Discrete-Convex-Analysis-Based Framework for Warm-Starting Algorithms with Predictions

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Abstract

Augmenting algorithms with learned predictions is a promising approach for going beyond worst-case bounds. Dinitz, Im, Lavastida, Moseley, and Vassilvitskii (2021) have demonstrated that warm-starts with learned dual solutions can improve the time complexity of the Hungarian method for weighted perfect bipartite matching. We extend and improve their framework in a principled manner via *discrete convex analysis* (DCA), a discrete analog of convex analysis. We show the usefulness of our DCA-based framework by applying it to weighted perfect bipartite matching, weighted matroid intersection, and discrete energy minimization for computer vision. Our DCA-based framework yields time complexity bounds that depend on the ℓ_∞-distance from a predicted solution to an optimal solution, which has two advantages relative to the previous ℓ_1-distance-dependent bounds: time complexity bounds are smaller, and learning of predictions is more sample efficient. We also discuss whether to learn primal or dual solutions from the DCA perspective.

1 Introduction

Discrete optimization algorithms are applied to many real-world instances that take place repetitively. For example, recommendation systems repeat to solve bipartite matching instances to match users with services, and we solve a series of pixel-labeling instances to process images of a movie. Since such instances arising in the same domain often have some tendencies, using predictions made from past instances to improve algorithms' performance is a natural and promising idea. A recent line of work [36, 41, 4, 34, 43, 2, 5] successfully combined online algorithms with predictions and showed that those algorithms perform provably better than known worst-case bounds if predictions are good while enjoying worst-case guarantees even if predictions are poor. See [37] for a survey.

Dinitz et al. [17] recently initiated the study of improving the time complexity of algorithms with predictions. They focused on warm-starting the well-known Hungarian method for the weighted perfect bipartite matching problem with predictions on dual solutions, and obtained the time complexity of $\mathrm{O}(\min\{m\sqrt{n}\|\hat{p} - p^*\|_1, mn\})$, where n and m are the number of vertices and edges, respectively, and $\hat{p} \in \mathbb{R}^n$ is a prediction on an optimal dual solution $p^* \in \mathbb{R}^n$. That is, while the Hungarian method takes $\mathrm{O}(mn)$ time in the worst case, it can run faster given a good prediction. Dinitz et al. [17] also presented an algorithm for converting infeasible learned dual solutions into initial feasible solutions, and proved an $\mathrm{O}(C^2 n^3 \log n)$ sample complexity bound for learning $\hat{p}$ that approximately minimizes the expected ℓ_1-error $\mathbb{E}\|\hat{p} - p^*\|_1$, assuming an optimal prediction is contained in $[-C, +C]^n$. They thus established an end-to-end framework for warm-starting the Hungarian method with predictions.

While Dinitz et al. [17] has shown that their prediction-based warm-start framework is effective for bipartite matching and b-matching, its application to other problems remains to be studied. Since their idea has the potential to yield strong *beyond-the-worst-case* time complexity bounds, the next question of theoretical interest is: *when and how can we warm-start algorithms with predictions?*

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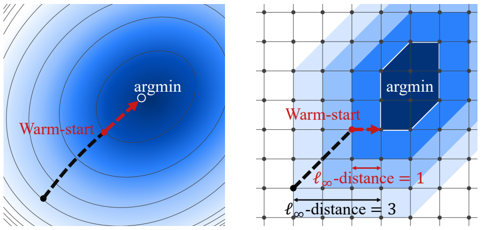
Table 1: Our results for weighted perfect bipartite matching (BM) on bipartite graphs with n vertices and at most m edges, weighted matroid intersection (MI) on pairs of rank-r matroids on an identical ground set of size n (τ is the running time of independence oracles), and discrete energy minimization on graphs with n vertices and at most m edges.

Problem	Local optimization problem	Time complexity	Prediction
Weighted perfect BM	Maximum cardinality BM	$\mathrm{O}(m\sqrt{n}\|p^* - \hat{p}\|_\infty)$	Dual
Weighted MI	Maximum cardinality MI	$\mathrm{O}(\tau n r^{1.5}\|p^* - \hat{p}\|_\infty)$	Dual
Discrete energy min.	Minimum cut	$\mathrm{O}(mn^2\|p^* - \hat{p}\|_\infty)$	Primal

Our contribution is to extend and improve the framework of [17] in a principled manner. Our idea comes from an intuition that the time complexity bound of [17] seems to be originating from some geometric property of the Hungarian method. In continuous optimization, warm-starting the gradient descent method reduces its running time, which we can see by simple geometric reasoning (see Figure 1a). We formalize this idea for discrete optimization via *discrete convex analysis* (DCA) by Murota [38], which is a discrete analog of convex analysis [42] and offers a gradient-descent-like interpretation of various discrete optimization algorithms, as in Figure 1b. Based on DCA, we show that warm-starting with predictions is effective for a large class of problems called *L/L$^\natural$-convex minimization*.



(a) Continuous

(b) Discrete

Figure 1: Images of warm-starts in (a) continuous and (b) discrete optimization, where darker colors indicate smaller objective values.

As with the framework of [17], given a prediction $\hat{p}$ on an optimal solution p^*, we convert $\hat{p}$ into an initial feasible solution p°. Then, starting from p°, we iteratively solve a local optimization problem to find a direction, along which we proceed. We will see that the number of iterations is $\mathrm{O}(\|p^* - \hat{p}\|_\infty)$ (see Figure 1b for an image). Thus, if a local optimization solver runs in T_{loc} time, the total time complexity is $\mathrm{O}(T_{\mathrm{loc}}\|p^* - \hat{p}\|_\infty)$ plus the time of converting $\hat{p}$ into p°, which is often shorter than T_{loc}. Table 1 summarizes the results obtained by applying our DCA-based framework to specific problems, where T_{loc} is replaced with the running time of standard local optimization solvers. As for bipartite matching, our bound is up to n times smaller than the $\mathrm{O}(m\sqrt{n}\|\hat{p} - p^*\|_1)$ bound of [17].[1]

We then provide an $\mathrm{O}(C^2 n)$ sample complexity bound for learning predictions that approximately minimize the expected ℓ_∞-error, $\mathbb{E}\|p^* - \hat{p}\|_\infty$, assuming an optimal prediction to be in $[-C, +C]^n$. Our bound is better than the previous $\mathrm{O}(C^2 n^3 \log n)$ bound of [17] for approximately minimizing the expected ℓ_1-error. Our method for learning predictions is based on a recent online-learning framework by Khodak et al. [31], and we also obtain an $\mathrm{O}(C\sqrt{nT})$ regret bound for the online setting.

We finally discuss whether to learn primal or dual solutions, which depends on problems as in Table 1 and has been decided in a somewhat ad-hoc manner in literature. We provide a guideline for choosing primal or dual from the DCA perspective by considering *path connectivity* of feasible regions.

1.1 Related work

Theoretically fast algorithms. We overview existing theoretically fast algorithms. We emphasize that, as with [17], our motivation is to accelerate simple algorithms, not to develop theoretically fast algorithms, which are often difficult to implement and empirically slow due to hidden large constants. For maximum cardinality bipartite matching, the standard Hopcroft–Karp algorithm [26] runs in $\mathrm{O}(m\sqrt{n})$ time, and a recent fast algorithm [47] takes $\mathrm{O}(m + n^{1.5})$ time (up to logarithmic factors). For weighted bipartite matching with non-negative integer weights bounded by W, a scaling-type algorithm [18] runs in $\mathrm{O}(m\sqrt{n}\log(W))$ time. Moreover, a recent almost-linear-time max/min-cost

[1] Unlike [17], our framework cannot yield worst-case bounds in general, which is a limitation of our work. This, however, does not matter in practice if we can run standard algorithms in parallel, as discussed in Section 6.

flow algorithm [14] implies an $\mathrm{O}(m^{1+\mathrm{o}(1)}\log^2 W)$-time algorithm for weighted bipartite matching. For maximum cardinality matroid intersection, a standard algorithm is Cunningham's $\mathrm{O}(\tau n r^{1.5})$-time algorithm [15], and a recent faster algorithm [12] runs in $\mathrm{O}(\tau n r \log r)$ time. For weighted matroid intersection, there is an $\mathrm{O}(\tau n r^2)$-time algorithm [9]. Moreover, if the maximum weight is bounded by W, there are $\mathrm{O}(\tau n^2 \log(nW))$-time [35] and $\mathrm{O}(\tau n r^{1.5} W)$-time [27] algorithms. For discrete energy minimization, there is an $\mathrm{O}(mn\log(n^2/m)\log(nW))$-time algorithm [1], where W is the number of possible vertex labels. For a particular case where smoothness of vertex labels is measured by linear deviation functions, there is an algorithm that takes almost the same time to solve a min-cut instance [25]; combined with the algorithm of [14], it runs in $\mathrm{O}(m^{1+\mathrm{o}(1)}\log^2 W)$ time.

Algorithms with predictions. Many recent studies [36, 41, 4, 34, 43, 2, 5] improved competitive ratios of online algorithms with predictions. Dinitz et al. [17] proposed to warm-start algorithms with predictions. Khodak et al. [31] developed an online-learning framework for learning predictions, and applied it to the ℓ_1-error minimization setting of [17] to obtain $\mathrm{O}(Cn\sqrt{T})$ regret and $\mathrm{O}(C^2n^2)$ sample complexity bounds. By contrast, we learn predictions to minimize the ℓ_∞-error, yielding better guarantees. *Data-driven algorithm design* [3] is closely related to algorithms with predictions. As discussed in [31], one distinction is that the former aims to tune algorithm parameters to optimize the expected performance, while the latter focuses on prediction-dependent bounds to quantify the improvement gained by using predictions. A very recent study [13] also extended and improved [17], which should be considered as independent of each other and employs a different approach from ours.

2 Preliminaries

Let $\lfloor\cdot\rceil$ denote the rounding to the nearest integer (0.5 is rounded down). We apply $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$, and $\lfloor\cdot\rceil$ to vectors in an element-wise manner. We use $V = \{1, 2, \dots, n\}$ to denote a finite ground set of size $n \in \mathbb{N}$. For any $X \subseteq V$, $\mathbf{1}_X \in \{0,1\}^V$ denotes an indicator vector whose entries corresponding to X are one and the others are zero; let $\mathbf{1} = \mathbf{1}_V$. Let $\vee$ and $\wedge$ denote the element-wise maximum and minimum operators, respectively. For any $S \subseteq \mathbb{Z}^V$, let $\mathrm{conv}(S) \subseteq \mathbb{R}^V$ denote the convex hull of S.

Given any function $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$, let $\mathrm{dom}\, g = \left\{ p \in \mathbb{Z}^V \;\middle|\; g(p) < +\infty \right\}$ be its *effective domain*, which indicates the feasible region of a minimization problem of form $\min_{p\in\mathbb{Z}^V} g(p)$. We say g is *proper* if $\mathrm{dom}\, g \neq \emptyset$. In this paper, we assume the following basic conditions to hold.

