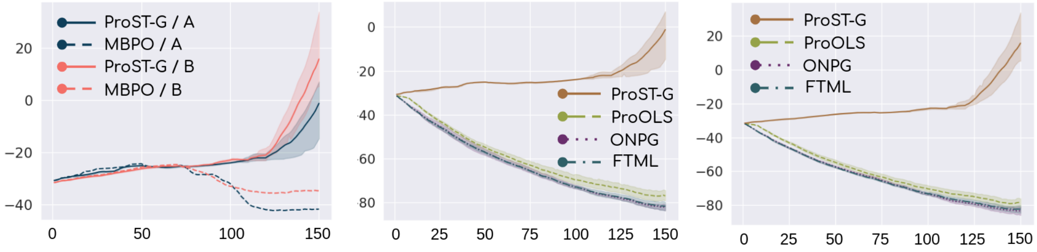
(1) Swimmer-v2



(a) (b) (c)

Figure 11: (a) average reward with `ProST-G` and MBPO on real data A,B (x-axis is episode). (b) average reward with ProST-G and three baselines on realdata A. (c) average reward with ProST-G and three baselines on realdata B.

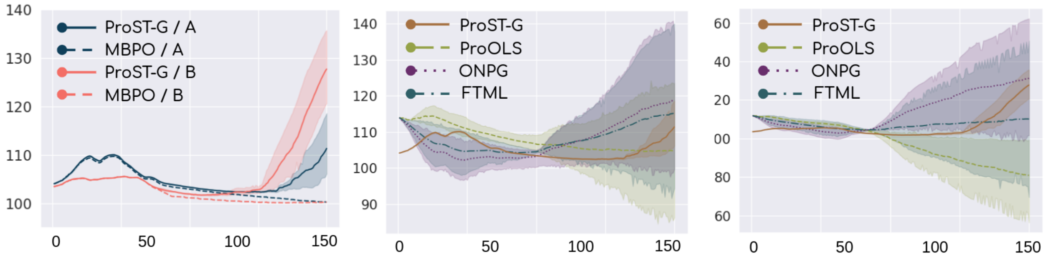
(2) Halfcheetah-v2



(a) (b) (c)

Figure 12: (a) average reward with `ProST-G` and MBPO on real data A,B (x-axis is episode). (b) average reward with ProST-G and three baselines on realdata A. (c) average reward with ProST-G and three baselines on realdata B.

(3) Hopper-v2



(a) (b) (c)

Figure 13: (a) average reward with `ProST-G` and MBPO on real data A,B (x-axis is episode). (b) average reward with ProST-G and three baselines on realdata A. (c) average reward with ProST-G and three baselines on realdata B.

F Algorithms

F.1 `ProST` framework

Algorithm 1: `ProST` framework

```
Set : $k_f = 1$
Init : policy $\pi^0$, forecaster $f_{\phi_f^0}$, model estimator $g_{\phi_g^0}$ , two dataset $\mathcal{D}_{env}, \mathcal{D}_{syn}$
for episode $k$ do
    Execute the agent with $\pi^k$ in a environment $\mathcal{M}_k$ and add a trajectory to $\mathcal{D}_{env}$.
    /* MDP forecaster $g \circ f$ */
    /* (1) Observe and forecast: */
    Observe a noisy non-stationary parameter $\hat{o}_k$
    Update $f_{\phi_f}, g_{\phi_g}$ using $\mathcal{D}_{env}$ and $\hat{o}_{k-(w-1):k}$.
    Use $f_{\phi_f^k}, g_{\phi_g^k}$ to predict the future $\widehat{\mathcal{P}}^{k+1}, \widehat{\mathcal{R}}^{k+1}$ and construct future MDP $\widehat{\mathcal{M}}_{k+1}$
    /* Baseline $A$ */
    /* (2) Optimize: */
    Roll out synthetic trajectories in $\widehat{\mathcal{M}}_{k+1}$ and add them to $\mathcal{D}_{syn}$
    Use $\mathcal{D}_{syn}$to evaluate and update $\pi^k$ to $\widehat{\pi}^{k+1}$
end for
```

