Supplementary Material

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1 Upper Bounds

1.1 Deterministic Elicitation, Deterministic Aggregation

Theorem 1. *For $t \in [m] \setminus \{1\}$ and $\ell \in \mathbb{N}$, we have*

$$C(\textsc{PrefThreshold}_{t,\ell}) = \log\left[\binom{m}{t} \cdot (\ell+1)^t\right] = \Theta\left(t \log \frac{m(\ell+1)}{t}\right),$$
$$\mathit{dist}(\textsc{PrefThreshold}_{t,\ell}) = O\left(m^{1+2/\ell}/t\right).$$

For $t = 1$ and $\ell \in \mathbb{N}$, we have

$$C(\textsc{PrefThreshold}_{1,\ell}) = \log(m\ell),\ \mathit{dist}(\textsc{PrefThreshold}_{t,\ell}) = O\left(m^{1+1/\ell}\right).$$

Proof. It is evident that the number of possible responses that a voter can provide under $\textsc{PrefThreshold}_{t,\ell}$ is $\binom{m}{t} \cdot (\ell+1)^t$ if $t > 1$, and $m\ell$ if $t = 1$. Taking the logarithm of this gives us the desired communication complexity.

We now establish the distortion of $\textsc{PrefThreshold}_{t,\ell}$. Let $\vec{v} = (v_1, \ldots, v_n)$ be the underlying valuations of voters. For alternative $a \in A$, recall that $\text{sw}(a, \vec{v}) = \sum_{i \in N} v_i(a)$, and

$$\widehat{\text{sw}}(a) = \sum_{i \in N} \widehat{v}_i(a) = \sum_{i \in N : a \in S_i^t} \widehat{v}_i(a) = \sum_{i \in N : a \in S_i^t} U_{p_{i,a}}.$$

Let $\widehat{a} \in \arg\max_{a \in A} \widehat{\text{sw}}(a)$ be the alternative chosen by the rule, and let $a^* \in \arg\max_{a \in A} \text{sw}(a, \vec{v})$ be an alternative maximizing social welfare.

We begin by finding an upper bound on $\text{sw}(a^*, \vec{v})$ in terms of $\widehat{\text{sw}}(\widehat{a})$.

$$\begin{aligned}
\text{sw}(a^*, \vec{v}) &= \sum_{i \in N} v_i(a^*) = \sum_{i \in N : a^* \in S_i^t} v_i(a^*) + \sum_{i \in N : a^* \notin S_i^t} v_i(a^*) \\
&\le \sum_{i \in N : a^* \in S_i^t} v_i(a^*) + \sum_{i \in N : a^* \notin S_i^t} \left(\frac{\sum_{a \in S_i^t} v_i(a)}{t}\right) \\
&\le \sum_{i \in N : a^* \in S_i^t} \widehat{v}_i(a^*) + \frac{\sum_{a \in A \setminus \{a^*\}} \sum_{i \in N : a^* \notin S_i^t \wedge a \in S_i^t} \widehat{v}_i(a)}{t} \\
&\le \widehat{\text{sw}}(a^*) + \frac{\sum_{a \in A \setminus \{a^*\}} \widehat{\text{sw}}(a)}{t} \le \widehat{\text{sw}}(\widehat{a}) + \frac{(m-1) \cdot \widehat{\text{sw}}(\widehat{a})}{t} = \frac{m+t-1}{t} \cdot \widehat{\text{sw}}(\widehat{a}),
\end{aligned} \tag{1}$$

where the third transition holds because for every $i \in N$ with $a^* \notin S_i^t$ and every $a \in S_i^t$, we have $v_i(a^*) \leq v_i(a)$; the fourth transition holds because for every $i \in N$ and $a \in S_i^t$, $v_i(a) \leq \widehat{v}_i(a)$; the fifth transition follows from the definition of $\widehat{\text{sw}}$; and the sixth transition holds because $\widehat{a}$ is a maximizer of $\widehat{\text{sw}}$.

We now establish the distortion for $t > 1$. The first step is to derive an upper bound on $\widehat{\text{sw}}(\widehat{a})$ in terms of $\text{sw}(\widehat{a}, \vec{v})$. Our bucketing implies that for all $i \in N$ and $a \in S_i^t$, we have $v_i(a) \leq \widehat{v}_i(a) \leq m^{2/\ell} v_i(a) + \frac{1}{m^2}$. Using this, we can derive the following.

$$\widehat{\text{sw}}(\widehat{a}) = \sum_{i \in N : \widehat{a} \in S_i^t} \widehat{v}_i(\widehat{a}) \leq \sum_{i \in N : \widehat{a} \in S_i^t} \left(m^{2/\ell} v_i(\widehat{a}) + \frac{1}{m^2} \right) \leq m^{2/\ell} \text{sw}(\widehat{a}, \vec{v}) + \frac{n}{m^2}. \tag{2}$$

