
Correlation Clustering with Local Objectives

Sanchit Kalhan

Konstantin Makarychev

Timothy Zhou

Abstract

Correlation Clustering is a powerful graph partitioning model that aims to cluster items based on the notion of similarity between items. An instance of the Correlation Clustering problem consists of a graph G (not necessarily complete) whose edges are labeled by a binary classifier as “similar” and “dissimilar”. An objective which has received a lot of attention in literature is that of minimizing the number of disagreements: an edge is in disagreement if it is a “similar” edge and is present across clusters or if it is a “dissimilar” edge and is present within a cluster. Define the disagreements vector to be an n dimensional vector indexed by the vertices, where the v -th index is the number of disagreements at vertex v . Recently, Puleo and Milenkovic (ICML ’16) initiated the study of the Correlation Clustering framework in which the objectives were more general functions of the disagreements vector. In this paper, we study algorithms for minimizing ℓ_q norms ($q \geq 1$) of the disagreements vector for both arbitrary and complete graphs. We present the first known algorithm for minimizing the ℓ_q norm of the disagreements vector on arbitrary graphs and also provide an improved algorithm for minimizing the ℓ_q norm ($q \geq 1$) of the disagreements vector on complete graphs. We also study an alternate cluster-wise local objective introduced by Ahmadi, Khuller and Saha (IPCO ’19), which aims to minimize the maximum number of disagreements associated with a cluster. We also present an improved $(2 + \varepsilon)$ -approximation algorithm for this objective. Finally, we compliment our algorithmic results for minimizing the ℓ_q norm of the disagreements vector with some hardness results.

1 Introduction

A basic task in machine learning is that of clustering items based on the similarity between them. This task can be elegantly captured by Correlation Clustering, a clustering framework first introduced by Bansal et al. [2004]. In this model, we are given access to items and the *similarity/dissimilarity* between them in the form of a graph G on n vertices. The edges of G represent whether the items are *similar* or *dissimilar* and are labelled as (“+”) and (“−”) respectively. The goal is to produce a clustering that agrees with the labeling of the edges as much as possible, i.e., to group positive edges in the same cluster and place negative edges across different clusters (a positive edge that is present across clusters or a negative edge that is present within the same cluster is said to be in disagreement). The Correlation Clustering problem can be viewed as an agnostic learning problem, where we are given noisy examples and the task is to fit a hypothesis as best as possible to these examples. Co-reference resolution (see e.g., Cohen and Richman [2001, 2002]), spam detection (see e.g., Ramachandran et al. [2007], Bonchi et al. [2014]) and image segmentation (see e.g., Wirth [2017]) are some of the applications to which Correlation Clustering has been applied to in practice.

This task is made trivial if the labeling given is consistent (transitive): if (u, v) and (v, w) are similar, then (u, w) is similar for all vertices u, v, w in G (the connected components on similar edges would give an optimal clustering). Instead, it is assumed that the given labeling is inconsistent, i.e., it is possible that (u, w) are dissimilar even though (u, v) and (v, w) are similar. For such a triplet u, v, w , every possible clustering incurs a disagreement on at least one edge and thus, no perfect clustering

exists. The optimal clustering is the one which minimizes the disagreements. Moreover, as the number of clusters is not predefined, the optimal clustering can use anywhere from 1 to n clusters.

Minimizing the total weight of edges in disagreement is the objective that has received the most consideration in literature. Define the disagreements vector be an n dimensional vector indexed by the vertices where the v -th coordinate equals the number of disagreements at v . Thus, minimizing the total number of disagreements is equivalent to minimizing the ℓ_1 norm of the disagreements vector. Puleo and Milenkovic [2016] initiated the study of local objectives in the Correlation Clustering framework. They focus on complete graphs and study the minimization of ℓ_q norms ($q \geq 1$) of the disagreements vector – for which they provided a 48-approximation algorithm. Charikar, Gupta, and Schwartz [2017] gave an improved 7-approximation algorithm for minimizing ℓ_q disagreements on complete graphs. They also studied the problem of minimizing the ℓ_∞ norm of the disagreements vector (also known as Min Max Correlation Clustering) for arbitrary graphs, for which they provided a $O(\sqrt{n})$ -approximation.

For higher values of q (particularly $q = \infty$), a clustering optimized for minimizing the ℓ_q norm prioritizes reducing the disagreements at vertices that are worst off. Thus, such metrics are very unforgiving in most cases as it is possible that in the optimal clustering there is only one vertex with high disagreements while every other vertex has low disagreements. Hence, one is forced to infer the most pessimistic picture about the overall clustering. The ℓ_2 norm is a solution to this tension between the ℓ_1 and ℓ_∞ objectives. The ℓ_2 norm of the disagreements vector takes into account the disagreements at each vertex while also penalizing the vertices with high disagreements more heavily. Thus, a clustering optimized for the minimum ℓ_2 norm gives a more balanced clustering as it takes into consideration both the global and local picture.

Recently, Ahmadi, Khuller, and Saha [2019b] introduced an alternative min max objective for correlation clustering (which we call AKS min max objective). For a cluster $C \subseteq V$, let us refer to similar edges with exactly one endpoint in C and dissimilar edges with both endpoints in C as edges in disagreements with respect to C . We call the weight of all edges in disagreement with C the cost of C . Then, the AKS min max objective asks to find a clustering C_1, \dots, C_T that minimizes the maximum cost C_i . Ahmadi et al. [2019b] gave an $O(\log n)$ -approximation algorithm for this objective. Ahmadi, Galhotra, Khuller, Saha, and Schwartz [2019a] improved the approximation factor to $O(\sqrt{\log n \cdot \max\{\log |E^-|, \log(k)\}})$.

Our contributions. In this paper, we provide positive and negative results for Correlation Clustering with the ℓ_q objective. We first study the problem of minimizing disagreements on arbitrary graphs. We present the first approximation algorithm minimizing any ℓ_q norm ($q \geq 1$) of the disagreements vector.

Theorem 1.1. *There exists a polynomial time $O(n^{\frac{1}{2} - \frac{1}{2q}} \cdot \log^{\frac{1}{2} + \frac{1}{2q}} n)$ -approximation algorithm for the minimum ℓ_q disagreements problem on general weighted graphs.*

For the ℓ_2 objective, the above algorithm leads to an approximation ratio of $\tilde{O}(n^{1/4})$, thus providing the first known approximation ratio for optimizing the clustering for this version of the objective. Note that the above algorithm matches the best approximation guarantee of $O(\log n)$ for the classical objective of minimizing the ℓ_1 norm of the disagreements vector. For the ℓ_∞ norm, our algorithm matches the guarantee of the algorithm by Charikar, Gupta, and Schwartz [2017] up to log factors. Fundamental combinatorial optimization problems like *Multicut*, *Multiway Cut* and *s-t Cut* can be framed as special cases of Correlation Clustering. Thus, Theorem 1.1 leads to the first known algorithms for *Multicut*, *Multiway Cut* and *s-t Cut* with the ℓ_q objective when $q \neq 1$ and $q \neq \infty$. We can also use the algorithm from Theorem 1.1 to obtain $O(n^{\frac{1}{2} - \frac{1}{2q}} \cdot \log^{\frac{1}{2} + \frac{1}{2q}} n)$ bi-criteria approximation for Min k -Balanced Partitioning with the ℓ_q objective (we omit details here).