F.2 `ProST-T` algorithm

Algorithm 2: `ProST-T`

```
Set : $k_f = 1$
Init : policy $\pi^k$ , forecaster $f_{\phi_f^k}$, tabular reward model $g_k^R$, tabular transition probability model
    $g_k^P$ , forecasted state-action value $\widehat{Q}^{\cdot,k+1}$, empty dataset $\mathcal{D}_{env}, \mathcal{D}_{syn}$
Explore $w$ episodes and add $(\tau^{-k}, \hat{o}_{-k})$ to $D_{env}$ where $k \in [w]$ before starts
for episodes $k = 1, .., K$ do
    Rollout a trajectory $\tau_k$ using $\pi^k$and $\mathcal{D}_{env} = \mathcal{D}_{env} \cup \{\tau_k\}$
    Observe a noisy non-stationary parameter $\hat{o}_k$
    /* MDP forecaster $g \circ f$: (1) update $f, g$ */
    Update $f_{\phi_f}$ : $\phi_f^k \leftarrow \arg\min_\phi \mathcal{L}_f(\hat{o}_{k-(w-1):k}; \phi)$
    Update $g_k^P(s', s, a, o)$
    Update $g_k^R(s, a, o)$
    /* MDP forecaster $g \circ f$: (2) predict $\widehat{\mathcal{P}}^{k+1}, \widehat{\mathcal{R}}^{k+1}$ */
    Forecast $1$ episode ahead non-stationary parameter: $\hat{o}_{k+1} = f_{\phi_f^k}(\hat{o}_{k-(w-1):k})$
    Forecast transition probability function: $\widehat{g}_{k+1}^P = g_k^P(\cdot, \hat{o}_{k+1})$
    Forecast reward function: $\widehat{g}_{k+1}^R = g_k^R(\cdot, \hat{o}_{k+1})$
    Reset $\mathcal{D}_{syn}$ to empty.
    /* Baseline $A$: NPG with entropy regularization */
    Set $\widehat{\pi}^{(0)} \leftarrow \pi^k$
    for $g = 0, .., G-1$ do
        Evaluate $Q_\tau^{\widehat{\pi}^{(g)}}$ using the rollouts from the future model $\widehat{g}_{k+1}^P, \widehat{g}_{k+1}^R$
        Update $\widehat{\pi}$ : $\widehat{\pi}^{(g+1)} \leftarrow 1/Z^{(t)} \cdot \left(\widehat{\pi}^{(g)}\right)^{1-\frac{\eta\tau}{1-\gamma}} \exp\left((\eta \widehat{Q}_\tau^{\widehat{\pi}^{(g)}})/(1-\gamma)\right)$
        where $Z^{(t)} = \sum_{a \in \mathcal{A}} \left(\widehat{\pi}^{(g)}\right)^{1-\frac{\eta\tau}{1-\gamma}} \exp\left((\eta \widehat{Q}_\tau^{\widehat{\pi}^{(g)}})/(1-\gamma)\right)$
    end for
    Set $\pi^{k+1} \leftarrow \widehat{\pi}^{(G)}$
end for
```

F.3 `ProST-G` algorithm

(1) Forecaster f. We adopt the ARIMA model to forecast $\hat{o}_{k+1}$ from the noisy observation $\widehat{o}_{k-(w-1):k}$. The ARIMA model is one of the most general classes of models for forecasting a time series, which can be made to be stationary by taking a difference among the data. For given time series data X_t, we define ARIMA(p,d,q) as given by $X_t - \alpha_1 X_{t-1} - \cdots - \alpha_p X_{t-p} = \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$, where α_i's are the parameters of the autoregressive part of the model, the θ_i's are the parameters of the moving average part, and ϵ_t's are the error terms that take d times difference between X_ts, which we assume to be independent and follow a normal distribution with a zero mean.