Assumption 1. *We assume that any objective function $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is proper and has at least one minimizer. Moreover, given any g, we uniquely associate $p^*(g) \in \mathrm{argmin}_{p\in\mathbb{Z}^V} g(p)$ with g by breaking ties with an arbitrary predefined rule (to deal with the case of multiple minimizers).*

2.1 Background on discrete convex analysis

We overview the basics of discrete convex analysis. We refer the reader to [38] for more information.

Considering functions on $\mathbb{Z}^V$, how to define convexity is already nontrivial. A well-known property of a continuous convex function f is midpoint convexity, i.e., $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$, and its natural discrete analog for a function $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ would be $g(p) + g(q) \geq g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right)$ for all $p, q \in \mathbb{Z}^V$. This condition is indeed equivalent to the following *$\mathrm{L}^\natural$-convexity* of g.

Definition 1. A proper function $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is *L-convex* if it has the following properties.

Submodularity: $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$ for all $p, q \in \mathbb{Z}^V$.

Linearity in the direction of $\mathbf{1}$: there exists $r \in \mathbb{R}$ such that $g(p + \mathbf{1}) = g(q) + r$ for all $p \in \mathbb{Z}^V$.

A proper function $g : \mathbb{Z}^V \to \mathbb{R} \cup \{\infty\}$ is *$\mathrm{L}^\natural$-convex* if a function $\tilde{g} : \mathbb{Z} \times \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ defined by $\tilde{g}(p_0, p) = g(p - p_0\mathbf{1})$ for all $p_0 \in \mathbb{Z}$ and $p \in \mathbb{Z}^V$ is L-convex.

While $\mathrm{L}^\natural$-convex functions form a wider class than L-convex functions, the two classes are essentially equivalent due to the one-to-one correspondence between $\mathrm{L}^\natural$-convex functions on $\mathbb{Z}^V$ and L-convex functions on $\mathbb{Z} \times \mathbb{Z}^V$. Thus, we can choose whichever is more convenient for modeling problems. Note that the sum of two $\mathrm{L}/\mathrm{L}^\natural$-convex functions, g_1 and g_2, is $\mathrm{L}/\mathrm{L}^\natural$-convex if it is proper.

Algorithm 1 Steepest descent method for L-convex (L$^\natural$-convex) function minimization

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1: $p \leftarrow p^\circ \in \mathrm{dom}\, g$                                  ▷ $p^\circ$ is an initial feasible solution.
2: while not converged :
3:     $d \leftarrow \mathrm{argmin}\{\, g(p + d') \mid d' \in \mathcal{N}_+ \,\}$            ▷ Replace $\mathcal{N}_+$ with $\mathcal{N}_\pm$ if $g$ is L$^\natural$-convex.
4:     if $g(p + d) = g(p)$ :
5:         return $p$
6:     $\lambda \leftarrow \sup\{\, \lambda' \in \mathbb{Z}_{>0} \mid g(p + \lambda' d) - g(p) = \lambda'(g(p + d) - g(p)) \,\}$ ▷ Alternatively, $\lambda \leftarrow 1$.
7:     $p \leftarrow p + \lambda d$
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We say a non-empty set $S \subseteq \mathbb{Z}^V$ is L/L$^\natural$-convex if its indicator function δ_S is L/L$^\natural$-convex, where $\delta_S(p) = 0$ if $p \in S$ and $+\infty$ otherwise. If $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is L/L$^\natural$-convex, then $\mathrm{dom}\, g \subseteq \mathbb{Z}^V$ is an L/L$^\natural$-convex set. The next proposition provides useful representations of L/L$^\natural$-convex sets.

Proposition 1. *A non-empty set $S \subseteq \mathbb{Z}^V$ is L$^\natural$-convex if and only if there exist $\alpha_i \in \mathbb{Z} \cup \{-\infty\}$, $\beta_i \in \mathbb{Z} \cup \{+\infty\}$, and $\gamma_{ij} \in \mathbb{Z} \cup \{+\infty\}$ ($i, j \in V; i \neq j$) such that*

$$S = \left\{\, p \in \mathbb{Z}^V \;\middle|\; \alpha_i \le p_i \le \beta_i,\ \ p_j - p_i \le \gamma_{ij} \ \text{ for } i, j \in V; i \neq j \,\right\},$$

and is L-convex if and only if S is written as above without the box constraints $\alpha_i \le p_i \le \beta_i$ ($i \in V$).

Therefore, the colored area in Figure 1b illustrates an example of L$^\natural$-convex sets in $\mathbb{Z}^2$. We can also represent $\mathrm{conv}(S) \subseteq \mathbb{R}^V$ as above by replacing $\mathbb{Z}^V$ with $\mathbb{R}^V$ (see [38, Section 5.5]). L/L$^\natural$-convex sets also have the following useful property, which we prove in Appendix A.

Lemma 1. *Let $S \subseteq \mathbb{Z}^V$ be an L/L$^\natural$-convex set and $p \in \mathrm{conv}(S)$. Then, it holds $\lfloor p \rceil \in S$.*

2.2 Steepest descent method for L/L$^\natural$-convex function minimization

An essential property of continuous convex functions is that the local optimality implies the global optimality, which enables the gradient descent method to reach a global optimum. L- and L$^\natural$-convexity inherits this property with respect to neighborhood $\mathcal{N}_+ = \{0, +1\}^V$ and $\mathcal{N}_\pm = \{0, -1\}^V \cup \{0, +1\}^V$, respectively, i.e., for any L-convex (L$^\natural$-convex) function $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ and $p \in \mathbb{Z}^V$, it holds

$$p \in \mathrm{argmin}\{\, g(q) \mid q \in \mathbb{Z}^V \,\} \quad \Longleftrightarrow \quad g(p) \le g(p + d) \quad \text{for every } d \in \mathcal{N}_+ \ (d \in \mathcal{N}_\pm).$$

This fact underpins the convergence of a steepest descent method (Algorithm 1) to a global minimum of any L/L$^\natural$-convex function $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$, as stated in Proposition 2 presented below.

Algorithm 1 is very simple: starting from an initial point $p^\circ \in \mathrm{dom}\, g$, it finds a steepest direction by solving a local optimization problem (Step 3), sets the step length λ, and updates the solution (Algorithm 1 is a *long-step* version [24, 46], and we can also let $\lambda = 1$ in Step 6). The algorithm terminates if Step 3 does not improve the objective value. The local optimization problem in Step 3 can be written as $\min_{X \subseteq V} f(X) = g(p + \mathbf{1}_X)$ (or $g(p \pm \mathbf{1}_X)$ if g is L$^\natural$-convex). From L/L$^\natural$-convexity of g, one can confirm that $f : 2^V \to \mathbb{R} \cup \{+\infty\}$ is a submodular function, which can be minimized in polynomial time in general, and more efficient local optimization solvers are available for many specific problems. Algorithm 1 thus provides an efficient way to minimize L/L$^\natural$-convex functions.

The number of iterations of Algorithm 1 is known to be bounded by the distance between an initial point and a global optimal solution. To describe this claim precisely, for any $p \in \mathbb{R}^V$, we define

$$\|p\|_\infty^\pm = \|p\|_\infty^+ + \|p\|_\infty^- \quad \text{where} \quad \|p\|_\infty^+ = \max_{i \in V} \max\{0, +p_i\} \quad \text{and} \quad \|p\|_\infty^- = \max_{i \in V} \max\{0, -p_i\}.$$

Note that $\|p\|_\infty \le \|p\|_\infty^\pm \le 2\|p\|_\infty$ holds, i.e., $\|p\|_\infty^\pm = \Theta(\|p\|_\infty)$. Moreover, $\|\cdot\|_\infty^\pm$ satisfies the axioms of norms, and hence we sometimes refer to it as the $\ell_\infty^\pm$-norm.

Proposition 2 ([39, Theorem 1.2] and [24, Theorem 6.2])**.** *Algorithm 1 returns a global optimal solution to $\min_{p \in \mathbb{Z}^V} g(p)$ in at most $\|p^*(g) - p^\circ\|_\infty^\pm + 1 = \mathrm{O}(\|p^*(g) - p^\circ\|_\infty)$ iterations.*

Note that Proposition 2 holds regardless of L/L$^\natural$ and the choice of λ in Step 6. The original results shown in [39, 24] provide stronger bounds on the number of iterations by using a global optimal solution p^* with the smallest $\|p^*(g) - p^\circ\|_\infty^\pm$ value, and we obtain Proposition 2 by replacing p^* with an optimal solution $p^*(g)$ that is uniquely associated with g by Assumption 1; we do this to get simple convex loss functions with which we can learn predictions (see Section 4).

3 DCA-based framework and its applications

We present our general DCA-based framework for L/L$^\natural$-convex minimization. Our basic idea is to warm-start Algorithm 1 with predictions and bound the number of iterations by using Proposition 2. Algorithm 1 has two parts that take considerable time: obtaining an initial feasible solution in Step 1 and solving a local optimization problem in Step 3. For now, we suppose that oracles for the two steps are available and present our main theorem, which, together with learning guarantees in Section 4, formalizes our DCA-based framework.