Next, we derive a lower bound on $\widehat{\text{sw}}(\widehat{a})$, which helps establish a lower bound on $\text{sw}(\widehat{a}, \vec{v})$. Note that for each voter $i \in N$, $\sum_{a \in S_i^t} v_i(a) \geq t/m$. Hence,

$$\sum_{a \in A} \widehat{\text{sw}}(a) = \sum_{i \in N} \sum_{a \in S_i^t} \widehat{v}_i(a) \geq \sum_{i \in N} \sum_{a \in S_i^t} v_i(a) \geq \frac{n \cdot t}{m}.$$

Because $\widehat{a}$ is a maximizer of $\widehat{\text{sw}}$, this yields $\widehat{\text{sw}}(\widehat{a}) \geq n \cdot t / m^2$. Substituting this into Equation (2), we get

$$\frac{n}{m^2} + \text{sw}(\widehat{a}, \vec{v}) \cdot m^{2/\ell} \geq \widehat{\text{sw}}(\widehat{a}) \geq \frac{n \cdot t}{m^2} \Rightarrow \text{sw}(\widehat{a}, \vec{v}) \geq \frac{n \cdot (t-1)}{m^2} \cdot m^{-2/\ell} \geq \frac{n}{m^2} \cdot m^{-2/\ell}. \tag{3}$$

Applying Equations (1), (2), and (3) in this order, we have

$$\begin{aligned}
\frac{\text{sw}(a^*, \vec{v})}{\text{sw}(\widehat{a}, \vec{v})} &\leq \frac{m+t-1}{t} \cdot \frac{\widehat{\text{sw}}(\widehat{a})}{\text{sw}(\widehat{a}, \vec{v})} \leq \frac{m+t-1}{t} \cdot \left(m^{2/\ell} + \frac{n}{m^2 \cdot \text{sw}(\widehat{a}, \vec{v})} \right) \\
&\leq \frac{m+t-1}{t} \cdot \left(m^{2/\ell} + m^{2/\ell} \right) \in O(m^{1+2/\ell}/t).
\end{aligned}$$

For $t = 1$, we have that for every $i \in N$ and $a \in S_i^t$, $v_i(a) \leq \widehat{v}_i(a) \leq m^{1/\ell} v_i(a)$. Hence, in Equation (2), the additive factor of n/m^2 disappears and the multiplicative factor of $m^{2/\ell}$ becomes $m^{1/\ell}$, yielding $\widehat{\text{sw}}(\widehat{a}) \leq \text{sw}(\widehat{a}, \vec{v}) \cdot m^{1/\ell}$. Similarly, Equation (3) becomes $\text{sw}(\widehat{a}, \vec{v}) \geq \frac{n}{m^2} \cdot m^{-1/\ell}$. Following the same line of proof as for the case of $t > 1$, we obtain

$$\frac{\text{sw}(a^*, \vec{v})}{\text{sw}(\widehat{a}, \vec{v})} \leq m \cdot \frac{\widehat{\text{sw}}(\widehat{a})}{\text{sw}(\widehat{a}, \vec{v})} \leq m \cdot m^{1/\ell},$$

which is the desired bound on distortion. □

1.2 Randomized Elicitation, Randomized Aggregation

Theorem 2. *For every voting rule f and $s \in [m]$, we have $C^m(\textsc{RandSubset}(f, s)) = C^s(f) + \log\lceil \log(4m) \rceil$ and dist$^m(\textsc{RandSubset}(f, s)) \leq \frac{4m}{s} \cdot$ dist$^s(f)$.*

Proof. Let $\vec{v} = (v_1, \ldots, v_n)$ denote the underlying valuations of voters. First, let us consider a fixed choice of $S \subseteq A$ with $|S| = s$. Due to our bucketing, we have that for every $i \in N$,

$$\frac{v_i(S)}{2} - \frac{1}{4m} \leq L_{p_i} \leq v_i(S). \tag{4}$$

Recall that in the input to the aggregation rule of f, we have $4m \cdot L_{p_i}$ copies of the response ρ_i of voter i. Hence, the social welfare function approximated by the aggregation rule of f is given by

$$\forall a \in S,\ \widehat{\text{sw}}(a, \vec{v}) = \sum_{i \in N} 4m \cdot L_{p_i} \cdot \frac{v_i(a)}{v_i(S)} = 4m \sum_{i \in N} v_i(a) \cdot \frac{L_{p_i}}{v_i(S)}.$$