Next, we study the case of complete graphs. For this case, we present an improved 5-approximation algorithm for minimizing any ℓ_q norm ($q \geq 1$) of the disagreements vector.

Theorem 1.2. *There exists a polynomial time 5-approximation algorithm for the minimum ℓ_q disagreements problem on complete graphs.*

We also study the case of complete bipartite graphs where disagreements need to be bounded for only one side of the bipartition, and not the whole vertex set. We give an improved 5-approximation algorithm for minimizing any ℓ_q norm ($q \geq 1$) of the disagreements vector.

Theorem 1.3. *There exists a polynomial time 5-approximation algorithm for the minimum ℓ_q disagreements problem on complete bipartite graphs where disagreements are measured for only one side of the bipartition.*

In this paper, we also consider the AKS min max objective. For this objective, we give a $(2 + \varepsilon)$ -approximation algorithm, which improves the approximation ratio of $O(\sqrt{\log n \cdot \max\{\log |E^-|, \log(k)\}})$ given by Ahmadi, Galhotra, Khuller, Saha, and Schwartz [2019a].

Theorem 1.4. *There exists a polynomial time $(2 + \varepsilon)$ -approximation algorithm for the AKS min max problem on arbitrary graphs.*

Finally, in the full version of this paper (see supplemental materials), we present an integrality gap of $\Omega(n^{\frac{1}{2} - \frac{1}{2q}})$ for minimum ℓ_q $s - t$ cut and prove a hardness of approximation of 2 for minimum ℓ_∞ $s - t$ cut.

Previous work. Bansal, Blum, and Chawla [2004] showed that it is NP-hard to find a clustering that minimizes the total disagreements, even on complete graphs. They give a constant-factor approximation algorithm to minimize disagreements and a PTAS to maximize agreements on complete graphs. For complete graphs, Ailon, Charikar, and Newman [2008] presented a randomized algorithm with an approximation guarantee of 3 to minimize total disagreements. They also gave a 2.5 approximation algorithm based on LP rounding. This factor was improved to slightly less than 2.06 by Chawla, Makarychev, Schramm, and Yaroslavtsev [2015]. Since, the natural LP is known to have an integrality gap of 2, the problem of optimizing the classical objective is almost settled with respect to the natural LP. For arbitrary graphs, the best known approximation ratio is $O(\log n)$ (see Charikar, Guruswami, and Wirth [2003], Demaine, Emanuel, Fiat, and Immorlica [2006]). Assuming the Unique Games Conjecture, there is no constant-factor approximation algorithm for minimizing ℓ_1 disagreements on arbitrary graphs (see Chawla et al. [2006]). Puleo and Milenkovic [2016] first studied Correlation Clustering with more local objectives. For minimizing ℓ_q ($q \geq 1$) norms of the disagreements vector on complete graphs, their algorithm achieves an approximation guarantee of 48. This was improved to 7 by Charikar, Gupta, and Schwartz [2017]. Charikar et al. [2017] also studied the problem of minimizing the ℓ_∞ norm of the disagreements vector on general graphs. They showed that the natural LP/SDP has an integrality gap of $n/2$ for this problem and provided a $O(\sqrt{n})$ -approximation algorithm for minimum ℓ_∞ disagreements. Puleo and Milenkovic [2016] also initiated the study of minimizing the ℓ_q norm of the disagreements vector (for one side of the bipartition) on complete bipartite graphs. They presented a 10-approximation algorithm for this problem, which was improved to 7 by Charikar, Gupta, and Schwartz [2017]. Recently, Ahmadi et al. [2019b] studied an alternative objective for the correlation clustering problem. Motivated by creating balanced communities for problems such as image segmentation and community detection in social networks, they propose a new cluster-wise min-max objective. This objective minimizes the maximum weight of edges in disagreement associated with a cluster, where an edge is in disagreement with respect to a cluster if it is a similar edge and has exactly one end point in the cluster or if it is a dissimilar edge and has both its endpoints in the cluster. They gave an $O(\sqrt{\log n \cdot \max\{\log |E^-|, \log(k)\}})$ -approximation algorithm for this objective. Moreover, they give a $O(r^2)$ -approximation algorithm for graphs that exclude a $K_{r,r}$ minor, and a 14-approximation algorithm for complete graphs.

2 Preliminaries

We now formally define the Correlation Clustering with ℓ_q objective problem. We will need the following definition. Consider a set of points V and two disjoint sets of edges on V : positive edges E^+ and negative edges E^- . We assume that every edge has a weight w_{uv} . For every partition \mathcal{P} of V , we say that a positive edge is in disagreement with \mathcal{P} if the endpoints u and v belongs to different parts of \mathcal{P} ; and a negative edge is in disagreement with \mathcal{P} if the endpoints u and v belongs to the same part of \mathcal{P} . The vector of disagreements, denoted by $\text{disagree}(\mathcal{P}, E^+, E^-)$, is a $|V|$ dimensional vector indexed by elements of V . Its coordinate u equals

$$\text{disagree}_u(\mathcal{P}, E^+, E^-) = \sum_{v: (u,v) \in E^+ \cup E^-} w_{uv} \mathbb{1}((u, v) \text{ is in disagreement with } \mathcal{P}).$$

$$\begin{aligned}
& \text{minimize} && \max \left(\|y\|_q, \left(\sum_{u \in V} z_u \right)^{\frac{1}{q}} \right) && \text{(P)} \\
& \text{subject to} && y_u = \sum_{v:(u,v) \in E^+} w_{uv} x_{uv} + \sum_{v:(u,v) \in E^-} w_{uv} (1 - x_{uv}) && \text{for all } u \in V && \text{(P1)} \\
& && z_u = \sum_{v:(u,v) \in E^+} w_{uv}^q x_{uv} + \sum_{v:(u,v) \in E^-} w_{uv}^q (1 - x_{uv}) && \text{for all } u \in V && \text{(P2)} \\
& && x_{v_1 v_2} + x_{v_2 v_3} \geq x_{v_1 v_3} && \text{for all } v_1, v_2, v_3 \in V && \text{(P3)} \\
& && x_{uv} = x_{vu} && \text{for all } u, v \in V && \text{(P4)} \\
& && x_{uv} \in [0, 1] && \text{for all } u, v \in V && \text{(P5)}
\end{aligned}$$

Figure 3.1: Convex relaxation for Correlation Clustering with $\min \ell_q$ objective for $q < \infty$.