Theorem 1. *Let $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be an L/L$^\natural$-convex function. If, for any $q \in \mathbb{R}^V$, we can compute an $\ell_\infty^\pm$-projection $\mathcal{P}(q) \in \operatorname{argmin}\{\, \|p - q\|_\infty^\pm \mid p \in \operatorname{conv}(\operatorname{dom} g)\,\}$ in T_{prj} time and we can solve a local optimization problem in Step 3 in T_{loc} time, the following guarantees hold.*

1. *Given a possibly infeasible prediction $\hat{p} \in \mathbb{R}^V$, we can obtain an initial feasible solution $p^\circ = \lfloor \mathcal{P}(\hat{p}) \rceil \in \operatorname{dom} g$ in $\mathrm{O}(T_{\mathrm{prj}} + |V|)$ time.*
2. *Given the initial feasible solution $p^\circ = \lfloor \mathcal{P}(\hat{p}) \rceil$, Algorithm 1 computes an optimal solution to $\min_{p \in \mathbb{Z}^V} g(p)$ in $\mathrm{O}(T_{\mathrm{loc}} \|p^*(g) - \hat{p}\|_\infty)$ time.*

Proof. Regarding the first claim, the time complexity follows from the T_{prj}-time projection and the $\mathrm{O}(|V|)$-time rounding, and $\lfloor \mathcal{P}(\hat{p}) \rceil \in \operatorname{dom} g$ follows from $\mathcal{P}(\hat{p}) \in \operatorname{conv}(\operatorname{dom} g)$ and Lemma 1. We below prove the second claim. Proposition 2 implies that we can compute an optimal solution in $T_{\mathrm{loc}}(\|p^*(g) - p^\circ\|_\infty^\pm + 1)$ time. We further bound $\|p^*(g) - p^\circ\|_\infty^\pm$ as follows. Since the rounding changes each entry up to $\pm 1/2$, $\|p^*(g) - p^\circ\|_\infty^\pm \le \|p^*(g) - \mathcal{P}(\hat{p})\|_\infty^\pm + 1$ holds. Furthermore, the triangle inequality of $\|\cdot\|_\infty^\pm$, $\mathcal{P}(\hat{p}) \in \operatorname{argmin}_{p \in \operatorname{conv}(\operatorname{dom} g)} \|p - \hat{p}\|_\infty^\pm$, and $p^*(g) \in \operatorname{conv}(\operatorname{dom} g)$ imply $\|p^*(g) - \mathcal{P}(\hat{p})\|_\infty^\pm \le \|p^*(g) - \hat{p}\|_\infty^\pm + \|\mathcal{P}(\hat{p}) - \hat{p}\|_\infty^\pm \le 2\|p^*(g) - \hat{p}\|_\infty^\pm \le 4\|p^*(g) - \hat{p}\|_\infty$. Thus, the time complexity is $\mathrm{O}(T_{\mathrm{loc}}\|p^*(g) - \hat{p}\|_\infty)$, as desired. □

That is, given a prediction $\hat{p} \in \mathbb{R}^V$, we can solve $\min_{p \in \mathbb{Z}^V} g(p)$ in $\mathrm{O}(T_{\mathrm{prj}} + |V| + T_{\mathrm{loc}}\|p^*(g) - \hat{p}\|_\infty)$ time. We below discuss how large T_{prj} and T_{loc} can be for bipartite matching, matroid intersection, and discrete energy minimization; we also mention general L$^\natural$-convex function minimization. In all the cases, it turns out $T_{\mathrm{prj}} + |V| \le T_{\mathrm{loc}}$, implying the total time complexity of $\mathrm{O}(T_{\mathrm{loc}}\|p^*(g) - \hat{p}\|_\infty)$. Thus, replacing T_{loc} with those of standard local optimization solvers, we obtain the results in Table 1.

3.1 Weighted perfect bipartite matching

We consider the weighted perfect bipartite matching problem studied in [17]. Let $G = (V, E)$ be a bipartite graph with bipartition $V = L \cup R$, $|L| = |R| = n/2$ (where n is even), and $|E| \le m$. Let $w \in \mathbb{Z}^E$ be edge weights. We assume that V is fixed and G has at least one perfect matching (which we can check by solving the maximum cardinality bipartite matching problem once). Under this condition, we allow both w and E to change over instances generated randomly or adversarially, as described in Section 4; this slightly extends the setting of [17], which fixes E.[2] The dual problem of the maximum weight perfect bipartite matching problem on G is given as follows:[3]

$$\underset{s \in \mathbb{Z}^L, t \in \mathbb{Z}^R}{\text{minimize}} \quad \sum_{i \in L} s_i - \sum_{j \in R} t_j \quad \text{subject to} \quad s_i - t_j \ge w_{ij} \quad \forall (i,j) \in E. \tag{1}$$

We below use $p = (s, t) \in \mathbb{Z}^L \times \mathbb{Z}^R = \mathbb{Z}^V$ to denote the dual variables. The objective function is linear (L-convex), and the inequalities defining the feasible region can be written as in Proposition 1, whose indicator function is L-convex. Thus, if we let $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be the sum of these two functions, we can write (1) as an L-convex function minimization problem of form $\min_{p \in \mathbb{Z}^V} g(p)$.

Projection. Given a prediction $\hat{p} = (\hat{s}, \hat{t}) \in \mathbb{R}^L \times \mathbb{R}^R$, we compute $\varepsilon = \max_{(i,j) \in E}(w_{ij} - \hat{s}_i + \hat{t}_j)$. If $\varepsilon \le 0$, $\hat{p}$ is already in $\operatorname{conv}(\operatorname{dom} g)$; otherwise, $\left(\hat{s} + \frac{\varepsilon}{2}\mathbf{1}, \hat{t} - \frac{\varepsilon}{2}\mathbf{1}\right)$ gives an $\ell_\infty^\pm$-projection $\mathcal{P}(\hat{p})$, as in the following Lemma 2. The total computation time of this projection step is $T_{\mathrm{prj}} = \mathrm{O}(m)$.

[2] While the minimum-weight setting is studied in [17], we consider the maximum-weight setting for convenience. Note that we can deal with the minimum-weight setting since w is allowed to have negative entries.

[3] Since edge weights w are integer and the constraint is given by a totally unimodular matrix, there is at least one integral optimal solution. Hence, we can restrict the domain to $\mathbb{Z}^L \times \mathbb{Z}^R$.

Lemma 2. *For any $\hat{p} = (\hat{s}, \hat{t}) \in \mathbb{R}^L \times \mathbb{R}^R$ such that $\varepsilon > 0$, $\left(\hat{s} + \frac{\varepsilon}{2}\mathbf{1}, \hat{t} - \frac{\varepsilon}{2}\mathbf{1}\right)$ gives an $\ell_\infty^\pm$-projection of $\hat{p}$ onto $\mathrm{conv}(\mathrm{dom}\, g)$, the convex hull of the feasible region of* (1).

Proof. For any $\Delta s \in \mathbb{R}^L$ and $\Delta t \in \mathbb{R}^R$, we have $(\hat{s} + \Delta s, \hat{t} + \Delta t) \in \mathrm{conv}(\mathrm{dom}\, g)$ if and only if $\Delta s_i - \Delta t_j \geq w_{ij} - \hat{s}_i + \hat{t}_j$ for $(i, j) \in E$; in particular, $\max_{(i,j)\in E}(\Delta s_i - \Delta t_j) \geq \varepsilon$ must hold. Thus, the $\ell_\infty^\pm$-distance from $(\hat{s}, \hat{t})$ to any point in $\mathrm{conv}(\mathrm{dom}\, g)$ is lower bounded by ε as follows:

$$\|(\Delta s, \Delta t)\|_\infty^\pm \geq \max_{i\in L} \max\{0, +\Delta s_i\} + \max_{j\in R} \max\{0, -\Delta t_j\} \geq \max_{(i,j)\in E}(\Delta s_i - \Delta t_j) \geq \varepsilon.$$

This lower bound is attained by setting $\Delta s = \frac{\varepsilon}{2}\mathbf{1}$ and $\Delta t = -\frac{\varepsilon}{2}\mathbf{1}$, as in the lemma statement. □

Local optimization. As in [17], local optimization reduces to the minimum vertex cover problem (the dual of the maximum cardinality matching problem). We below briefly describe this reduction. Given a current solution $p = (s, t) \in \mathrm{dom}\, g$, Step 3 asks to find an optimal direction $d = (\mathbf{1}_X, \mathbf{1}_Y)$ over $X \subseteq L$ and $Y \subseteq R$. Letting $S = X$ and $T = R \setminus Y$, we can formulate this problem as

$$\underset{S\subseteq L, T\subseteq R}{\text{minimize}} \ |S| + |T| + \text{const.} \quad \text{subject to } \mathbb{1}_{i\in S} + \mathbb{1}_{j\in T} \geq w_{ij} - s_i + t_j + 1 \quad \forall (i,j) \in E, \tag{2}$$

where $\mathbb{1}_{\{\cdot\}} = 1$ (0) if the argument is true (false). The constraint for each $(i, j) \in E$ matters only when (i, j) is *tight*, i.e., $s_i - t_j = w_{ij}$. Thus, letting E^* be the set of tight edges, we can write (2) as

$$\underset{S\subseteq L, T\subseteq R}{\text{minimize}} \ |S| + |T| + \text{const.} \qquad \text{subject to } \ i \in S \text{ or } j \in T \quad \forall (i,j) \in E^*,$$

which is the minimum vertex cover problem on (V, E^*), the dual of maximum cardinality matching. If we solve it with the Hopcroft–Karp algorithm [26], we have $T_{\mathrm{loc}} = \mathrm{O}(m\sqrt{n})$.

By the Kőnig–Egerváry theorem [44, Theorem 16.2], there exists a vertex cover (S, T) with $|S| + |T| < n/2$ if and only if (V, E^*) has no perfect matching. Once a minimum vertex cover (S, T) with $|S| + |T| < n/2$ is found, we update (s, t) to $(s + \lambda\mathbf{1}_S, t + \lambda\mathbf{1}_{R\setminus T})$, where the step length λ in Step 6 is given by $\min_{i\in L\setminus S, j\in R\setminus T}\{s_i - t_j - w_{ij}\}$. If we find a vertex cover (S, T) such that $|S| + |T| = n/2$ on (V, E^*), any maximum-cardinality matching on (V, E^*) is a maximum-weight matching on G by complementary slackness [44, Section 18.5b], thus solving the original problem.

3.2 Weighted matroid intersection

We next consider the weighted matroid intersection problem, a generalization of various problems such as bipartite matchings, packing spanning trees, and arborescences in a directed graph. Due to this broad coverage, the discussion here would serve as a role model for applying our DCA-based framework to various problems, even though the general result by itself may not immediately provide practical algorithms for specific problems.

A *matroid* $\mathbf{M}$ consists of a finite set V and a non-empty set family $\mathcal{B} \subseteq 2^V$ satisfying the following exchange axiom: for any $B_1, B_2 \in \mathcal{B}$ and $i \in B_1 \setminus B_2$, there exists $j \in B_2 \setminus B_1$ such that $B_1 \setminus \{i\} \cup \{j\} \in \mathcal{B}$ and $B_2 \setminus \{j\} \cup \{i\} \in \mathcal{B}$. Elements in $\mathcal{B}$ are called *bases*, and an *independent set* is a subset of a base. The *rank function* $\rho : 2^V \to \mathbb{Z}$ of $\mathbf{M}$ is defined as $\rho(X) = \max_{B\in\mathcal{B}} |X \cap B|$. The *rank* of $\mathbf{M}$ is defined by $\rho(V)$, which coincides with the cardinality of any base $B \in \mathcal{B}$.

Let $\mathbf{M}_1 = (V, \mathcal{B}_1)$ and $\mathbf{M}_2 = (V, \mathcal{B}_2)$ be rank-r matroids on an identical fixed ground set V of size n, equipped with a weight vector $w \in \mathbb{Z}^V$. We assume that the matroids are given as independence oracles, which test whether an input set is independent or not in τ time. We also assume $\mathcal{B}_1 \cap \mathcal{B}_2 \neq \emptyset$ (which we can check by solving the maximum cardinality matroid intersection problem once). The weighted matroid intersection problem on $(\mathbf{M}_1, \mathbf{M}_2)$ asks to find $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ that maximizes $w(B)$, where $v(X) = \sum_{i\in X} v_i$ for any $v \in \mathbb{Z}^V$ and $X \subseteq V$. Its dual structure can be captured via the weight-splitting theorem [22, Theorem 13.2.4] and the dual problem is written as

$$\underset{p\in\mathbb{Z}^V}{\text{minimize}} \quad g(p) = \max_{B\in\mathcal{B}_1} p(B) + \max_{B\in\mathcal{B}_2} (w - p)(B). \tag{3}$$

The objective function g in (3) is known to be L-convex.[4] Since (3) is an unconstrained problem, nothing is needed for the projection. We below focus on the local optimization step (Step 3).