Combining this with Equation (4), we have that for each $a \in S$,

$$\widehat{\mathrm{sw}}(a, \vec{v}) \geq 4m \sum_{i \in N} v_i(a) \cdot \left(\frac{1}{2} - \frac{1}{4m \cdot v_i(S)} \right) = 2m \cdot \mathrm{sw}(a, \vec{v}) - \sum_{i \in N} \frac{v_i(a)}{v_i(S)} \geq 2m \cdot \mathrm{sw}(a, \vec{v}) - n, \tag{5}$$

as well as

$$\widehat{\mathrm{sw}}(a, \vec{v}) \leq 4m \sum_{i \in N} v_i(a) \cdot 1 = 4m \cdot \mathrm{sw}(a, \vec{v}). \tag{6}$$

Let $\widehat{a}$ denote the alternative chosen by our rule. Because the distortion of f for choosing an alternative from S is $\mathrm{dist}^s(f)$, we have that $\mathbb{E}[\widehat{\mathrm{sw}}(\widehat{a}, \vec{v})] \geq \max_{a \in S} \widehat{\mathrm{sw}}(a, \vec{v}) / \mathrm{dist}^s(f)$. Note that so far, we have fixed S. The expectation on the left hand side is due to the fact that even for fixed S, $\widehat{a}$ can be randomized if f is randomized.

Next, we take expectation over the choice of S, and use the fact that the optimal alternative $a^* \in \arg\max_{a \in A} \mathrm{sw}(a, \vec{v})$ belongs to S with probability s/m. We obtain

$$\mathbb{E}[\widehat{\mathrm{sw}}(\widehat{a}, \vec{v})] \geq \frac{\mathbb{E}[\max_{a \in S} \widehat{\mathrm{sw}}(a, \vec{v})]}{\mathrm{dist}^s(f)} \geq \frac{\frac{s}{m} \cdot \widehat{\mathrm{sw}}(a^*, \vec{v})}{\mathrm{dist}^s(f)} \geq \frac{\frac{s}{m} \left(2m \cdot \mathrm{sw}(a^*, \vec{v}) - n\right)}{\mathrm{dist}^s(f)}, \tag{7}$$

where the final transition follows from Equation (5). On the other hand, from Equation (6), we have

$$\mathbb{E}[\widehat{\mathrm{sw}}(\widehat{a}, \vec{v})] \leq 4m \, \mathbb{E}[\mathrm{sw}(\widehat{a}, \vec{v})]. \tag{8}$$

Combining Equations (7) and (8), we have that

$$\mathrm{dist}^m(\textsc{RandSubset}(f, s)) = \frac{\mathrm{sw}(a^*, \vec{v})}{\mathbb{E}[\mathrm{sw}(\widehat{a}, \vec{v})]} \leq \frac{\mathrm{sw}(a^*, \vec{v})}{\frac{\mathrm{sw}(a^*, \vec{v})}{2} - \frac{n}{4m}} \cdot \frac{m}{s} \cdot \mathrm{dist}^s(f) \leq \frac{4m}{s} \cdot \mathrm{dist}^s(f),$$

where the final transition uses the fact that $\mathrm{sw}(a^*, \vec{v}) \geq (1/m) \cdot \sum_{a \in A} \mathrm{sw}(a, \vec{v}) = n/m$. This establishes the desired distortion bound. Since each voter answers the query of f for s alternatives and chooses one of $\lceil \log(4m) \rceil$ buckets, we get $\mathrm{C}^m(\textsc{RandSubset}(f, s)) = \mathrm{C}^s(f) + \log\lceil \log(4m) \rceil$, as desired. □

2 Lower Bounds

2.1 Direct Lower Bounds for Deterministic Elicitation

We start by establishing a straightforward lemma. Recall that for a valuation $v \in \Delta^m$, $\mathrm{supp}(v)$ denotes the support of v.

Lemma 1. *Let f be a voting rule which uses deterministic elicitation and deterministic aggregation. Let q^* be the query used by f. If some compartment of q^* contains two valuations v^1 and v^2 such that $\mathrm{supp}(v^1) \cap \mathrm{supp}(v^2) = \emptyset$, then the distortion of f is unbounded.*

Proof. Suppose compartment P contains valuations v^1 and v^2 such that $\mathrm{supp}(v^1) \cap \mathrm{supp}(v^2) = \emptyset$. Let $\widehat{a}$ be the alternative returned by f when all voters pick compartment P. Pick $t \in \{1, 2\}$ such that $\widehat{a} \notin \mathrm{supp}(v^t)$. Note that $v^t(\widehat{a}) = 0$, but there exists $a^* \in \mathrm{supp}(v^t)$ such that $v^t(a^*) > 0$.