That is, $\text{disagree}_u(\mathcal{P}, E^+, E^-)$ is the weight of disagreeing edges incident to u . We similarly define a cut vector for a set of edges E :

$$\text{cut}_u(\mathcal{P}, E) = \sum_{v:(u,v) \in E} w_{uv} \mathbf{1}(u \text{ and } v \text{ are separated by } \mathcal{P}).$$

We use the standard definition for the ℓ_q norm of a vector x : $\|x\|_q = (\sum_u x_u^q)^{\frac{1}{q}}$ and $\|x\|_\infty = \max_u x_u$. For a partition \mathcal{P} , we denote by $\mathcal{P}(u)$ the piece that contains vertex u .

Definition 1. *In the Correlation Clustering problem with ℓ_q objective, we are given a graph G on a set V with two disjoint sets of edges E^+ and E^- and a set of weights w_{uv} . The goal is find a partition \mathcal{P} that minimizes the ℓ_q norm of the disagreements vector, $\|\text{disagree}(\mathcal{P}, E^+, E^-)\|_q$.*

In our algorithm for Correlation Clustering on arbitrary graphs, we will use a powerful technique of padded metric space decompositions (see e.g., Bartal [1996], Rao [1999], Fakcharoenphol and Talwar [2003], Gupta, Krauthgamer, and Lee [2003]).

Definition 2 (Padded Decomposition). *Let (X, d) be a metric space on n points, and let $\Delta > 0$. A probabilistic distribution of partitions \mathcal{P} of X is called a padded decomposition if it satisfies the following properties:*

- Each cluster $C \in \mathcal{P}$ has diameter at most Δ .
- For every $u \in X$ and $\varepsilon > 0$, $\Pr(\text{Ball}(u, \delta) \not\subseteq \mathcal{P}(u)) \leq D \cdot \frac{\delta}{\Delta}$ where $\text{Ball}(u, \delta) = \{v \in X : d(u, v) \leq \delta\}$

Theorem 2.1 (Fakcharoenphol, Rao, and Talwar [2004]). *Every metric space (X, d) on n points admits a $D = O(\log n)$ separating padded decomposition. Moreover, there is a polynomial-time algorithm that samples a partition from this distribution.*

3 Convex Relaxation

In our algorithms for minimizing ℓ_q disagreements in arbitrary and complete graphs, we use a convex relaxation given in Figure 3.1. Our convex relaxation for Correlation Clustering is fairly standard. It is similar to relaxations used in the papers by Garg, Vazirani, and Yannakakis [1996], Demaine, Emanuel, Fiat, and Immorlica [2006], Charikar, Guruswami, and Wirth [2003]. For every pair of vertices u and v , we have a variable x_{uv} that is equal to the distance between u and v in the “multicut metric”. Variables x_{uv} satisfy the triangle inequality constraints (P3). They are also symmetric (P4) and $x_{uv} \in [0, 1]$ (P5). Thus, the set of vertices V equipped with the distance function $d(u, v) = x_{uv}$ is a metric space.

Additionally, for every vertex $u \in V$, we have variables y_u and z_u (see constraints (P1) and (P2)) that lower bound the number of disagreeing edges incident to u . The objective of our convex program is to minimize $\max(\|y\|_q, (\sum_u z_u)^{\frac{1}{q}})$. Note that all constraints in the program (P) are linear; however, the objective function of (P) is not convex as is. So in order to find the optimal solution, we

raise the objective function to the power of q and find feasible x, y, z that minimizes the objective $\max(\|y\|_q^q, \sum_u z_u)$.

This program has a polynomial number of linear constraints, and its objective function is convex: This is because the objective function, $\max(\|y\|_q^q, \sum_u z_u)$, is the maximum of two convex functions. The first function, $\|y\|_q^q$ is the sum of q -th powers of the variables y_u which are positive. Thus, $\|y\|_q^q$ is convex and differentiable. The second function, $\sum_u z_u$ is a linear function. Therefore, we can use off-the-shelf convex solvers (quadratic solvers for ℓ_2) to get an optimal solution to (P).

Let us verify that program (P) is a relaxation for Correlation Clustering. Consider an arbitrary partitioning \mathcal{P} of V . In the integral solution corresponding to \mathcal{P} , we set $x_{uv} = 0$ if u and v are in the same cluster in \mathcal{P} ; and $x_{uv} = 1$ if u and v are in different clusters in \mathcal{P} . In this solution, distances x_{uv} satisfy triangle inequality constraints (P3) and $x_{uv} = x_{vu}$ (P4). Observe that a positive edge $(u, v) \in E^+$ is in disagreement with \mathcal{P} if $x_{uv} = 1$; and a negative edge $(u, v) \in E^-$ is in disagreement if $x_{uv} = 0$. Thus, in this integral solution, $y_u = \text{disagree}_u(\mathcal{P}, E^+, E^-)$ and moreover, $z_u \leq y_u^q$. Therefore, in the integral solution corresponding to \mathcal{P} , the objective function of (P) equals $\|\text{disagree}_u(\mathcal{P}, E^+, E^-)\|_q$. Of course, the cost of the optimal fractional solution to the problem may be less than the cost of the optimal integral solution. Thus, (P) is a relaxation for our problem. Below, we denote the cost of the optimal fraction solution to (P) by LP .

We remark that we can get a simpler relaxation by removing variables z and changing the objective function to $\|y\|_q$. This relaxation also works for ℓ_∞ norm. We use it in our 5-approximation algorithm.

4 Overview of Algorithms

We note that some proofs from Subsections 4.1, 4.2 and 4.3 have been deferred to Sections A, B and C respectively (in the supplementary material). These Lemmas and their proofs have been referenced appropriately.

4.1 Correlation Clustering on arbitrary graphs

In this section, we describe our algorithm for minimizing ℓ_q disagreements on arbitrary graphs. We will prove the following main theorem.

Theorem 4.1. *There exists a randomized polynomial-time $O(n^{\frac{q-1}{2q}} \log^{\frac{q+1}{2q}} n)$ -approximation algorithm for Correlation Clustering with the ℓ_q objective ($q \geq 1$).*

We remark that the same algorithm gives $O(\sqrt{n \log n})$ -approximation for the ℓ_∞ norm. We omit the details in the conference version of the paper.

Our algorithm relies on a procedure for partitioning arbitrary metric spaces into pieces of small diameter. In particular, we prove the following theorem,

Theorem 4.2. *There exists a polynomial-time randomized algorithm that given a metric space (X, d) on n points and parameter Δ returns a random partition \mathcal{P} of X such that the diameter of every set P in \mathcal{P} is at most Δ and for every $q \geq 1$ ($q \neq \infty$) and every weighted graph $G = (X, E, w)$, we have*

$$\mathbb{E}[\|\text{cut}(\mathcal{P}, E)\|_q] \leq C n^{\frac{q-1}{2q}} \log^{\frac{q+1}{2q}} n \cdot \left[\left(\sum_{u \in X} \sum_{v: (u,v) \in E} w_{uv}^q \frac{d(u,v)}{\Delta} \right)^{1/q} + \left(\sum_{u \in X} \left(\sum_{v: (u,v) \in E} w_{uv} \frac{d(u,v)}{\Delta} \right)^q \right)^{1/q} \right], \quad (1)$$

for some absolute constant C .