[4] For $k = 1, 2$, $g_k(p) = \max_{B\in\mathcal{B}_k} p(B)$ is an L-convex function obtained as the *discrete Legendre–Fenchel conjugate* of the indicator function $\delta_{\mathcal{B}_k}$ of $\mathcal{B}_k$ (regarded as the collection of $\mathbf{1}_B$ for $B \in \mathcal{B}_k$), which is *M-convex*. As g_1 and g_2 are L-convex, so is $g(p) = g_1(p) + g_2(w - p)$. See [38] for details.

Local optimization. We see that the local optimization reduces to the maximum cardinality matroid intersection problem, which asks to find a maximum-cardinality common independent set of two matroids. Letting $p \in \mathbb{Z}^V$ be a current feasible solution to (3) and $q = w - p$, the problem of finding an optimal direction $d = \mathbf{1}_X$ over $X \subseteq V$ is written as

$$\underset{X \subseteq V}{\text{minimize}} \quad \max_{B \in \mathcal{B}_1} (p + \mathbf{1}_X)(B) + \max_{B \in \mathcal{B}_2} (q - \mathbf{1}_X)(B). \tag{4}$$

For a matroid $\mathbf{M} = (V, \mathcal{B})$ and $v \in \mathbb{Z}^V$, let $\mathcal{B}^v = \operatorname{argmax}_{B \in \mathcal{B}} v(B)$ and $\mathbf{M}^v = (V, \mathcal{B}^v)$. Then, $\mathbf{M}^v$ forms a matroid [20], and its rank function is given as follows (see Appendix B.1 for the proof).

Lemma 3. *The rank function of* $\mathbf{M}^v$ *is given by* $\rho^v(X) = \max_{B \in \mathcal{B}} (v + \mathbf{1}_X)(B) - \max_{B \in \mathcal{B}} v(B)$.

From Lemma 3, by using rank functions ρ_1^p and ρ_2^q of $\mathbf{M}_1^p$ and $\mathbf{M}_2^q$, respectively, we can rewrite (4) as

$$\underset{X \subseteq V}{\text{minimize}} \quad \rho_1^p(X) + \rho_2^q(V \setminus X) + \text{const.}, \tag{5}$$

where $\text{const.} = \max_{B \in \mathcal{B}_1} p(B) + \max_{B \in \mathcal{B}_2} q(B) - r$. The problem (5) (without the constant term) is the dual of the maximum cardinality matroid intersection problem on $(\mathbf{M}_1^p, \mathbf{M}_2^q)$ (Edmonds' matroid intersection theorem [19]), thus completing the reduction. The standard augmenting-path algorithm by Cunningham [15] makes $\mathrm{O}(nr^{1.5})$ calls to independence oracles of $\mathbf{M}_1^p$ and $\mathbf{M}_2^q$, and every independence testing on $\mathbf{M}_1^p$ and $\mathbf{M}_2^q$ queried by the algorithm can be implemented with a single call to independence oracles of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively, which takes τ time (see Appendix B.2).

Lemma 4. *There is an algorithm that solves the maximum cardinality matroid intersection problem on* $(\mathbf{M}_1^p, \mathbf{M}_2^q)$ *by making* $\mathrm{O}(nr^{1.5})$ *calls to independence oracles of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$.

Once a maximum common independent set I of $(\mathbf{M}_1^p, \mathbf{M}_2^q)$ is found, we can obtain optimal solution $X \subseteq V$ to (5) by traversing the auxiliary graph constructed in the augmenting-path algorithm (see [44, Theorem 41.3]). Thus, the total time complexity of local optimization is $T_{\text{loc}} = \mathrm{O}(\tau n r^{1.5})$. If $|I| < r$, we update p to $p + \lambda \mathbf{1}_X$, where the step length λ is determined by binary search [46] (or let $\lambda = 1$). If $|I| = r$, current p is optimal to (3) and I is a maximum weight common base of $(\mathbf{M}_1, \mathbf{M}_2)$ [21].

3.3 Discrete energy minimization

We consider discrete energy minimization problems on a fixed vertex set V. Let $G = (V, E)$ be an undirected graph with $|V| = n$ and $|E| \le m$. Given univariate convex functions $\phi_i : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ and $\psi_{ij} : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ for $i, j \in V$ with $i \ne j$, the energy minimization problem is written as

$$\underset{p \in \mathbb{Z}^V}{\text{minimize}} \quad g(p) = \sum_{i \in V} \phi_i(p_i) + \sum_{(i,j) \in E} \psi_{ij}(p_j - p_i). \tag{6}$$

This problem appears in many computer-vision (CV) applications [8, 25, 28] and belongs to L$^\natural$-convex minimization [32]. In CV settings, ϕ_i measures how well label p_i fits pixel i, and ψ_{ij} quantifies smoothness of labels of adjacent pixels i and j.

Projection. In CV applications, we usually have a box constraint representing the range of pixel values. Also, we may have an acceptable range of non-smoothness of adjacent pixels. To deal with such constraints, we suppose ϕ_i and ψ_{ij} to return $+\infty$ if input variables are out of the ranges. The resulting feasible region can be represented as in Proposition 1. If we only have box constraints, we can easily obtain an $\ell_\infty^\pm$-projection of $\hat{p} \in \mathbb{R}^V$ by computing $\max\{\alpha_i, \min\{\hat{p}_i, \beta_i\}\}$ for each $i \in V$, which takes $T_{\text{prj}} = \mathrm{O}(n)$ time. When imposing the smoothness constraints, we need to compute an $\ell_\infty^\pm$-projection onto a general L$^\natural$-convex set. This is reduced to the shortest path problem, and we can compute the projection in $T_{\text{prj}} = \mathrm{O}(mn)$ time with the Bellman–Ford algorithm (see Appendix D).

Local optimization. Since the steepest descent method for problem (6) is already studied in [32], we here briefly describe key points. Let $p \in \operatorname{dom} g$ be a current solution and consider finding a steepest direction $d \in \mathcal{N}_\pm = \{0, +1\}^V \cup \{0, -1\}^V$. We focus on exploring $\{0, +1\}^V$; the case of $\{0, -1\}^V$ is analogous. Letting $\phi_i^{(p)}(d_i) = \phi_i(p_i + d_i)$ for $i \in V$ and $\psi_{ij}^{(p)}(d_i, d_j) = \psi_{ij}(p_j + d_j - p_i - d_i)$ for $(i, j) \in E$, the local optimization problem on $\{0, +1\}^V$ is written as

$$\underset{d \in \{0,+1\}^V}{\text{minimize}} \quad g^{(p)}(d) = \sum_{i \in V} \phi_i^{(p)}(d_i) + \sum_{(i,j) \in E} \psi_{ij}^{(p)}(d_i, d_j).$$

Since convexity of ψ_{ij} implies $\psi_{ij}^{(p)}(1,0)+\psi_{ij}^{(p)}(0,1) \geq \psi_{ij}^{(p)}(1,1)+\psi_{ij}^{(p)}(0,0)$, $g^{(p)}(d)$ is a submodular function that can be written as a sum of pseudo-boolean functions with at most two variables. Minimization of such a submodular function is reduced to a min-cut problem on a graph with $n+2$ vertices and $3m+n$ edges [29, 6]. If we solve the min-cut problem with the Dinic algorithm [16], it takes $T_{\mathrm{loc}} = \mathrm{O}(mn^2)$ time. We can also use empirically fast min-cut algorithms for CV settings [7]. Once a direction d is found, the step length λ can be determined by binary search [46] (or let $\lambda = 1$).

3.4 General L$^\natural$-convex function minimization

We consider general L$^\natural$-convex function minimization $\min_{p \in \mathbb{Z}^V} g(p)$, assuming $\mathrm{dom}\, g$ to be represented as in Proposition 1. Given a prediction $\hat{p} \in \mathbb{R}^V$, we can compute an $\ell_\infty^\pm$-projection $\mathcal{P}(\hat{p})$ onto $\mathrm{conv}(\mathrm{dom}\, g)$ with the Bellman–Ford algorithm in $T_{\mathrm{prj}} = \mathrm{O}(n^3)$ time (since $m \leq n^2$), as mentioned in Section 3.3. As for the local optimization, given a current solution $p \in \mathrm{dom}\, g$, we can find a steepest direction by solving $\min_{X \subseteq V} f(X)$, where $f(X) = g(p + \mathbf{1}_X)$ or $g(p - \mathbf{1}_X)$ is a submodular function, as mentioned in Section 2.2. An empirically fast algorithm for submodular function minimization is the Fujishige–Wolfe algorithm [23], which solves the local optimization problem in $T_{\mathrm{loc}} = \mathrm{O}((n^4\mathrm{EO} + n^5)F^2)$ time [11, 33], where EO is the time of evaluating $f(X)$ for any $X \subseteq V$ and $F = \max_{i \in V} \max\{|f(\{i\})|, |f(V) - f(V \setminus \{i\})|\}$ (and a theoretically faster strongly polynomial-time algorithm runs in $\mathrm{O}(n^3 \log^2 n \cdot \mathrm{EO} + n^4 \log^{\mathrm{O}(1)} n)$ time [35]). In total, we can minimize a general L$^\natural$-convex function in $\mathrm{O}((n^4\mathrm{EO} + n^5)F^2 \cdot \|p^*(g) - \hat{p}\|_\infty)$ time.

4 Learning predictions

We detail how to learn predictions $\hat{p} \in \mathbb{R}^V$. We consider online and batch learning settings where instances of form $\min_{p \in \mathbb{Z}^V} g_t(p)$ for $t = 1, \dots, T$ are given adversarially and randomly, respectively. We make the following assumption, which is common with the previous studies [17, 31].

Assumption 2. *Functions $g_1, \dots, g_T$ are defined on the same ground set V and satisfy Assumption 1.*

Furthermore, as in Theorem 2, we assume that a benchmark (optimal) prediction $\hat{p}^*$ is included in $[-C, +C]^V$ for some $C > 0$, which is also common with [17, 31]. As for bipartite matching, there always exists an optimal dual solution included in $[-C, +C]^V$ if $C \geq n\|w\|_\infty$ [30, Lemma 4.5],[5] and the same condition holds for matroid intersection with $C \geq r\|w\|_\infty$ (see Appendix B.3). This condition is also natural in discrete energy minimization since the range of pixel values is bounded. Also, we can choose any suboptimal solution in $[-C, +C]^V$ as a benchmark prediction at the cost of increasing the bounds in Theorem 2.