Define voter valuations $\vec{v} = (v_1, \ldots, v_n)$ such that $v_i = v^t$ for each $i \in N$. This yields $\mathrm{sw}(\widehat{a}, \vec{v}) = 0$ and $\mathrm{sw}(a^*, \vec{v}) > 0$, which implies that f must have infinite distortion. □

Theorem 3. *Every voting rule that has deterministic elicitation, deterministic aggregation, and communication complexity strictly less than $\log m$ has unbounded distortion.*

Proof. We need the following definition. For $a \in A$, we say that the *unit valuation* corresponding to a is the valuation $v^a \in \Delta^m$ for which $v^a(a) = 1$. Let f be a voting rule that has deterministic elicitation and deterministic aggregation, and let $\mathrm{C}(f) < \log m$. Hence, the query used by f must partition Δ^m into less than m compartments.

Because there are m unit valuations, by the pigeonhole principle there must exist distinct $a, b \in A$ such that v^a and v^b belong to the same compartment. Because $\mathrm{supp}(v^a) \cap \mathrm{supp}(v^b) = \emptyset$, Lemma 1 implies that the distortion of f must be infinite. □

Theorem 4. *Let f be a voting rule which uses deterministic elicitation and has $\mathrm{C}(f) \le \log m$. If f uses deterministic aggregation, then dist$(f) = \Omega(m^2)$. If f uses randomized aggregation, then dist$(f) = \Omega(m)$.*

Proof. Let f be a voting rule which has deterministic elicitation and $\mathrm{C}(f) \le \log m$. As argued above, we can assume $\mathrm{C}(f) = \log m$ without loss of generality. Hence, the query q^* used by f partitions Δ^m into m compartments. Let $\mathcal{P} = (P_1, \ldots, P_m)$ denote the set of compartments. If f has unbounded distortion, we are done. Suppose f has bounded distortion.

Due to Lemma 1, each of m unit vectors must belong to a different compartment. Since there are m compartments, we identify each compartment by the unit valuation it contains. For $a \in A$, let P^a denote the compartment containing unit valuation v^a. Before we construct adversarial valuations, we need to define *low valuations* and *high valuations*.

Low valuations: We say that a valuation $v \in \Delta^m$ is a *low valuation* if $|\mathrm{supp}(v)| = m/5$ and $v(a) = 5/m$ for every $a \in \mathrm{supp}(v)$. Let $\Delta^{m,\text{low}}$ denote the set of all low valuations. Due to Lemma 1, we have

$$v \in \Delta^{m,\text{low}} \cap P^a \Rightarrow a \in \mathrm{supp}(v) \wedge v(a) = \frac{5}{m}. \tag{9}$$

Let $\mathcal{L} = \{P \in \mathcal{P} : P \cap \Delta^{m,\text{low}} \neq \emptyset\}$ be the set of compartments containing at least one low valuation, and $A^{\mathcal{L}} = \{a \in A : P^a \in \mathcal{L}\}$ be the set of alternatives corresponding to these compartments.

We claim that $|A^{\mathcal{L}}| = |\mathcal{L}| \ge 4m/5 + 1$. Suppose for contradiction that $|A^{\mathcal{L}}| \le 4m/5$. Then, $|A \setminus A^{\mathcal{L}}| \ge m/5$. Hence, there exists a low valuation $v \in \Delta^{m,\text{low}}$ such that $\mathrm{supp}(v) \subseteq A \setminus A^{\mathcal{L}}$. Let $a \in A$ be the alternative for which $v \in P^a$. Because P^a contains a low valuation, $a \in A^{\mathcal{L}}$ by definition. Thus, the construction of v ensures $v(a) = 0$. We have $v \in \Delta^{m,\text{low}} \cap P^a$ with $v(a) = 0$, which contradicts Equation (9). Hence, $|A^{\mathcal{L}}| \ge 4m/5 + 1$.

High valuations: We say that a valuation $v \in \Delta^m$ is a *high valuation* if $|\mathrm{supp}(v)| = 2$ and $v(a) = 1/2$ for each $a \in \mathrm{supp}(v)$. Let $\Delta^{m,\text{high}}$ denote the set of high valuations. Note that $|\Delta^{m,\text{high}}| = \binom{m}{2}$. Similarly to the case of low valuations, we can apply Lemma 1, and obtain that

$$v \in \Delta^{m,\text{high}} \cap P^a \Rightarrow a \in \mathrm{supp}(v) \wedge v(a) = \frac{1}{2}. \tag{10}$$

For $a \in A$, let $\mathcal{H}^a = \{P \in \mathcal{L} : \exists v \in \Delta^{m,\text{high}} \cap P \text{ s.t. } a \in \mathrm{supp}(v)\}$. In words, $\mathcal{H}^a$ is the set of compartments from $\mathcal{L}$ which contain at least one high valuation v for which $v(a) = 1/2$. Let $A^{\text{high}} = \{a \in A : |\mathcal{H}^a| \ge m/5\}$. We claim that $|A^{\text{high}}| \ge m/6$.