We defer the proof of the above theorem to Section A.

We now show how to use the above metric space partitioning scheme to obtain an approximation algorithm for Correlation Clustering. Note that this proves Theorem 4.1.

Proof of Theorem 4.1. Our algorithm first finds the optimal solution x, y, z to the convex relaxation (P) presented in Section 3. Then, it defines a metric $d(u, v) = x_{uv}$ on the vertices of the graph. Finally, it runs the metric space partitioning algorithm with $\Delta = 1/2$ from Section A (see Theorem 4.2) and outputs the obtained partitioning \mathcal{P} .

Let us analyze the performance of this algorithm. Denote the cost of the optimal solution x, y, z by LP . We know that the cost of the optimal solution OPT is lower bounded by LP (see Section 3 for details). By Theorem 4.2, applied to the graph $G = (V, E^+)$ (note: we ignore negative edges for now),

$$\mathbb{E}[\|\text{cut}(\mathcal{P}, E^+)\|_q] \leq \frac{C}{\Delta} n^{\frac{q-1}{2q}} \log^{\frac{q+1}{2q}} n \cdot \left(\left(\sum_{u \in V} y_u^q \right)^{\frac{1}{q}} + \left(\sum_{u \in V} z_u \right)^{\frac{1}{q}} \right) \leq 4C n^{\frac{q-1}{2q}} \log^{\frac{q+1}{2q}} n \cdot LP. \quad (2)$$

Recall that a positive edge is not in agreement if and only if it is cut. Hence, $\text{disagree}(\mathcal{P}, E^+, \emptyset) = \text{cut}(\mathcal{P}, E^+)$, and the bound above holds for $\mathbb{E}\|\text{disagree}(\mathcal{P}, E^+, \emptyset)\|_q$. By the triangle inequality, $\mathbb{E}\|\text{disagree}(\mathcal{P}, E^+, E^-)\|_q \leq \mathbb{E}\|\text{disagree}(\mathcal{P}, E^+, \emptyset)\|_q + \mathbb{E}\|\text{disagree}(\mathcal{P}, \emptyset, E^-)\|_q$. Hence, to finish the proof, it remains to upper bound $\mathbb{E}\|\text{disagree}(\mathcal{P}, \emptyset, E^-)\|_q$.

Observe that the diameter of every cluster returned by the algorithm is at most $\Delta = 1/2$. For all disagreeing negative edges $(u, v) \in E^-$, we have $x_{uv} \leq 1/2$ and $1 - x_{uv} \geq 1/2$. Thus, $\text{disagree}_u(\mathcal{P}, \emptyset, E^-) \leq 2y_u$ for every u , and $\mathbb{E}\|\text{disagree}(\mathcal{P}, \emptyset, E^-)\|_q \leq 2\|y\|_q \leq 2LP$. This completes the proof. \square

4.2 Correlation Clustering on complete graphs

In this section, we present our algorithm for Correlation Clustering on complete graphs and its analysis. Our algorithm achieves an approximation ratio of 5 and is an improvement over the approximation ratio of 7 by Charikar, Gupta, and Schwartz [2017].

4.2.1 Summary of the algorithm

Our algorithm is based on rounding an optimal solution to the convex relaxation (P). Recall that for complete graphs, we can get a simpler relaxation by removing the variables z in our convex programming formulation. We start with considering the entire vertex set of unclustered vertices. At each step t of the algorithm, we select a subset of vertices as a cluster C_t and remove it from the set of unclustered vertices. Thus, each vertex is assigned to a cluster exactly once and is never removed from a cluster once it is assigned.

For each vertex $w \in V$, let $\text{Ball}(w, \rho) = \{u \in V : x_{uw} \leq \rho\}$ be the set of vertices within a distance of ρ from w . For $r = 1/5$ the quantity $r - x_{uw}$ where $u \in \text{Ball}(w, r)$ represents the distance from u to the boundary of the ball of radius $1/5$ around w . Let $V_t \subseteq V$ be the set of unclustered vertices at step t , and define

$$L_t(w) = \sum_{u \in \text{Ball}(w, r) \cap V_t} r - x_{uw}.$$

At each step t , we select the vertex w_t that maximizes the quantity $L_t(w)$ over all unclustered vertices $w \in V_t$ and select the set $\text{Ball}(w_t, 2r)$ as a cluster. We repeat this step until all the nodes have been clustered. A complete description of the algorithm can be found in Figure B.1 (supplementary material).

4.2.2 Overview of the analysis

Our main result for complete graphs is the following, which proves Theorem 1.3.

Theorem 4.3. *Algorithm 2 is a 5-approximation algorithm for Correlation Clustering on complete graphs.*

For an edge $(u, v) \in E$, let $LP(u, v)$ be the LP cost of the edge (u, v) : $LP(u, v) = x_{uv}$ if $(u, v) \in E^+$ and $LP(u, v) = 1 - x_{uv}$ if $(u, v) \in E^-$. Let $ALG(u, v) = \mathbb{1}((u, v) \text{ is in disagreement})$.

Define

$$\text{profit}(u) = \sum_{(u, v) \in E} LP(u, v) - r \sum_{(u, v) \in E} ALG(u, v),$$

where $r = 1/5$. We show that for each vertex $u \in V$, we have $\text{profit}(u) \geq 0$ (see Lemma 4.4) and, therefore, the number of disagreeing edges incident to u is upper bounded by $5y(u)$:

$$ALG(u) = \sum_{v:(u,v) \in E} ALG(u, v) \leq \frac{1}{r} \sum_{v:(u,v) \in E} LP(u, v) = 5y(u).$$

Thus, $\|ALG\|_q \leq 5\|y\|_q$ for any $q \geq 1$. Consequently, the approximation ratio of the algorithm is at most 5 for any norm ℓ_q .

Lemma 4.4. *For every $u \in V$, we have $\text{profit}(u) \geq 0$.*

At each step t of the algorithm, we create a new cluster C_t and remove it from the graph. We also remove all edges with at least one endpoint in C_t . Denote this set of edges by

$$\Delta E_t = \{(u, v) : u \in C_t \text{ or } v \in C_t\}.$$

Now let

$$\begin{aligned} \text{profit}_t(u, v) &= \begin{cases} LP(u, v) - rALG(u, v), & \text{if } (u, v) \in \Delta E \\ 0, & \text{otherwise} \end{cases} \\ \text{profit}_t(u) &= \sum_{v \in V_t} \text{profit}_t(u, v) = \sum_{(u,v) \in \Delta E_t} LP(u, v) - r \sum_{(u,v) \in \Delta E_t} ALG(u, v). \end{aligned} \quad (3)$$

As all sets ΔE_t are disjoint, $\text{profit}(u) = \sum_t \text{profit}_t(u)$. Thus, to prove Lemma 4.4, it is sufficient to show that $\text{profit}_t(u) \geq 0$ for all t . Note that we only need to consider $u \in V_t$ as $\text{profit}_t(u) = 0$ for $u \notin V_t$.