We use the online-learning framework by Khodak et al. [31], and thus the learning method itself is not novel. Nevertheless, the improvement in the bounds is considerable. As with [31], the regret bound is obtained from that of the online gradient descent method (OGD) [45], and the sample complexity bound follows from online-to-batch conversion [10]. The key to the improvement is that we apply OGD to 1-Lipschitz convex loss functions $L_t(p) = \|p^*(g_t) - p\|_\infty$,[6] while Khodak et al. [31] applied OGD to $L_t(p) = \|p^*(g_t) - p\|_1$, which is $\sqrt{n}$-Lipschitz, following the setting of [17].

Theorem 2. *Let V be a finite set of size n and $C > 0$. For any sequence of functions $g_1, \dots, g_T$ that map $\mathbb{Z}^V$ to $\mathbb{R} \cup \{+\infty\}$, there is an online learning algorithm that returns $\hat{p}_1, \dots, \hat{p}_T \in \mathbb{R}^V$ with the following regret bound for any $\hat{p}^* \in [-C, +C]^V$:*

$$\sum_{t=1}^{T} \|p^*(g_t) - \hat{p}_t\|_\infty \leq \sum_{t=1}^{T} \|p^*(g_t) - \hat{p}^*\|_\infty + C\sqrt{2nT}.$$

Moreover, for any distribution $\mathcal{D}$ over functions $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$, $\delta \in (0,1]$, and $\varepsilon > 0$, if $T = \Omega\left(\left(\frac{C}{\varepsilon}\right)^2 \left(n + \log\frac{1}{\delta}\right)\right)$ i.i.d. samples $g_1, \dots, g_T$ from $\mathcal{D}$ are given, we can compute $\hat{p} \in \mathbb{R}^V$ that satisfies the following condition with a probability of at least $1 - \delta$ for all $\hat{p}^* \in [-C, +C]^V$:

$$\mathbb{E}_{g \sim \mathcal{D}} \|p^*(g) - \hat{p}\|_\infty \leq \mathbb{E}_{g \sim \mathcal{D}} \|p^*(g) - \hat{p}^*\|_\infty + \varepsilon.$$

[5] In [17], it is stated that $C \geq \|w\|_\infty$ is sufficient, but this is not true as shown in Appendix C.

[6] We can also use $L_t(p) = \|p^*(g_t) - p\|_\infty^\pm$, which is convex and $\sqrt{2}$-Lipschitz; this may yield shaper bounds on the number of iterations, as in Proposition 2. We here present the ℓ_∞-loss version for ease of exposition.

Proof. We regard $L_t(p) = \|p^*(g_t) - p\|_\infty$ for $t = 1, \dots, T$ as loss functions of $p \in \mathbb{R}^V$ and use the online gradient descent method (OGD). Note that $L_t(p)$ is convex since

$$L_t\left(\frac{p+q}{2}\right) = \left\|p^*(g_t) - \frac{p+q}{2}\right\|_\infty \le \left\|\frac{p^*(g_t)}{2} - \frac{p}{2}\right\|_\infty + \left\|\frac{p^*(g_t)}{2} - \frac{q}{2}\right\|_\infty = \frac{L_t(p) + L_t(q)}{2}$$

for any $p, q \in \mathbb{R}^V$ due to the triangle inequality. Furthermore, $L_t(p)$ is 1-Lipschitz since

$$L_t(p) - L_t(q) = \|p^*(g_t) - p\|_\infty - \|p^*(g_t) - q\|_\infty \le \|p - q\|_\infty \le \|p - q\|_2$$

for any $p, q \in \mathbb{R}^V$ due to the triangle inequality and $\|x\|_\infty \le \|x\|_2$ for any $x \in \mathbb{R}^V$. Since L_t is a 1-Lipschitz convex loss function and the ℓ_2-norm of any vector in $[-C, +C]^V$ is at most $C\sqrt{n}$, the regret of OGD is at most $C\sqrt{2nT}$ (see [45, Corollary 2.7]), thus obtaining the first claim. The second claim is obtained by using online-to-batch conversion. Specifically, since the loss function value is at most $2C$, if we feed sampled $g_1, \dots, g_T$ to OGD and let $\hat{p} = \frac{1}{T}\sum_{t=1}^{T} \hat{p}_t$, then [10, Proposition 1] implies that the following inequality holds with a probability of at least $1 - \delta$:

$$\mathbb{E}_{g \sim \mathcal{D}} \|p^*(g) - \hat{p}\|_\infty \le \min_{p \in [-C, +C]^V} \mathbb{E}_{g \sim \mathcal{D}} \|p^*(g) - p\|_\infty + C\sqrt{\frac{2n}{T}} + 2C\sqrt{\frac{2}{T} \log \frac{1}{\delta}}.$$

Therefore, setting $T = 32\left(\frac{C}{\varepsilon}\right)^2 \left(n + \log \frac{1}{\delta}\right)$ is sufficient for bounding the sum of the last two terms in the right-hand side by ε from above. □

5 Whether to learn primal or dual solutions

We discuss whether to learn primal or dual solutions for successfully warm-starting algorithms with predictions from the DCA perspective (we refer the reader to [38] for more information). We expect that the discussion here is also useful in the context of augmenting online algorithms with predictions.

We consider a minimization problem of form $\min_{p \in \mathbb{Z}^V} g(p)$, where $g : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is a general objective function. As we have seen above, iterative algorithms look for an optimal solution by alternately exploring a neighborhood to find a steepest direction and proceeding along the direction. Intuitively, such an iterative algorithm can benefit from a prediction that is close to an optimum if the feasible region, $\operatorname{dom} g$, is *path connected* with respect to the neighborhood; conversely, if the feasible region is not connected, good predictions are not always helpful. We below elaborate more on this idea for the case of L-convex minimization (the L$^\natural$-convex case is analogous).

First, we need to define a neighborhood $\mathcal{N}$ so that we can efficiently solve local optimization problems on $\mathcal{N}$. In the case of L-convex minimization, the local optimization on $\mathcal{N} = \mathcal{N}_+ = \{0, +1\}^V$ reduces to submodular function minimization in general, which we can solve in polynomial time (and more efficient methods are available for many specific problems). Once a neighborhood $\mathcal{N}$ is defined, the next important requirement is the *path connectivity* of the feasible region $\operatorname{dom} g$ with respect to $\mathcal{N}$, i.e., given any $p, q \in \operatorname{dom} g$, we can move from p to q by iteratively finding an appropriate direction $d \in \mathcal{N}$ and proceeding along d. L-convex sets are path connected with respect to $\mathcal{N}_+$, and this property is necessary for ensuring that the steepest descent method converges to an optimum.

Under the above conditions, a prediction close to an optimum is expected to be beneficial since it provides a short path to an optimum. In other words, if feasible regions are not path connected, predictions close to an optimum do not always improve the performance of iterative algorithms. This observation gives a guideline for judging whether to learn primal or dual solutions: we should choose the one such that a prediction can be converted into a solution in a path-connected feasible region.

In DCA, there are other convexity notions than L-convexity: *M*-, *L*$_2$-, and *M*$_2$*-convexity* (and their L$^\natural$ counterparts). In these classes, L-convex sets are path connected with respect to $\mathcal{N}_+$, and so are M-convex sets with respect to $\mathcal{N} = \left\{ \mathbf{1}_{\{i\}} - \mathbf{1}_{\{j\}} \;\middle|\; i, j \in V \right\}$. By contrast, no appropriate neighborhoods are known that make L$_2$- and M$_2$-convex sets path connected. In the cases of bipartite matching and matroid intersection, the primal problems belong to M$_2$-concave maximization, while their dual problems are L-convex minimization. Thus, learning dual solutions for bipartite matching and matroid intersection is reasonable. On the other hand, the primal problem of discrete energy minimization is L-convex, and thus we do not need to consider its dual.

6 Conclusion

We have developed a framework for warm-starting the steepest descent method for L/L$^\natural$-convex function minimization with predictions, thus bridging between discrete convex analysis and algorithms with predictions. We have demonstrated its effectiveness for weighted bipartite matching, weighted matroid intersection, and discrete energy minimization. We have also presented regret and sample complexity bounds for learning of predictions and discussed whether to learn primal or dual solutions. As for the practical aspect, experiments in Appendix F show that our DCA-based approach performs comparably to (or slightly better than) the method of [17].

A limitation of our work is that it does not yield prediction-independent worst-case bounds in general. This, however, is often not a serious matter since we can run standard algorithms with worst-case guarantees in parallel and terminate both once either one returns a solution. This point should be contrasted with the situation of augmenting online algorithms with predictions, where every decision made over time is irrevocable and thus attaining good prediction-dependent and worst-case guarantees simultaneously by a single algorithm is crucial. In Appendix E, we further discuss worst-cae guarantees of our DCA-based framework. In addition, we cannot deal with problems without L/L$^\natural$-convexity, and extending our framework to M/M$^\natural$-convex function minimization, which also enjoys the path connectivity as mentioned in Section 5, will be an interesting future direction.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] See Section 1 for details of our contributions.
   (b) Did you describe the limitations of your work? [Yes] See Section 6.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] This is a theoretical paper and no negative societal impacts are expected.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Assumptions 1 and 2
   (b) Did you include complete proofs of all theoretical results? [Yes] Due to the space limitation, some proofs are shown in the supplementary.

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No] All details are described in Appendix F.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

Appendix

A Proof of Lemma 1

Lemma 1. *Let $S \subseteq \mathbb{Z}^V$ be an L/L$^\natural$-convex set and $p \in \operatorname{conv}(S)$. Then, it holds $\lfloor p \rceil \in S$.*

Proof. We focus on the case where S is L$^\natural$-convex; the following proof subsumes the case where S is L-convex. From Proposition 1, S can be written as

$$S = \left\{ p \in \mathbb{Z}^V \;\middle|\; \alpha_i \le p_i \le \beta_i,\;\; p_j - p_i \le \gamma_{ij} \;\text{ for } i, j \in V; i \ne j \right\}$$

with some $\alpha_i \in \mathbb{Z} \cup \{-\infty\}$, $\beta_i \in \mathbb{Z} \cup \{+\infty\}$, and $\gamma_{ij} \in \mathbb{Z} \cup \{+\infty\}$. Also, $\operatorname{conv}(S)$ has the same representation except for the replacement of $\mathbb{Z}^V$ with $\mathbb{R}^V$. We prove that for any $p \in \mathbb{R}^V$ satisfying the inequality constraints, $\lfloor p \rceil \in \mathbb{Z}^V$ also satisfies the constraints, which implies the lemma statement.