(2) Model predictor g. We use a bootstrap ensemble of dynamic models $\{g^1_{\phi_g}, g^2_{\phi_g}, ..., g^M_{\phi_g}\}$. Each ensemble model is a probabilistic neural network whose output is parameterized by the mean vector μ and the diagonal vector of the standard deviation Diag(Σ) of a Gaussian distribution, namely $g^i_{\phi_g}(s_{h+1}, r_h | s_h, a_h, \widehat{o}_{k+1}) = \mathcal{N}(\mu^i_{\phi_g}(s_h, a_h), \Sigma^i_{\phi_g}(s_h, a_h))$. To efficiently handle uncertainty due to the non-stationary environment, we design each neural network to be a probabilistic model to capture the aleatoric uncertainty, i.e. the noise of the output, and learn multiple models as bootstrap ensemble to handle the epistemic uncertainty, i.e. the uncertainty in the model parameters. Then we predict s_{h+1} and r_h from a model uniformly chosen from its ensemble randomly that admits different transitions along a single model rollout to be sampled from different dynamics modes.

(3) Baseline algorithm A. We adopt soft-actor critic (SAC) as our policy optimization algorithm. SAC alternates the policy evaluation step and the policy optimization step. For a given policy $\widehat{\pi}$, it estimates the forecasted $\widehat{Q}^{\widehat{\pi},k+1}$ value using the Bellman backup operator and optimizes the policy that minimizes the expected KL-divergence between π and the exponential of the difference $\widehat{Q}^{\widehat{\pi},k+1} - \widehat{V}^{\widehat{\pi},k+1} : \mathbb{E}_{s\sim D_{syn}}[D_{KL}(\widehat{\pi} \| \exp(\widehat{Q}^{\widehat{\pi},k+1} - \widehat{V}^{\widehat{\pi},k+1}))]$.

Algorithm 3: `ProST-G`

```
Set : $k_f = 1$
Init : policy $\pi^k$, forecaster $f_{\phi_f^k}$, model estimator $g_{\phi_g^k}$ , two dataset $\mathcal{D}_{env}, \mathcal{D}_{syn}$
Explore $w$ episodes and add $(\tau^{-k}, \hat{o}_{-k})$ to $D_{env}$ where $k \in [w]$ before starts
for episodes $k = 1, .., K$ do
    Execute the agent with $\pi^k$ in a environment $\mathcal{M}_k$ and add a trajectory to $\mathcal{D}_{env}$.
    /* MDP forecaster $g \circ f$:  (1) update $f, g$ */
    Observe a noisy non-stationary variable $\hat{o}_k$
    Optimize $f_{\phi_f^k}$ on $\hat{o}_{k-(w-1):k}$
    Optimize $g_{\phi_g^k}$ on $\mathcal{D}_{env}$
    /* MDP forecaster $g \circ f$:  (2) predict $f, g$ */
    Forecast $\hat{o}_{k+1} = f_{\phi_f^k}(\hat{o}_{k-(w-1):k})$
    Forecast model : $\widehat{g}_{k+1} = g_{\phi_g^k}(\cdot, \hat{o}_{k+1})$
    Reset $\mathcal{D}_{syn}$ to empty.
    /* Baseline $A$:  SAC */
    Set $\widehat{\pi}^{k+1} \leftarrow \pi^k$
    for epochs $n = 1, ..., N$ do
        for model rollouts $m = 1, .., M$ do
            Sample $\hat{s}^m_0$ uniformly from $D_{env}$.
            Perform a $\widehat{H}$-step model rollout using $\hat{a}^m_h = \widehat{\pi}^{k+1}(\hat{s}^m_h)$, $\hat{s}^m_{h+1} = \widehat{g}_{k+1}(\hat{s}^m_h, \hat{a}^m_h)$ and
             add a rollout to $\mathcal{D}_{syn}$ .
        end for
        for updates $g = 1, .., G$ do
            Evaluate and update forecasted policy $\widehat{\pi}^{k+1}$ on $\mathcal{D}_{syn}$
        end for
    end for
    Set $\pi_{k+1} \leftarrow \widehat{\pi}^{k+1}$
end for
```

G Experiment Platforms and Licenses

G.1 Platforms

All experiments are done on 12 Intel Xeon CPU E5-2690 v4 and 2 Tesla V100 GPUs.

G.2 Licenses

We have used the following libraries/ repos for our python codes:

- Pytorch (BSD 3-Clause "New" or "Revised" License).
- OpenAI Gym (MIT License).
- Numpy (BSD 3-Clause "New" or "Revised" License).
- Official codes distributed from the paper [7]: to compare the four baselines.
- Official codes distributed from the paper [24]: to build `PMT-G`.