Consider a step t of the algorithm and vertex $u \in V_t$. Let $w = w_t$ be the center of the cluster chosen at this step. First, we show that since the diameter of the cluster C_t is $4r$, for all negative edges $(u, v) \in E^-$ with $u, v \in C_t$, we can charge the cost of disagreement to the edge itself, that is, $\text{profit}_t(u, v)$ is nonnegative for $(u, v) \in E^-$ (see Lemma B.3). We then consider two cases: $x_{uw} \in [0, r] \cup [3r, 1]$ and $x_{uw} \in (r, 3r]$.

The former case is fairly simple since disagreeing positive edges $(u, v) \in E^+$ (with $x_{uw} \in [0, r] \cup [3r, 1]$) have a “large” LP cost. In Lemma B.4 and Lemma B.5, we prove that the cost of disagreement can be charged to the edge itself and hence $\text{profit}_t(u) \geq 0$.

We then consider the latter case. For vertices u with $x_{uw} \in (r, 3r]$, $\text{profit}_t(u, v)$ for some disagreeing positive edges (u, v) might be negative. Thus, we split the profit at step t for such vertices u into the profit they get from edges (u, v) with v in $\text{Ball}(w, r) \cap V_t$ and from edges with v in $V_t \setminus \text{Ball}(w, r)$. That is,

$$\text{profit}_t(u) = \underbrace{\sum_{v \in \text{Ball}(w, r)} \text{profit}_t(u, v)}_{P_{\text{high}}(u)} + \underbrace{\sum_{v \in V_t \setminus \text{Ball}(w, r)} \text{profit}_t(u, v)}_{P_{\text{low}}(u)}.$$

Denote the first term by $P_{\text{high}}(u)$ and the second term by $P_{\text{low}}(u)$. We show that $P_{\text{low}}(u) \geq -L_t(u)$ (see Claim B.6 and Lemma B.7) and $P_{\text{high}} \geq L_t(w)$ (see Claim B.8 and Lemma B.9) and conclude that $\text{profit}_t(u) = P_{\text{high}}(u) + P_{\text{low}}(u) \geq L_t(w) - L_t(u) \geq 0$ since $L_t(w) = \max_{w' \in V_t} L_t(w') \geq L_t(u)$.

This finishes the proof of Lemma 4.4.

4.3 Correlation Clustering with AKS Min Max Objective

In this section, we present our improved algorithm for Correlation Clustering with AKS Min Max Objective. Our algorithm produces a clustering of cost at most $(2 + \varepsilon)OPT$, which improves upon the bound of $O(\sqrt{\log n \cdot \max\{\log |E^-|, \log(k)\}})$ —approximation algorithm studied by Ahmadi, Galhotra, Khuller, Saha, and Schwartz [2019a].

For a subset $S \subseteq V$ of vertices, we use $\text{cost}^+(S)$ to refer to the weight of positive edges “associated” with S that are in disagreement. These are the edges with exactly one end point in S . Thus, $\text{cost}^+(S) = \sum_{(u,v) \in E^+, u \in S, v \notin S} w_{uv}$. Similarly, we use $\text{cost}^-(S)$ to refer to the weight of dissimilar edges “associated” with S that are in disagreement. These are the edges with both endpoints in

S . Thus, $\text{cost}^-(S) = \sum_{(u,v) \in E^-, u,v \in S} w_{uv}$. The total cost of the set S is $\text{cost}(S) = \text{cost}^+(S) + \text{cost}^-(S)$.

Similar to the algorithm of Ahmadi et al. [2019b], our algorithm works in two phases. In the first phase, the algorithm covers all vertices of the graph with (possibly overlapping) sets S_1, \dots, S_k such that the cost of each set S_i is at most $2OPT$ (i.e., $\text{cost}(S_i) \leq 2OPT$ for each $i \in \{1, \dots, k\}$). In the second phase, the algorithm finds sets P_1, \dots, P_k such that: (1) P_1, \dots, P_k are disjoint and cover the vertex set; (2) $P_i \subseteq S_i$ (and, consequently, $\text{cost}^-(P_i) \leq \text{cost}^-(S_i)$); (3) $\text{cost}^+(P_i) \leq (1 + \varepsilon) \text{cost}^+(S_i)$.

The sets P_1, \dots, P_k are obtained from S_1, \dots, S_k using an uncrossing procedure of Bansal et al. [2011]. Hence the clustering that is output is $\mathcal{P} = (P_1, \dots, P_k)$. The improvement in the approximation factor comes from the first phase of the algorithm.

4.3.1 Summary of the algorithm

At the core of our algorithm is a simple subproblem: For a given vertex $z \in V$, find a subset $S \subseteq V$ containing z such that $\text{cost}(S)$ is minimized. We solve this subproblem using a linear programming relaxation, which is formulated as follows: The LP has a variable x_u for each vertex $u \in V$. In the intended integral solution, we have $x_u = 1$ if u is in the set S , and $x_u = 0$, otherwise. That is, x_u is the indicator of the event “ $u \in S$ ”. The LP has only one constraint: $x_z = 1$. A complete description of the LP can be found in Figure C.1. In Claim C.1 we show that this LP is indeed a valid relaxation for our subproblem.

Moreover we prove that this LP is half-integral, please see section C.1 for details. We now present our algorithm which gives a 2-approximation to the subproblem.

Rounding algorithm for subproblem. We present a simple rounding algorithm. Let x^* be an optimal half-integral LP solution to the problem. We obtain an integral solution x by rounding down x^* , that is $x_u = \lfloor x_u^* \rfloor$ for all u . Thus, $\mu_{uv} \leq 2 \cdot \mu_{uv}^*$ and $\eta_{uv} \leq \eta_{uv}^*$ for all positive and negative edges respectively. Thus, the cost of the rounded solution x is at most $2OPT$.

Rounding algorithm for AKS Min Max Correlation Clustering. To obtain a cover of all the vertices, we pick yet uncovered vertices $z \in V$ one by one and for each z , find a set $S(z)$ as described above. Then, we remove those sets $S(z)$ that are completely covered by other sets. The obtained family of sets $\mathcal{S} = \{S(z)\}$ satisfies the following properties: (1) Sets in \mathcal{S} cover the entire set V ; (2) $\text{cost}(S) \leq 2OPT$ for each $S \in \mathcal{S}$; (3) Each set $S \in \mathcal{S}$ is not covered by the other sets in \mathcal{S} (that is, for each $S \in \mathcal{S}$, $S \not\subseteq \cup_{S' \in (\mathcal{S} \setminus \{S\})} S'$). However, sets S in \mathcal{S} are not necessarily disjoint.