Since $\alpha_i \in \mathbb{Z} \cup \{-\infty\}$ and $\beta_i \in \mathbb{Z} \cup \{+\infty\}$, $\alpha_i \le p_i \le \beta_i$ implies $\alpha_i \le \lfloor p_i \rceil \le \beta_i$. We below discuss the remaining constraints of form $p_j - p_i \le \gamma_{ij}$. Since $p_j - p_i \le \gamma_{ij}$ and $\gamma_{ij} \in \mathbb{Z} \cup \{+\infty\}$ imply $\lceil p_j - p_i \rceil \le \gamma_{ij}$, it suffices to prove $\lfloor p_j \rceil - \lfloor p_i \rceil \le \lceil p_j - p_i \rceil$. Let $p_i = a + r$ and $p_j = b + s$, where $a, b \in \mathbb{Z}$ and $r, s \in [0, 1)$. If $s > r$, since $\lfloor p_i \rceil \ge a$ and $\lfloor p_j \rceil \le b + 1$, we have

$$\lfloor p_j \rceil - \lfloor p_i \rceil \le b + 1 - a = \lceil b - a + s - r \rceil = \lceil p_j - p_i \rceil.$$

If $s \le r$, we have

$$\lfloor p_j \rceil - \lfloor p_i \rceil = \lfloor b + s \rceil - \lfloor a + r \rceil \le \lfloor b + r \rceil - \lfloor a + r \rceil = b - a = \lceil b - a + s - r \rceil = \lceil p_j - p_i \rceil.$$

In any case, we have $\lfloor p_j \rceil - \lfloor p_i \rceil \le \lceil p_j - p_i \rceil \le \gamma_{ij}$. Hence $p \in \operatorname{conv}(S)$ implies $\lfloor p \rceil \in S$. □

B Details of matroid intersection

We describe details of the weighted matroid intersection algorithm discussed in Section 3.2 and present proofs of Lemmas 3 and 4. We also give a bound on dual optimal solutions to the weighted matroid intersection problem. Readers interested in matroid theory are referred to [40].

B.1 Proof of Lemma 3

Let $\mathbf{M} = (V, \mathcal{B})$ be a matroid and $v \in \mathbb{Z}^V$ be a weight vector. Recall that $\mathbf{M}^v = (V, \mathcal{B}^v)$ denotes the matroid with $\mathcal{B}^v = \operatorname{argmax}_{B \in \mathcal{B}} v(B)$. A maximum-weight base $B \in \mathcal{B}^v$ can be obtained by the following greedy algorithm [20]. First, sort elements of V in a non-increasing order of v as $i_1, \dots, i_{|V|}$, i.e., $v_{i_1} \ge \dots \ge v_{i_{|V|}}$. Starting with $B = \emptyset$, from $k = 1$ to $|V|$, add i_k to B if $B \cup \{i_k\}$ is independent. The resulting B is a base of $\mathbf{M}$ that maximizes $v(B)$. The correctness of this greedy algorithm justifies Lemma 3 as follows.

Lemma 3. *The rank function of $\mathbf{M}^v$ is given by $\rho^v(X) = \max_{B \in \mathcal{B}} (v + \mathbf{1}_X)(B) - \max_{B \in \mathcal{B}} v(B)$.*

Proof. By the definitions of the rank function and $\mathcal{B}^v$, we have

$$\rho^v(X) = \max_{B \in \mathcal{B}^v} |X \cap B| = \max_{B \in \mathcal{B}^v} \mathbf{1}_X(B) = \max_{B \in \mathcal{B}^v} (v + \mathbf{1}_X)(B) - \max_{B \in \mathcal{B}} v(B).$$

Therefore, our goal is to show $\max_{B \in \mathcal{B}} (v + \mathbf{1}_X)(B) = \max_{B \in \mathcal{B}^v} (v + \mathbf{1}_X)(B)$. Consider an ordering of elements in V that is non-increasing with respect to $v + \frac{1}{2}\mathbf{1}_X$. Since v is an integer vector, this order is non-increasing with respect to both v and $v + \mathbf{1}_X$, meaning that there exists $B \in \mathcal{B}$ that maximizes v and $v + \mathbf{1}_X$ simultaneously, hence $\max_{B \in \mathcal{B}} (v + \mathbf{1}_X)(B) = \max_{B \in \mathcal{B}^v} (v + \mathbf{1}_X)(B)$. □

B.2 Proof of Lemma 4

We give preliminaries on matroid theory (see [40] for details). Let $\mathbf{M} = (V, \mathcal{B})$ be a matroid and X be a subset of V. The *restriction* of $\mathbf{M}$ to X, denoted by $\mathbf{M} \mid X$, is the matroid on X whose independent sets are subsets of X that are independent in $\mathbf{M}$. The *contraction* of $\mathbf{M}$ with respect to X is the matroid $\mathbf{M} / X = (V \setminus X, \mathcal{B} / X)$ with $\mathcal{B} / X = \{ B \subseteq V \setminus X \mid B \cup B' \in \mathcal{B} \}$, where B' is any base of $\mathbf{M} \mid X$ (indeed, $\mathbf{M} / X$ does not depend on the choice of B'). The *direct sum* of two matroids

$\mathbf{M}_1 = (V_1, \mathcal{B}_1)$ and $\mathbf{M}_2 = (V_2, \mathcal{B}_2)$ with $V_1 \cap V_2 = \emptyset$ is the matroid $\mathbf{M}_1 \oplus \mathbf{M}_2 = (V_1 \cup V_2, \mathcal{B}_1 \oplus \mathcal{B}_2)$ with $\mathcal{B}_1 \oplus \mathcal{B}_2 = \{\, B_1 \cup B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2 \,\}$.

Define $V^v_{\geq}(t) = \{\, i \in V \mid v_i \geq t \,\}$ for any $t \in \mathbb{Z}$. The correctness of the greedy algorithm for $v \in \mathbb{Z}^V$ also gives the following well-known decomposition theorem of $\mathbf{M}^v$ (see, e.g., [27, Lemma 1]).

Lemma 5. *It holds* $\mathbf{M}^v = \bigoplus_{t=-\infty}^{\infty} \mathbf{M}^v(t)$, *where* $\mathbf{M}^v(t) = \left(\mathbf{M} \mid V^v_{\geq}(t)\right) / V^v_{\geq}(t+1)$.

We are ready to prove Lemma 4.

Lemma 4. *There is an algorithm that solves the maximum cardinality matroid intersection problem on* $(\mathbf{M}^p_1, \mathbf{M}^q_2)$ *by making* $\mathrm{O}(nr^{1.5})$ *calls to independence oracles of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$.

Proof. Cunningham's algorithm [15] asks the independence of sets in the form $I \cup \{j\}$ or $I \setminus \{i\} \cup \{j\}$, where I is a common independent set of $(\mathbf{M}^p_1, \mathbf{M}^q_2)$, $i \in I$, and $j \in V \setminus I$. Thus, it suffices to show the following: for a matroid $\mathbf{M} = (V, \mathcal{B})$, a weight vector $v \in \mathbb{Z}^V$, an independent set I of $\mathbf{M}^v$, $i \in I$, and $j \in V \setminus I$, one can check whether $I \cup \{j\}$ and $I \setminus \{i\} \cup \{j\}$ are independent or not in $\mathbf{M}^v$ by a single call to the independence oracle of $\mathbf{M}$.

Let $V^v(t) = \{\, i \in V \mid v_i = t \,\}$ for $t \in \mathbb{Z}$. By Lemma 5, a set $X \subseteq V$ is independent in $\mathbf{M}^v$ if and only if $X \cap V^v(t)$ is independent in $\mathbf{M}^v(t)$ for every $t \in \mathbb{Z}$. This and the independence of I in $\mathbf{M}^v$ imply that $I \cup \{j\}$ and $I \setminus \{i\} \cup \{j\}$ are independent in $\mathbf{M}^v$ if and only if $(I \cup \{j\}) \cap V^v(v_j)$ and $(I \setminus \{i\} \cup \{j\}) \cap V^v(v_j)$, respectively, are independent in $\mathbf{M}^v(v_j)$. The independence of a set $X \subseteq V^v(t)$ in $\mathbf{M}^v(t)$ is equivalent to that of $X \cup (B \cap V^v_{\geq}(t+1))$ in $\mathbf{M}$, where B is an arbitrary base of $\mathbf{M}^v$. Therefore, we can reduce the independence testing of $I \cup \{j\}$ and $I \setminus \{i\} \cup \{j\}$ in $\mathbf{M}^v$ to that of $(I \cup \{j\}) \cup (B \cap V^v_{\geq}(v_j + 1))$ and $(I \setminus \{i\} \cup \{j\}) \cup (B \cap V^v_{\geq}(v_j + 1))$, respectively, in $\mathbf{M}$. We can obtain $B \in \mathcal{B}^v$ by running the greedy algorithm once in advance, which makes $\mathrm{O}(n)$ calls to the independence oracle of $\mathbf{M}$. Thus, we obtain the desired oracle complexity. □

B.3 Bound on dual optimal solutions

Let $\mathbf{M}_1 = (V, \mathcal{B}_1)$ and $\mathbf{M}_2 = (V, \mathcal{B}_2)$ be matroids and $w \in \mathbb{Z}^V$ be a weight vector. Fixing any maximum-weight common base $B \in \mathcal{B}_1 \cap \mathcal{B}_2$, let $D = (V, A)$ be a directed bipartite graph with bipartition $V = B \cup (V \setminus B)$ and arc set $A = A_1 \cup A_2$ given by

$$\begin{aligned} A_1 &= \{\, (i,j) \mid i \in B, j \in V \setminus B, B \setminus \{i\} \cup \{j\} \in \mathcal{B}_1 \,\}, \\ A_2 &= \{\, (j,i) \mid i \in B, j \in V \setminus B, B \setminus \{i\} \cup \{j\} \in \mathcal{B}_2 \,\}. \end{aligned} \tag{7}$$

We define an arc-length vector $\gamma \in \mathbb{Z}^A$ by $\gamma_{ij} = -w_j$ for $(i,j) \in A_1$ and $\gamma_{ji} = w_i$ for $(j,i) \in A_2$. Then, D has no negative cycles [44, Theorem 41.5] and thus has a *potential* $p \in \mathbb{Z}^V$, which is a vector satisfying $p_j - p_i \leq \gamma_{ij}$ for $(i,j) \in A$. Indeed, the potentials coincide with the dual optimal solutions, as in the following lemma.

Lemma 6 ([44, Theorem 41.9])**.** *A vector* $p \in \mathbb{Z}^V$ *is optimal to* (3) *if and only if* p *is a potential of* D *with respect to the arc-length vector* $\gamma \in \mathbb{Z}^A$.

This lemma gives a bound on an optimal dual solution to the weighted matroid intersection problem.