Suppose this is not true. Let $B = |A \setminus A^{\text{high}}|$. Then, $|B| \ge 5m/6$. Consider $a \in B$. Each of the $m-1$ high valuations which contain a in their support must belong to some compartments in $\mathcal{H}^a \cup (\mathcal{P} \setminus \mathcal{L})$. Since $|\mathcal{H}^a| \le m/5 - 1$ for $a \in B$ and $|\mathcal{P} \setminus \mathcal{L}| \le m/5 - 1$, the $m-1$ high valuations containing a in their support are distributed across at most $2m/5 - 2$ compartments. However, due to Lemma 1, a compartment other than P^a can contain at most one high valuation with a in its support. Hence, P^a must contain at least $m - 1 - (2m/5 - 3) = 3m/5 + 2$ high valuations. Thus, we have established that $|B| \ge 5m/6$ and for each $a \in B$, P^a contains at least $3m/5 + 2$ high valuations. Thus, the number of high valuations is at least $(5m/6) \cdot (3m/5 + 2) > m^2/2 > \binom{m}{2}$, which is a contradiction. Thus, we have $|A^{\text{high}}| \ge m/6$.

We are now ready to prove the desired result for both deterministic and randomized aggregation.

Voter responses: When responding to the query q^*, suppose each compartment $P \in \mathcal{L}$ is picked by a set N_P of $n/|\mathcal{L}|$ voters.

Deterministic aggregation: Let $\widehat{a}$ denote the alternative picked by f. We claim that $\widehat{a} \in A^{\mathcal{L}}$. If $\widehat{a} \notin A^{\mathcal{L}}$, consider voter valuations $\vec{v}$ such that every voter i picking compartment $P^a \in \mathcal{L}$ has valuation $v_i = v^a$. Since $\widehat{a} \notin A^{\mathcal{L}}$, we have $v_i(\widehat{a}) = 0$ for each $i \in N$, i.e., $\mathrm{sw}(\widehat{a}, \vec{v}) = 0$. Since $\mathrm{sw}(a, \vec{v}) > 0$ for some $a \in A$, f has infinite distortion, which is a contradiction. Thus, we must have $\widehat{a} \in A^{\mathcal{L}}$.

Now, let us construct the voter valuations as follows. Pick a low valuation $\widehat{v} \in P^{\widehat{a}} \cap \Delta^{m,\text{low}}$, which exists because we have established $\widehat{a} \in A^{\mathcal{L}}$. Note that $\widehat{v}(\widehat{a}) = 5/m$. For each $i \in N_{P^{\widehat{a}}}$, let $v_i = \widehat{v}$.

Pick $a^* \in A^{\text{high}} \setminus \{\widehat{a}\}$. Let $\bar{P}$ be the compartment containing the high valuation under which both $\widehat{a}$ and a^* have utility $1/2$. For each $P \in \mathcal{H}^{a^*} \setminus \{P^{\widehat{a}}, \bar{P}\}$, and for each $i \in N_P$, let v_i be the high valuation in P such that $v_i(a^*) = 1/2$ and $v_i(\widehat{a}) = 0$. For every other $P^a \in \mathcal{L}$ and every $i \in N_{P^a}$, let $v_i = v^a$.

Observe that under these valuations, $\text{sw}(\widehat{a}, \vec{v}) = \Theta(n/m^2)$, whereas, since $|\mathcal{H}^{a^*}| \geq m/5$ and $|\mathcal{L}| \leq |\mathcal{P}| = m$, $\text{sw}(a^*, \vec{v}) = \Theta(n)$. We conclude that $\text{dist}(f) = \Omega(m^2)$.