Following Ahmadi et al. [2019b], we then apply an uncrossing procedure developed by Bansal et al. [2011] to the sets S_i in \mathcal{S} and obtain disjoint sets P_i such that (1) $P_i \subseteq S_i$ and (2) $\text{cost}^+(P_i) \leq \text{cost}^+(S_i) + \varepsilon OPT$ for each i (see Lemma C.3 in Section C.2). We have $\text{cost}^+(P_i) \leq \text{cost}^+(S_i) + \varepsilon OPT$ and $\text{cost}^-(P_i) \leq \text{cost}^-(S_i)$, since P_i is a subset of S_i . Thus, $\text{cost}(P_i) \leq \text{cost}(S_i) + \varepsilon OPT$ and, consequently, P_1, \dots, P_k is a $2(1 + \varepsilon)$ -approximation for Correlation Clustering with the AKS Min Max objective. We note that by slightly modifying our algorithm we can obtain a 2-approximation.

Finally, we show that AKS Min-Max Correlation Clustering is at least as hard as Vertex Cover (see C.3 for details). Vertex Cover is NP-hard to approximate within any constant factor better than 2 assuming the Unique Games conjecture (UGC) (see Khot and Regev [2008]). Thus, our algorithm gives the best possible approximation if UGC holds.

References

- Saba Ahmadi, Sainyam Galhotra, Samir Khuller, Barna Saha, and Roy Schwartz. Min-max correlation clustering via multicut, 2019a.
- Saba Ahmadi, Samir Khuller, and Barna Saha. Min-max correlation clustering via multicut. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 13–26. Springer, 2019b.
- Nir Ailon, Moses Charikar, and Alantha Newman. Aggregating inconsistent information: ranking and clustering. *Journal of the ACM (JACM)*, 55(5):23, 2008.

- Nikhil Bansal, Avrim Blum, and Shuchi Chawla. Correlation clustering. *Machine learning*, 56(1-3): 89–113, 2004.
- Nikhil Bansal, Uriel Feige, Robert Krauthgamer, Konstantin Makarychev, Viswanath Nagarajan, Joseph Naor, and Roy Schwartz. Min-max graph partitioning and small set expansion. In *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*, pages 17–26. IEEE, 2011.
- Yair Bartal. Probabilistic approximation of metric spaces and its algorithmic applications. In *Foundations of Computer Science, 1996. Proceedings., 37th Annual Symposium on*, pages 184–193. IEEE, 1996.
- Francesco Bonchi, David Garcia-Soriano, and Edo Liberty. Correlation clustering: from theory to practice. In *KDD*, page 1972. Citeseer, 2014.
- Moses Charikar, Venkatesan Guruswami, and Anthony Wirth. Clustering with qualitative information. In *IEEE Symposium on Foundations of Computer Science (FOCS)*. Citeseer, 2003.
- Moses Charikar, Neha Gupta, and Roy Schwartz. Local guarantees in graph cuts and clustering. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 136–147. Springer, 2017.
- Shuchi Chawla, Robert Krauthgamer, Ravi Kumar, Yuval Rabani, and D Sivakumar. On the hardness of approximating multicut and sparsest-cut. *computational complexity*, 15(2):94–114, 2006.
- Shuchi Chawla, Konstantin Makarychev, Tselil Schramm, and Grigory Yaroslavtsev. Near optimal lp rounding algorithm for correlation clustering on complete and complete k-partite graphs. In *Proceedings of the forty-seventh annual ACM symposium on Theory of computing*, pages 219–228. ACM, 2015.
- William Cohen and Jacob Richman. Learning to match and cluster entity names. In *ACM SIGIR-2001 Workshop on Mathematical/Formal Methods in Information Retrieval*, 2001.
- William W Cohen and Jacob Richman. Learning to match and cluster large high-dimensional data sets for data integration. In *Proceedings of the eighth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 475–480. ACM, 2002.
- Erik D Demaine, Dotan Emanuel, Amos Fiat, and Nicole Immorlica. Correlation clustering in general weighted graphs. *Theoretical Computer Science*, 361(2-3):172–187, 2006.
- Jittat Fakcharoenphol and Kunal Talwar. Improved decompositions of graphs with forbidden minors. In *6th International workshop on Approximation algorithms for combinatorial optimization*, pages 36–46, 2003.
- Jittat Fakcharoenphol, Satish Rao, and Kunal Talwar. A tight bound on approximating arbitrary metrics by tree metrics. *Journal of Computer and System Sciences*, 69(3):485–497, 2004.
- Naveen Garg, Vijay V Vazirani, and Mihalis Yannakakis. Approximate max-flow min-(multi) cut theorems and their applications. *SIAM Journal on Computing*, 25(2):235–251, 1996.
- Anupam Gupta, Robert Krauthgamer, and James R Lee. Bounded geometries, fractals, and low-distortion embeddings. In *null*, page 534. IEEE, 2003.
- Subhash Khot and Oded Regev. Vertex cover might be hard to approximate to within $2 - \epsilon$. *Journal of Computer and System Sciences*, 74(3):335–349, 2008.
- Gregory Puleo and Olgica Milenkovic. Correlation clustering and biclustering with locally bounded errors. In *International Conference on Machine Learning*, pages 869–877, 2016.
- Anirudh Ramachandran, Nick Feamster, and Santosh Vempala. Filtering spam with behavioral blacklisting. In *Proceedings of the 14th ACM conference on Computer and communications security*, pages 342–351. ACM, 2007.
- Satish Rao. Small distortion and volume preserving embeddings for planar and euclidean metrics. In *Proceedings of the fifteenth annual symposium on Computational geometry*, pages 300–306. ACM, 1999.

Anthony Wirth. *Correlation Clustering*, pages 280–284. Springer US, Boston, MA, 2017. ISBN 978-1-4899-7687-1. doi: 10.1007/978-1-4899-7687-1_176. URL https://doi.org/10.1007/978-1-4899-7687-1_176.

A Algorithm for Partitioning Metric Spaces

In this section, we will prove the following main theorem,

Theorem A.1. *There exists a polynomial-time randomized algorithm that given a metric space (X, d) on n points and parameter Δ returns a random partition \mathcal{P} of X such that the diameter of every set P in \mathcal{P} is at most Δ and for every $q \geq 1$ ($q \neq \infty$) and every weighted graph $G = (X, E, w)$, we have*

$$\mathbb{E}[\|\text{cut}(\mathcal{P}, E)\|_q] \leq C n^{\frac{q-1}{2q}} \log^{\frac{q+1}{2q}} n \cdot \left[\left(\sum_{u \in X} \sum_{v: (u,v) \in E} w_{uv}^q \frac{d(u,v)}{\Delta} \right)^{1/q} + \left(\sum_{u \in X} \left(\sum_{v: (u,v) \in E} w_{uv} \frac{d(u,v)}{\Delta} \right)^q \right)^{1/q} \right], \quad (4)$$

for some absolute constant C .