Proposition 3. *There exists a dual optimal solution* $p^* \in \mathbb{Z}^V$ *such that* $\|p^*\|_\infty \leq rW$, *where* r *is the rank of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *and* $W = \|w\|_\infty$.

Proof. Let $p \in \mathbb{Z}^V$ be the vector whose ith component p_i is the minimum length of any directed paths on D that ends at i (starting from wherever). Then, p is a potential of D [44, Theorem 8.2] and thus is a dual optimal solution by Lemma 6. The number of arcs in a simple path on D is at most $2r$, and each arc has a length at least $-W$, hence $-2rW \leq p_i \leq 0$. Let $p^* = p + rW\mathbf{1}$, which is also a dual optimal solution and satisfies $-rW \leq p^*_i \leq rW$ for $i \in V$. Hence, we have $\|p^*\|_\infty \leq rW$. □

The following example shows that the bound given in Proposition 3 is tight.

Example 1. Let $n \in \mathbb{N}$ be odd. Define $\mathbf{M}_1 = (V, \mathcal{B}_1)$ and $\mathbf{M}_2 = (V, \mathcal{B}_2)$ by $V = \{1, \dots, n\}$ and

$$\begin{aligned} \mathcal{B}_1 &= \{\, B \subseteq V \mid 1 \notin B \text{ and } |B \cap \{i, i+1\}| = 1 \text{ for even } i \in V \,\}, \\ \mathcal{B}_2 &= \{\, B \subseteq V \mid n \notin B \text{ and } |B \cap \{i, i+1\}| = 1 \text{ for odd } i \in V \,\}. \end{aligned}$$

Then, $\mathbf{M}_1$ and $\mathbf{M}_2$ are matroids (*partition matroids*) of rank $r = (n-1)/2$ having a unique common base $B^* = \{2, 4, \dots, n-3, n-1\}$. The sets A_1 and A_2 defined in (7) with respect to $B = B^*$ are $A_1 = \{(2,3), (4,5), \dots, (n-1, n)\}$ and $A_2 = \{(1,2), (3,4), \dots, (n-2, n-1)\}$, respectively, meaning that $D = (V, A)$ with $A = A_1 \cup A_2$ is the directed path graph from 1 to n. Let $W \in \mathbb{N}$ and $w \in \mathbb{Z}^V$ be a weight vector defined by $w_i = (-1)^{i+1} W$ for $i \in V$. Then, the corresponding arc length $\gamma \in \mathbb{Z}^A$ is $\gamma_{i,i+1} = -W$ for every arc $(i, i+1) \in A$.

Let $p^* \in \mathbb{Z}^V$ be a dual optimal solution. By Lemma 6, p^* is a potential of D with respect to γ. Thus, we have $p^*_{i+1} - p^*_i \leq -W$ for $i = 1, \dots, n-1$, hence $p^*_i \leq p^*_1 - (i-1)W$ for $i \in V$. The ℓ_∞-norm of p^* is minimized when $p^*_i = ((n+1)/2 - i)W$ for $i \in V$ with the minimum value of $(n-1)W/2 = rW$, thus implying the tightness of the bound in Proposition 3.

C Counterexample to the bound on dual solutions for bipartite matching

In the proof of [17, Lemma 22], the following claim is used: if all edge costs c_e of a bipartite graph (V, E) are non-negative and at most C, there is an optimal dual solution $y^* \in \mathbb{R}^V$ to the *minimum* cost perfect bipartite matching problem such that $\|y^*\|_\infty \leq C$, where the dual problem is given as

$$\underset{y \in \mathbb{R}^V}{\text{maximize}} \quad \sum_{i \in V} y_i \quad \text{subject to} \quad y_i + y_j \leq c_e \quad (e = \{i, j\} \in E).$$

We give a counterexample to this claim.

Example 2. Let $n \in \mathbb{N}$ be even and $P_n = (V, E)$ be the path graph with vertices $V = \{1, \dots, n\}$ and edges $E = \{\,\{i, i+1\} \mid i = 1, \dots, n-1\,\}$. Then, P_n is a bipartite graph with bipartition $\{L, R\}$ of V, where L and R consist of the odd and even numbers in V, respectively. Set an edge cost c_e of $e = \{i, i+1\} \in E$ to $C > 0$ if $i \in L$ and to 0 if $i \in R \setminus \{n\}$. Since $M^* = \{\,\{i, i+1\} \mid i \in L\,\}$ is a unique perfect matching, any optimal dual solution y^* must satisfy the tightness condition for all edges in M^*, i.e., $y^*_i + y^*_{i+1} = C$ for $i \in L$. In addition, the feasibility of y^* implies $y^*_i + y^*_{i+1} \leq 0$ for $i \in R \setminus \{n\}$. Thus, we have $y^*_i \geq y^*_{i+2} + C$ for $i \in L \setminus \{n-1\}$, hence $y^*_1 \geq y^*_{n-1} + (n/2 - 1)C$. For $n \geq 8$, $y^*_{n-1} \in [-C, +C]$ implies $y^*_1 \geq 2C > C$, contradicting the claim.

D Projection onto general L$^\natural$-convex sets

Let $S \subseteq \mathbb{Z}^V$ be a non-empty L$^\natural$-convex set. We assume that S is represented as in Proposition 1, i.e.,

$$S = \left\{\, p \in \mathbb{Z}^V \;\middle|\; \alpha_i \leq p_i \leq \beta_i, \;\; p_j - p_i \leq \gamma_{ij} \;\text{ for } i, j \in V; i \neq j \,\right\}$$

where $\alpha_i \in \mathbb{Z} \cup \{-\infty\}$, $\beta_i \in \mathbb{Z} \cup \{+\infty\}$, and $\gamma_{ij} \in \mathbb{Z} \cup \{+\infty\}$. We define an edge set $E = \{\,(i, j) \mid i, j \in V; i \neq j, \; \gamma_{ij} < +\infty\,\}$ and let $m = |E|$ (note that constraints with $\gamma_{ij} = +\infty$ can be ignored). Given any $\hat{p} \in \mathbb{R}^V$, we show that an $\ell^{\pm}_\infty$-projection $\mathcal{P}(\hat{p}) \in \operatorname{argmin}_{p \in \operatorname{conv}(S)} \|p - \hat{p}\|^{\pm}_\infty$ can be computed in $\mathrm{O}(mn)$ time, where $n = |V|$.

Using variables $q = p - \hat{p} \in \mathbb{R}^V$, we rewrite the problem of computing $\mathcal{P}(\hat{p})$ as

$$\begin{array}{ll} \text{minimize} & \|q\|^{\pm}_\infty = \max\limits_{i \in V} \max\{0, +q_i\} + \max\limits_{i \in V} \max\{0, -q_i\} \\ \text{subject to} & \alpha_i - \hat{p}_i \leq q_i \leq \beta_i - \hat{p}_i \qquad \forall i \in V \\ & q_j - q_i \leq \gamma_{ij} - \hat{p}_j + \hat{p}_i \qquad \forall i, j \in V; i \neq j. \end{array}$$

For convenience, let $\hat{\gamma}_{i0} = -\alpha_i + \hat{p}_i$, $\hat{\gamma}_{0i} = \beta_i - \hat{p}_i$, and $\hat{\gamma}_{ij} = \gamma_{ij} - \hat{p}_j + \hat{p}_i$ for $i, j \in V$ such that $i \neq j$, $V_0 = \{0\} \cup V$, and $E_0 = E \cup \{\,(i, 0) \mid i \in V, \; \alpha_i > -\infty\,\} \cup \{\,(0, i) \mid i \in V, \; \beta_i < +\infty\,\}$. Then, using variables $(q_0, q) \in \mathbb{R} \times \mathbb{R}^V$, we can rewrite the problem as

$$\begin{array}{ll} \text{minimize} & \max\limits_{i \in V_0} q_i - \min\limits_{i \in V_0} q_i \\ \text{subject to} & q_j - q_i \leq \hat{\gamma}_{ij} \quad \forall (i, j) \in E_0 \\ & q_0 = 0. \end{array}$$

We further rewrite this problem as a linear programming problem with additional variables $q_s, q_t \in \mathbb{R}$ ($s, t \notin V_0$). Letting q_s and q_t represent $\max_{i \in V_0} q_i$ and $\min_{i \in V_0} q_i$, respectively, the objective function is written as $q_s - q_t$, and this yields additional constraints $q_i - q_s \leq 0$ and $q_t - q_i \leq 0$ for $i \in V_0$. Thus, the negative of the above problem is written as

$$\begin{array}{lll} \text{maximize} & q_t - q_s & \\ \text{subject to} & q_j - q_i \leq \hat{\gamma}_{ij} & \forall (i,j) \in E_0 \\ & q_i - q_s \leq 0 & \forall i \in V_0 \\ & q_t - q_i \leq 0 & \forall i \in V_0 \\ & q_0 = 0. & \end{array} \tag{8}$$

If we do not have the last constraint, $q_0 = 0$, (8) is the dual of the shortest s–t path problem on a graph $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{w})$, where $\tilde{V} = V_0 \cup \{s, t\}$, $\tilde{E} = E_0 \cup \{\, (s, i) \mid i \in V_0 \,\} \cup \{\, (i, t) \mid i \in V_0 \,\}$, and

$$\tilde{w}_{ij} = \begin{cases} \hat{\gamma}_{ij} & \text{for } (i,j) \in E_0 \\ 0 & \text{otherwise} \end{cases}$$

for $(i,j) \in \tilde{E}$. Moreover, given any optimal solution $q' \in \mathbb{R}^{\tilde{V}}$ to (8) without the last constraint, $q^* = q' - q'_0\mathbf{1}$ is also optimal and satisfies $q^*_0 = 0$. Hence, the remaining task is to solve the shortest path problem on $\tilde{G}$. Since $|\tilde{V}| = n + 3$, $|\tilde{E}| \leq m + 4n + 2$, and the $\mathrm{L}^\natural$-convexity of $S \neq \emptyset$ rules out the presence of negative cycles, the Bellman–Ford algorithm can solve the shortest path problem on $\tilde{G}$ in $\mathrm{O}(mn)$ time. More precisely, to obtain an optimal solution $q^* \in \mathbb{R}^{\tilde{V}}$ to (8), we find shortest paths from s to all vertices in $\tilde{V} \setminus \{s\}$ with the Bellman–Ford algorithm, and we set the potential q^* along the found paths so that $q^*_0 = 0$ holds. We obtain a desired projection as $p = \hat{p} + q^*_V$, where $q^*_V \in \mathbb{R}^V$ is the restriction of $q^* \in \mathbb{R}^{\tilde{V}}$ to V.

E Discussion on worst-case guarantees

While we have focused on prediction-dependent bounds, we can bound the worst-case time complexity of the DCA-based algorithms for bipartite matching and matroid intersection, as described below. The following bounds are, however, weaker than those of standard algorithms; therefore, we should run standard algorithms in parallel to obtain better worst-case guarantees, as discussed in Section 6.