Randomized aggregation: Note that f must select at least one alternative $a^* \in A^{\text{high}}$ with probability at most $1/|A^{\text{high}}| \leq 6/m$. Construct voter valuations such that for every $P \in \mathcal{H}^{a^*}$ and every $i \in N_P$, v_i is the high valuation under which $v_i(a^*) = 1/2$. For every $P^a \in \mathcal{L} \setminus \mathcal{H}^{a^*}$, and for every $i \in N_{P^a}$, let $v_i = v^a$. It holds that $\text{sw}(a^*, \vec{v}) = \Theta(n)$ (as before), whereas $\text{sw}(a, \vec{v}) = O(n/m)$ for every $a \in A \setminus \{a^*\}$. Because f selects a^* with probability at most $6/m$, we have $\mathbb{E}_{\widehat{a} \sim f(\vec{v})}[\text{sw}(\widehat{a}, \vec{v})] = O(n/m)$, implying $\text{dist}(f) = \Omega(m)$, as required. □

2.2 Lower Bound for Plurality Votes

In this section, we show that eliciting plurality votes (whereby each voter picks her most favorite alternative) results in $\Omega(m)$ distortion, even with randomized aggregation. This is implied by Theorem 4, which proves this for any elicitation that has at most $\log m$ communication complexity. However, for the special case of plurality votes, we can provide a much simpler proof.

Theorem 5. *Every voting rule which elicits plurality votes incurs $\Omega(m)$ distortion.*

Proof. For simplicity, let the number of voters n be divisible by the number of alternatives m. Consider an input profile in which the set of voters N is partitioned into equal-size sets $\{N_a\}_{a \in A}$ such that for each $a \in A$, a is the most favorite alternative of every voter in N_a.

Take any voting rule f. It must return some alternative $a^* \in A$ with probability at most $1/m$. Now, construct adversarial valuations of voters $\vec{v}$ as follows.

- For all $i \in N_{a^*}$, $v_i(a^*) = 1$ and $v_i(a) = 0$ for all $a \in A \setminus \{a^*\}$.
- For all $\widehat{a} \in A \setminus \{a^*\}$ and $i \in N_{\widehat{a}}$, $v_i(\widehat{a}) = v_i(a^*) = 1/2$ and $v_i(a) = 0$ for all $a \in A \setminus \{a^*, \widehat{a}\}$.

Under these valuations, we have $\text{sw}(a^*, \vec{v}) \geq n/2$, while $\text{sw}(a, \vec{v}) = (n/m) \cdot (1/2)$ for every $a \in A \setminus \{a^*\}$. Hence, the distortion of f is

$$\text{dist}(f) \geq \frac{\text{sw}(a^*, \vec{v})}{\frac{1}{m}\text{sw}(a^*, \vec{v}) + \frac{m-1}{m}\frac{n}{2m}} = \Omega(m),$$

where the final transition holds when substituting $\text{sw}(a^*, \vec{v}) \geq n/2$. □

3 Lower Bounds Through Multi-Party Communication Complexity

3.1 Lower Bound on the Communication Complexity of $\textbf{FDISJ}_{m,s,t}$

In this section, we prove a lower bound on the communication complexity of multi-party fixed-size set-disjointness. Let us recall Theorem 6.

Theorem 6. *For a sufficiently small constant $\delta > 0$ and $m \geq (3/2)st$, $R_\delta(\text{FDISJ}_{m,s,t}) = \Omega(s)$.*

Proof. Suppose there is a δ-error protocol Π for $\text{FDISJ}_{m,s,t}$. We use it to construct a 2δ-error protocol Π' for $\text{DISJ}_{m',t'}$, where $m' = st/2$ and $t' = 2t$.

Consider an instance $(S'_1, \ldots, S'_{t'})$ of $\text{DISJ}_{m',t'}$. Due to the promise that the sets are either pairwise disjoint or pairwise uniquely intersecting, we have that at most one of the m' elements can appear in multiple sets. Hence, $\sum_{i=1}^{t'} |S'_i| \leq m' - 1 + t'$. Due to the pigeonhole principle, there must exist at least $t'/2 = t$ sets of size at most $2(m' + t' - 1)/t'$. Note that

$$\frac{2(m' + t' - 1)}{t'} = \frac{st/2 + 2t - 1}{t} = \frac{s}{2} + 2 - \frac{1}{t} \leq s.$$

The final transition holds when $s \geq 4$. When $s < 4$, the lower bound of $\Omega(s)$ is trivial.

Consider a set of t players $\{i_1, \ldots, i_t\}$ such that $|S'_{i_k}| \leq s$ for each $k \in [t]$. Suppose that each such player i_k adds $s - |S'_{i_k}|$ unique elements to S'_{i_k} and creates a set S_{i_k} with $|S_{i_k}| = s$. The number of unique elements required is at most st. Hence, the total number of elements used in sets $S_{i_1}, \ldots, S_{i_t}$ is at most $m' + st = (3/2)st \leq m$. In other words, these sets can be created using the m-element universe of $\text{FDISJ}_{m,s,t}$. Further, it is easy to check that sets $S_{i_1}, \ldots, S_{i_t}$ are pairwise disjoint (resp. pairwise uniquely intersecting) if and only if sets $S'_1, \ldots, S'_{t'}$ are pairwise disjoint (resp. pairwise uniquely intersecting). Thus, $(S_{i_1}, \ldots, S_{i_t})$ is a valid instance of $\text{FDISJ}_{m,s,t}$ and has the same solution as the instance $(S'_1, \ldots, S'_{t'})$ of $\text{DISJ}_{m',t'}$.