We remark that our algorithm also works for $q = \infty$. Indeed, the behaviour of the algorithm does not depend on q (in fact, q is not even a part of the algorithm's input). Hence, inequality (1) holds for any $q < \infty$. In the limit as q tends to infinity, we get the following result. We provide the details in the full version of the paper (see supplemental materials for details).

We will need the following definition.

Definition 3. *Let (X, d) be a metric space. The ε -neighborhood of a set $S \subset X$ is the set of points at distance at most ε from S :*

$$N_\varepsilon(S) = \{u \in X : \exists v \in S \text{ such that } d(u, v) \leq \varepsilon\}.$$

The ε -neighborhood of the boundary of a partition \mathcal{P} is the set of points

$$N_\varepsilon(\partial\mathcal{P}) = \bigcup_{P \in \mathcal{P}} (N_\varepsilon(P) \setminus P) = \{u \in X : \exists v \in X \text{ s.t. } d(u, v) \leq \varepsilon \text{ and } \mathcal{P}(u) \neq \mathcal{P}(v)\}.$$

We first describe an algorithm which succeeds with probability at least $1/2$ and fails with probability at most $1/2$. If the algorithm succeeds it outputs a random partition \mathcal{P} of X such that the diameter of every set P in \mathcal{P} is at most Δ and for every q and every weighted graph $G = (X, E, w)$, we have

$$\mathbb{E}[\|\text{cut}(\mathcal{P}, E)\|_q \mid \text{algorithm succeeds}] \leq C' n^{\frac{q-1}{2q}} \log^{\frac{q+1}{2q}} n \cdot \left(\sum_{u \in X} \sum_{v: (u,v) \in E} w_{uv}^q \frac{d(u,v)}{\Delta} \right)^{1/q} + \left(\sum_{u \in X} \left(\sum_{v: (u,v) \in E} w_{uv} \frac{d(u,v)}{\Delta} \right)^q \right)^{1/q}. \quad (5)$$

To obtain a valid partition with probability 1, we repeat our algorithm for at most $\lceil \log_2 n \rceil$ iterations till it succeeds and output the obtained solution. If the algorithm does not succeed after $\lceil \log_2 n \rceil$ iterations (which happens with probability at most $1/n$), we partition the graph using a simple deterministic procedure which we describe in the end of this section.

Our algorithm is based on the procedure for generating bounded padded stochastic decompositions (see Section 2). First, the algorithm picks a random padded decomposition \mathcal{P} of the metric space X . Then, it finds the ε -neighborhood $N_\varepsilon(\partial\mathcal{P})$ of the boundary of \mathcal{P} . Finally, it outputs \mathcal{P} if $|N_\varepsilon(\partial\mathcal{P})| \leq 2D\varepsilon/\Delta$ and fails otherwise. We present a pseudo-code for our algorithm in Figure A.1.

A.1 Analysis

Our algorithm is scale invariant i.e., its output does not change if we multiply all distances in the metric space (X, d) and the parameter Δ by some positive number λ . Thus, for the sake of analysis, we assume that $\Delta = 1$. Algorithm A.1 succeeds when $N_\varepsilon(\mathcal{P})$ has size at most M . Denote this event by \mathcal{E} . We first show that $\Pr(\mathcal{E}) \geq 1/2$.

Lemma A.2. *Algorithm A.1 succeeds with probability at least $1/2$.*

Input: metric space (X, d) and parameter $\Delta > 0$.
Output: a random partition \mathcal{P} of X .

1. Let $D = O(\log n)$ be the parameter from Theorem 2.1, $\varepsilon = 1/\sqrt{2Dn}$ and $M = 2D\varepsilon n/\Delta$.
 2. Draw a random padded decomposition \mathcal{P} of the metric space (X, d) with parameter Δ using Theorem 2.1.
 3. Find the neighborhood $N_\varepsilon(\partial\mathcal{P})$ of the partition boundary.
 4. If $|N_\varepsilon(\mathcal{P})| \leq M$ then output \mathcal{P} ; else fail.
-

Figure A.1: Metric decomposition algorithm.

Proof. Let $\bar{\mathcal{E}}$ be the complement of the event \mathcal{E} . We need to show that $\Pr(\bar{\mathcal{E}}) \leq 1/2$. To this end, we bound the expected size of the set $N_\varepsilon(\mathcal{P})$ using the second property of padded decompositions:

$$\begin{aligned} \mathbb{E}[|N_\varepsilon(\partial\mathcal{P})|] &= \sum_{u \in X} \Pr(u \in N_\varepsilon(\partial\mathcal{P})) \\ &= \sum_{u \in X} \Pr(\text{Ball}(u, \varepsilon) \not\subset \mathcal{P}(u)) \\ &\leq \sum_{u \in X} D\varepsilon = D\varepsilon n. \end{aligned}$$

Here, we used that $u \in N_\varepsilon(\partial\mathcal{P})$ if and only if $\text{Ball}(u, \varepsilon) \not\subset \mathcal{P}(u)$. Now, by Markov's inequality,

$$\Pr(\bar{\mathcal{E}}) = \Pr(|N_\varepsilon(\partial\mathcal{P})| > \underbrace{2D\varepsilon n}_M) \leq \frac{D\varepsilon n}{2D\varepsilon n} = \frac{1}{2}.$$

□

Let X_{uv} be the indicator of the event $\{\mathcal{P}(u) \neq \mathcal{P}(v)\}$ i.e., the event that points u and v are separated by the partition \mathcal{P} . By the second property of padded stochastic decompositions, we have $\mathbb{E}(X_{uv}) = \Pr(\mathcal{P}(u) \neq \mathcal{P}(v)) \leq D \cdot d(u, v)$. Since $\Pr(\mathcal{E}) \geq 1/2$, for each $(u, v) \in E$, we have

$$\mathbb{E}[X_{uv} \mid \mathcal{E}] \leq \frac{\mathbb{E}[X_{uv}]}{\Pr(\mathcal{E})} \leq 2\mathbb{E}[X_{uv}] \leq 2D \cdot d(u, v).$$

Consequently,

$$\mathbb{E}[w_{uv}X_{uv} \mid \mathcal{E}] \leq 2D \cdot w_{uv}d(u, v) \quad \text{and} \quad (6)$$

$$\mathbb{E}[w_{uv}^q X_{uv}^q \mid \mathcal{E}] \leq 2D \cdot w_{uv}^q d(u, v). \quad (7)$$

We split all edges E into two groups: short edges, which we denote by $E_{\leq \varepsilon}$, and long edges, which we denote by $E_{> \varepsilon}$. Short edges are edges of length at most ε ; long edges are edges of length greater than ε . Note that $\text{cut}(\mathcal{P}, E) = \text{cut}(\mathcal{P}, E_{\leq \varepsilon}) + \text{cut}(\mathcal{P}, E_{> \varepsilon})$.