As shown in [46, Theorem 4.17], the long-step steepest descent algorithm for an L-convex function $g : \mathbb{Z}^V \to Z \cup \{+\infty\}$ converges in $n \cdot \max\bigl\{\, g(p) - g(p + d) \bigm| p \in \mathbb{Z}^V, d \in \mathcal{N}_+ \,\bigr\}$ iterations if the minimum minimizer $d \in \mathcal{N}_+$ is chosen in every local optimization. Thus, the DCA-based algorithm for bipartite matching (resp. matroid intersection) terminates in $\mathrm{O}(n^2)$ (resp. $\mathrm{O}(nr)$) iterations. Since a single iteration takes $\mathrm{O}(m\sqrt{n})$ (resp. $\mathrm{O}(\tau n r^{1.5})$) time, the total worst-case time complexity is $\mathrm{O}(mn^{2.5})$ (resp. $\mathrm{O}(\tau n^2 r^{2.5})$). We can also obtain a similar worst-case bound for discrete energy minimization if the derivatives of ϕ_i and ψ_{ij} are bounded. We, however, do not know whether those bounds are tight; i.e., we do not know whether there is, for example, a worst-case bipartite-matching instance such that the DCA-based algorithm incurs $\Theta(mn^{2.5})$ time.

E.1 Difficulty in recovering $\tilde{\mathrm{O}}(mn)$ time guarantee for bipartite matching

We also explain why the DCA-based algorithm cannot immediately recover the $\tilde{\mathrm{O}}(mn)$-time bound for bipartite matching ($\tilde{\mathrm{O}}$ hides logarithmic factors). In short, this is because the DCA-based algorithm is subtly different from the standard $\tilde{\mathrm{O}}(mn)$-time Hungarian method, as briefly described below (see [44, Section 18.5b] for more details).

Let $G = (L \cup R, E)$ be a bipartite graph with edge weights $w \in \mathbb{Z}^E$. Keeping a dual solution $p = (s, t) \in \mathbb{R}^{L \cup R}$, the DCA-based algorithm alternately computes a maximum matching of the tight subgraph $G^* = (L \cup R, E^*)$ with respect to (s, t) from scratch using the Hopcroft–Karp algorithm and updates (s, t). This results in the $\mathrm{O}(mn^{2.5})$ time complexity bound, as described above. By contrast, the standard Hungarian method keeps a maximum matching M of G^* in addition to a dual solution (s, t). In every iteration, it augments M or updates (s, t) by searching an augmenting path in an orientation D^*_M of G^* defined from M. Moreover, the sequence of dual updates between two augmentations of M can be aggregated into the single shortest-path searching on D_M, the orientation of G by M defined similarly to D^*_M, where edge lengths are given by $l_{ij} := s_i - t_j - w_{ij}$ for $ij \in E$. Since the edge lengths are non-negative, we can use Dijkstra's $\tilde{\mathrm{O}}(m)$-time algorithm to find a shortest path on D_M, thus achieving the $\tilde{\mathrm{O}}(mn)$ time.

To recover this $\tilde{\mathrm{O}}(mn)$-time bound with the DCA-based algorithm, we need to convert it into the above standard Hungarian method. This modification, however, is specific to the bipartite-matching case and is not covered by the general DCA theory. Furthermore, the modification may worsen the prediction-dependent bound. This is because, while the DCA-based algorithm always updates a dual solution in every iteration, the standard Hungarian method may not when M is augmented, implying that the bound based on $\|p^* - \hat{p}\|_\infty$ does not follow immediately.

F Experiments

We compare our DCA-based method with that of Dinitz et al. [17] using synthetic weighted bipartite matching instances. First, we summarize the differences between the two methods. Both methods are based on the same basic methodology: predict an (infeasible) dual solution $\hat{p}$, convert $\hat{p}$ into an initial feasible solution p°, and warm-start the bipartite-matching solver with p°. As for the weighted-bipartite-matching solver, both iteratively call the Hopcroft–Karp algorithm [26] to compute an optimal dual solution; hence the bipartite-matching solver is identical. The differences between our method and [17] lie in how to learn (infeasible) predictions $\hat{p}$ and how to obtain initial feasible solutions p°. Below we detail these two differences.

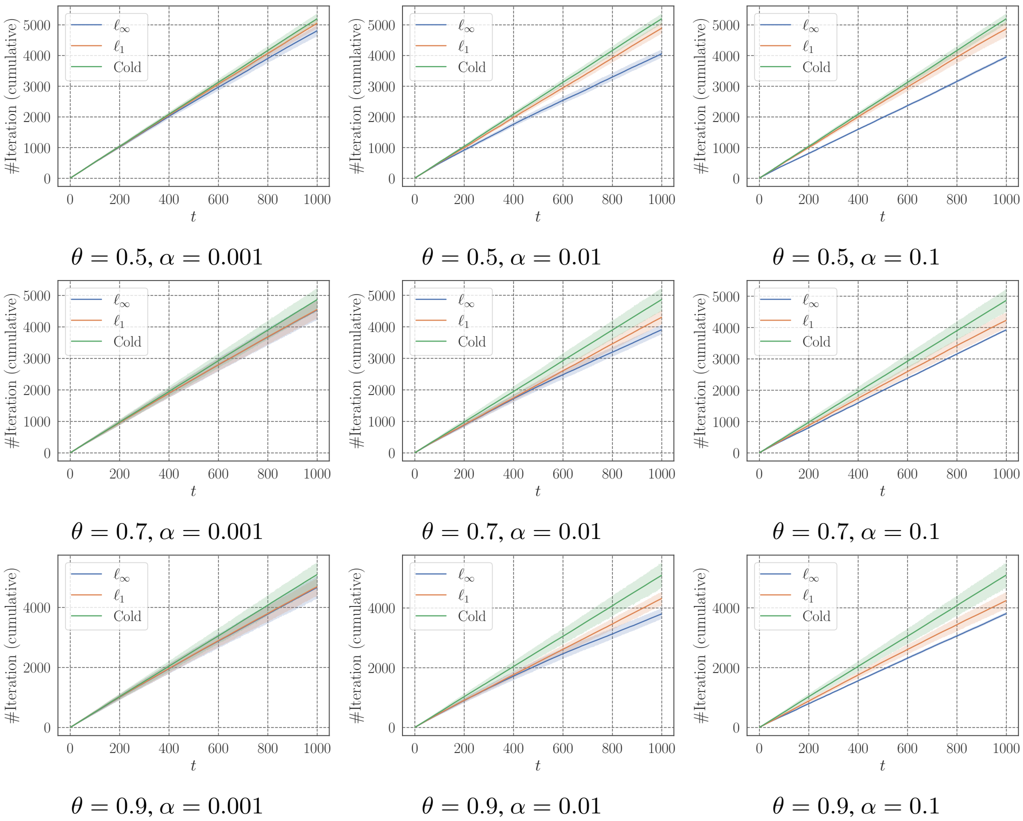
Learning predictions. Dinitz et al. [17] learned predictions $\hat{p}$ to minimize the empirical ℓ_1-loss, while we have replaced the ℓ_1-loss with the ℓ_∞-loss. In the following experiment, as described in Section 4 and [31], both methods learn predictions using the online gradient descent method (OGD), where the ℓ_1- and ℓ_∞-losses are used for the method of [17] and ours, respectively.

Obtaining feasible solutions. Dinitz et al. [17] converted an infeasible prediction $\hat{p}$ into feasible p° using a greedy approximation algorithm tailored to the ℓ_1-loss. By contrast, we obtain feasible solutions p° by minimally shifting $\hat{p}$ in the all-one direction as in Section 3.1 and rounding the resulting vector to the nearest integer point.

Due to the above differences, the method of [17] and ours yield different initial feasible solutions p°, which are fed to the common bipartite-matching solver. The following experiment examines how this difference affects the number of iterations taken by the bipartite-matching solver.

Settings. We generated three random bipartite graphs (V, E) such that $V = L \cup R$ and $|L| = |R| = 5$ (i.e., $n = 10$) with three probability values $\theta = 0.5, 0.7, 0.9$ for edge creation. For each graph, we generated $T = 1000$ sets $\{w_e\}_{e \in E}$ of random edge weights by setting $w_e = 5 + \lfloor u \rceil$ for each $e \in E$, where u is drawn from the standard normal distribution. We thus created 1000 bipartite-matching instances with various edge weights for each graph. For each sequence of the 1000 instances, we iteratively predicted a dual solution $\hat{p}$ with OGD, converted $\hat{p}$ into a feasible solution p°, and solved the instance by warm-starting the bipartite-matching solver with p°. We implemented OGD based on that of [45, Section 2.4]. The original OGD uses the step size of $\frac{B}{L\sqrt{2T}}$, where B is the radius of the ℓ_2-ball that contains optimal predictions and L is the Lipschitz constant of the loss functions. In our setting, we have $B = C\sqrt{n} = n^{1.5}\|w\|_\infty$, where we set $\|w\|_\infty = 9$ since all the instances satisfied $|w_e| \le 9$ for all $e \in E$. The Lipschitz constant L was set to 1 for the ℓ_∞-loss and $\sqrt{n}$ for the ℓ_1-loss. We, however, observed that the step size of $\frac{B}{L\sqrt{2T}}$ was too large to achieve better performances than a baseline method with random initial solutions p°, which we call the cold-start baseline. Thus, we rescaled the step size as $\alpha \times \frac{B}{L\sqrt{2T}}$, where α is the scaling parameter. We let $\alpha = 0.001, 0.01, 0.1$.

Results. Figure 2 presents the cumulative number of iterations taken by the common weighted-bipartite-matching solver for ours (ℓ_∞), the method of [17] (ℓ_1), and the cold-start baseline (Cold). We conducted 10 independent trials with random initial points of OGD drawn from the n-dimensional standard normal distribution; the error band indicates the 95% confidence interval of the 10 random trials. Both ours and the method of [17] took fewer iterations than the cold-start baseline, and ours with $\alpha = 0.01, 0.1$ outperformed the method of [17]. The results suggest the practical usefulness of our DCA-based framework.



$\theta = 0.5$, $\alpha = 0.001$

$\theta = 0.5$, $\alpha = 0.01$

$\theta = 0.5$, $\alpha = 0.1$

$\theta = 0.7$, $\alpha = 0.001$

$\theta = 0.7$, $\alpha = 0.01$

$\theta = 0.7$, $\alpha = 0.1$

$\theta = 0.9$, $\alpha = 0.001$

$\theta = 0.9$, $\alpha = 0.01$

$\theta = 0.9$, $\alpha = 0.1$

Figure 2: The cumulative number of iterations taken by the bipartite-matching solver for each method. The parameter θ is the probability of edge creation, and α is the scaling parameter to control the step size of OGD. The error band indicates the 95% confidence interval over the 10 random trials.