Our goal is to construct a 2δ-error protocol Π' for $\text{DISJ}_{m',t'}$ that solves $(S'_1, \ldots, S'_{t'})$ by effectively running the given δ-error protocol Π for $\text{FDISJ}_{m,s,t}$ on $(S'_{i_1}, \ldots, S'_{i_t})$. We could ask each player i to report a single bit indicating whether $|S'_i| \leq s$, determine t players for which this holds, and then run Π on them. However, this would add a t'-bit overhead. Instead, we would like to bound the overhead in terms of the communication cost of Π, denoted $|\Pi|$, which could be significantly smaller.

This is achieved as follows. We first order the players according to a uniformly random permutation σ. Then, we simulate Π. Every time Π wants to interact with a new player, we ask players that we have not interacted with so far, in the order in which they appear in σ, whether their sets have size at most s, until we find one such player. Then, we let Π interact with this player. Protocol Π' terminates naturally when protocol Π terminates (and returns the same answer), but terminates abruptly if, at any point, it has interacted with more than $2|\Pi|/\delta$ players (and returns an arbitrary answer).

Note that $|\Pi|$ is also an upper bound with the number of players that Π needs to interact with. Let X be the smallest index such that there are at least $|\Pi|$ players having sets of size at most s among the first X players in σ. Then, because at least half of the players have sets of size at most s, we have $\mathbb{E}[X] \leq 2 \cdot |\Pi|$. Due to Markov's inequality, we have that $\Pr[X > 2|\Pi|/\delta] \leq \delta$. Hence, the probability that Π' terminates abruptly is at most δ. When it does not terminate abruptly, it returns the wrong answer with probability at most δ (as Π is a δ-error protocol). Hence, due to the union bound, we conclude that Π' is a 2δ-error protocol for $\text{DISJ}_{m',t'}$.

Finally, we have that $|\Pi'| \leq 2|\Pi|/\delta + |\Pi| = |\Pi|(1 + 2/\delta)$. When δ is sufficiently small, Gronemeier [1] showed that $|\Pi'| \geq R_{2\delta}(\text{DISJ}_{m',t'}) = \Omega(m'/t') = \Omega(s)$. Hence, we have that $|\Pi| = \Omega(s)$. Since this holds for every δ-error protocol Π for $\text{FDISJ}_{m,s,t}$, we have $R_\delta(\text{FDISJ}_{m,s,t}) = \Omega(s)$. $\square$

3.2 Lower Bounds on the Communication Complexity of Voting Rules

Theorem 7. *For a voting rule f with elicitation rule Π_f and dist$(f) = d$, the following hold.*

- *If Π_f is deterministic, then $C(f) \geq \Omega\left(m/d^2\right)$.*
- *If Π_f is randomized, then $C(f) \geq \Omega\left(m/d^3\right)$.*

Proof. Let $t = 2 \cdot \text{dist}(f)$ and $s = 2m/(3t)$. Note that for these parameters, we have $R_\delta(\text{FDISJ}_{m,s,t}) = \Omega(s)$ from Theorem 6.

Consider an input $(S_1, \ldots, S_t)$ to $\text{FDISJ}_{m,s,t}$ with a universe U of size m. Let us create an instance of the voting problem with a set of n voters N and a set of m alternatives A. Each alternative in A corresponds to a unique element of U. Partition the set of voters N into t equal-size buckets $\{N_1, \ldots, N_t\}$. Here, bucket N_i corresponds to player i, and consists of n/t voters that each have valuation v^{S_i} given by $v^{S_i}(a) = 1/s$ for each $a \in S_i$ and $v^{S_i}(a) = 0$ for each $a \notin S_i$. Let $\vec{v}$ denote the resulting profile of voter valuations. Note that under these valuations, $\text{sw}(a, \vec{v}) = \frac{n}{ts} \sum_{i=1}^{t} \mathbb{1}[a \in S_i]$, where $\mathbb{1}$ is the indicator variable. Due to the promise that an element either belongs to at most one set or belongs to every set, we have $\text{sw}(a, \vec{v}) \in \{0, n/(ts), n/s\}$. We say that a is a "good" alternative if $\text{sw}(a, \vec{v}) = n/s$ and a "bad" alternative otherwise.

We define two processes that will help covert our voting rule f into a protocol for $\text{FDISJ}_{m,s,t}$.

Process E: In this process, we ask each player i to respond to the query posed by voting rule f (possibly selected in a randomized manner) according to valuation v^{S_i}. We note that this requires a total of $t \cdot \text{C}(f)$ bits of communication from the players.

Process A: We take players' responses from process E, create n/t copies of the response of each player, and pass the resulting profile as input to the aggregation rule Γ_f to obtain the returned alternative $\widehat{a}$ (possibly selected in a randomized manner). We end the process by determining if $\widehat{a}$ is a good alternative or a bad alternative. This requires eliciting 2 extra bits of information: we can ask any two players i and j whether their sets contain $\widehat{a}$, and due to the promise of $\mathrm{FDISJ}_{m,s,t}$, we know that $\widehat{a}$ is good if and only if it belongs to both S_i and S_j.

Knowing whether $\widehat{a}$ is good or bad is useful for solving the given instance of $\mathrm{FDISJ}_{m,s,t}$ due to the following reason.

1. If $(S_1, \ldots, S_t)$ is a "NO input", then we know that every alternative is a bad alternative. Hence, $\mathrm{sw}(a, \vec{v}) \leq (n/t) \cdot (1/s) = n/(ts)$ for each $a \in A$. In particular, this implies $\mathrm{sw}(\widehat{a}, \vec{v}) \leq n/(ts)$ with probability 1.
2. If $(S_1, \ldots, S_t)$ is a "YES input", then there exists a unique good alternative $a^* \in A$ with $\mathrm{sw}(a^*, \vec{v}) = n/s$, and every other alternative a is a bad alternative with $\mathrm{sw}(a, \vec{v}) \leq n/(ts)$. Because $\mathrm{dist}(f) = t/2$, we have that $\mathbb{E}[\mathrm{sw}(\widehat{a}, \vec{v})] \geq \frac{n/s}{t/2} = \frac{2n}{ts}$. This implies that $\Pr[\mathrm{sw}(\widehat{a}, \vec{v}) = n/s] = \Pr[\widehat{a} = a^*] \geq 1/t$ because if $\Pr[\widehat{a} = a^*] < 1/t$, then $\mathbb{E}[\mathrm{sw}(\widehat{a}, \vec{v})] < (1/t) \cdot (n/s) + 1 \cdot n/(ts) = 2n/(ts)$, which is a contradiction.

We are now ready to use f to construct a protocol for $\mathrm{FDISJ}_{m,s,t}$, and use Theorem 6 to derive a lower bound on $\mathrm{C}(f)$. We consider two cases depending on whether the elicitation rule Π_f is deterministic or randomized.

1. *Deterministic elicitation*: In this case, we run process E once and then run process A $t\ln(1/\delta)$ times. In a NO input, we always get a bad alternative. In a YES input, each run of process A returns a good alternative with probability at least $1/t$. Hence, the probability that we get a good alternative at least once is at least $1-(1-1/t)^{t\ln(1/\delta)} \geq 1-\delta$. Hence, this is a δ-error protocol for $\mathrm{FDISJ}_{m,s,t}$ which requires $t \cdot \mathrm{C}(f) + t\ln(1/\delta) \cdot 2$ bits of total communication from the players. Using Theorem 6, we have that $t \cdot (\mathrm{C}(f) + 2\ln(1/\delta)) = \Omega(s)$. Using $s = 2m/(3t)$ and $t = 2d$, we have $\mathrm{C}(f) = \Omega(m/d^2)$.
2. *Randomized elicitation*: In this case, we run E once followed by running A once. And we repeat this entire process $t\ln(1/\delta)$ times. Note that we need to repeat process E because the elicitation is also randomized. Like in the previous case, we always get a bad alternative in a NO input, and get a good alternative with probability at least $1/t$ in each run in a YES input. Hence, in a YES input, we get a good alternative in at least one run with probability at least $1-(1-1/t)^{t\ln(1/\delta)} \geq 1-\delta$. This results in a δ-error protocol for $\mathrm{FDISJ}_{m,s,t}$ which requires $t\ln(1/\delta) \cdot (t \cdot \mathrm{C}(f) + 2)$ bits of total communication from the players. Using Theorem 6, we have $t\ln(1/\delta) \cdot (t \cdot \mathrm{C}(f) + 2) = \Omega(s)$. Using $s = 2m/(3t)$ and $t = 2d$, we have $\mathrm{C}(f) = \Omega(m/d^3)$.

These are the desired lower bounds on $\mathrm{C}(f)$. □

References

[1] A. Gronemeier. Asymptotically optimal lower bounds on the NIH-multi-party information complexity of the AND-function and disjointness. In *Proceedings of the 26th International Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 505–516, 2009.