For every subset $E' \subset E$ (in particular, for $E' = E_{\leq \varepsilon}$ and $E' = E_{> \varepsilon}$), we have

$$\mathbb{E}\left[\|\text{cut}(\mathcal{P}, E')\|_q^q \mid \mathcal{E}\right] = \sum_{u \in X} \mathbb{E}\left[\left(\sum_{v: (u,v) \in E'} w_{uv}X_{uv}\right)^q \mid \mathcal{E}\right]. \quad (8)$$

We separately upper bound $\mathbb{E}[\|\text{cut}(\mathcal{P}, E_{\leq \varepsilon})\|_q^q \mid \mathcal{E}]$ and $\mathbb{E}[\|\text{cut}(\mathcal{P}, E_{> \varepsilon})\|_q^q \mid \mathcal{E}]$ using the formula above and inequalities (6), (7) and then use the triangle inequality for ℓ_q norms to bound $\mathbb{E}[\|\text{cut}(\mathcal{P}, E)\|_q \mid \mathcal{E}]$.

Long edges. Fix a vertex u and consider long edges incident to u . Their total weight is upper bounded by

$$\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} \leq \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} \underbrace{\frac{d(u,v)}{\varepsilon}}_{\geq 1}.$$

Thus,

$$\begin{aligned} \left(\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \right)^q &\leq \left(\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} \right)^{q-1} \left(\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \right) \\ &\leq \left(\sum_{v:(u,v) \in E_{>\varepsilon}} \frac{w_{uv} d(u,v)}{\varepsilon} \right)^{q-1} \left(\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \right). \end{aligned}$$

Plugging this expression into formula (8) with $E' = E_{>\varepsilon}$ and using inequality (6), we get the following upper bound on $\mathbb{E}[\|\text{cut}(\mathcal{P}, E_{>\varepsilon})\|_q^q \mid \mathcal{E}]$:

$$\sum_{u \in X} \left(\sum_{v:(u,v) \in E_{>\varepsilon}} \frac{w_{uv} d(u,v)}{\varepsilon} \right)^{q-1} \mathbb{E} \left[\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \mid \mathcal{E} \right] \leq \frac{2D}{\varepsilon^{q-1}} \sum_{u \in X} \left(\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} d(u,v) \right)^q.$$

Finally, by Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}[\|\text{cut}(\mathcal{P}, E_{>\varepsilon})\|_q \mid \mathcal{E}] &= \mathbb{E}[(\|\text{cut}(\mathcal{P}, E_{>\varepsilon})\|_q^q)^{\frac{1}{q}} \mid \mathcal{E}] \\ &\leq \left(\mathbb{E}[\|\text{cut}(\mathcal{P}, E_{>\varepsilon})\|_q^q \mid \mathcal{E}] \right)^{\frac{1}{q}} \\ &\leq \left(\frac{2D}{\varepsilon^{q-1}} \sum_{u \in X} \left(\sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} d(u,v) \right)^q \right)^{\frac{1}{q}}. \end{aligned} \quad (9)$$

Short edges. To bound $\|\text{cut}(\mathcal{P}, E_{\text{short}})\|_q$, we will make use of the following lemma.

Lemma A.3. *Consider non-negative (dependent) random variables X_1, \dots, X_n . Suppose that at most M of them are non-zero with probability 1. Then, for every $q \geq 1$, the following bound holds:*

$$\mathbb{E}[(X_1 + \dots + X_n)^q] \leq M^{q-1} \sum_{i=1}^n \mathbb{E}[X_i^q].$$

Proof. Let x_{i_1}, \dots, x_{i_m} be the non-zero random variables in a certain sampling of X_1, \dots, X_n for some $m \leq M$. Suppose that $m \neq 0$. Using Jensen's inequality, we have

$$\left(\frac{x_{i_1} + \dots + x_{i_m}}{m} \right)^q \leq \frac{1}{m} \sum_{j=1}^m x_{i_j}^q,$$

and, therefore,

$$\left(x_{i_1} + \dots + x_{i_m} \right)^q \leq m^{q-1} \sum_{j=1}^m x_{i_j}^q \leq M^{q-1} \sum_{j=1}^m x_{i_j}^q.$$

The inequality above also holds when $m = 0$. Thus, the expectation of the left hand side is upper bounded by the expectation of the right hand side. This concludes the proof. \square

Fix a vertex u . Observe that if (u, v) is a short edge which is cut by \mathcal{P} then v must belong to $N_\varepsilon(\partial\mathcal{P})$. Thus, the number of non-zero random variables X_{uv} for a given u and $(u, v) \in E_{\leq\varepsilon}$ is upper bounded by $|N_\varepsilon(\partial\mathcal{P})|$. If the algorithm succeeds, then $|N_\varepsilon(\partial\mathcal{P})| \leq M$. Thus, by Lemma A.3,

$$\mathbb{E} \left[\left(\sum_{v:(u,v) \in E_{\leq\varepsilon}} w_{uv} X_{uv} \right)^q \mid \mathcal{E} \right] \leq M^{q-1} \sum_{v:(u,v) \in E_{\leq\varepsilon}} \mathbb{E}[w_{uv}^q X_{uv}^q \mid \mathcal{E}].$$

Plugging this bound into formula (8) with $E' = E_{\leq \varepsilon}$ and using inequality (7), we get the following upper bound on $\mathbb{E}[\|\text{cut}(\mathcal{P}, E_{\leq \varepsilon})\|_q^q \mid \mathcal{E}]$:

$$\sum_{u \in X} \left(M^{q-1} \sum_{v: (u,v) \in E_{\leq \varepsilon}} \mathbb{E}[w_{uv}^q X_{uv}^q \mid \mathcal{E}] \right) \leq 2D M^{q-1} \sum_{u \in X} \sum_{v: (u,v) \in E_{\leq \varepsilon}} w_{uv}^q d(u, v).$$

Finally, by Jensen's inequality, we have

$$\mathbb{E}[\|\text{cut}(\mathcal{P}, E_{\leq \varepsilon})\|_q \mid \mathcal{E}] \leq \left(2D M^{q-1} \right)^{1/q} \left(\sum_{u \in X} \sum_{v: (u,v) \in E_{\leq \varepsilon}} w_{uv}^q d(u, v) \right)^{1/q}. \quad (10)$$

To obtain the desired bound (5), we substitute $D = O(\log n)$, $\varepsilon = 1/\sqrt{2Dn}$, and $M = 2D\varepsilon n/\Delta$ in bounds (9) and (10) and then apply the triangle inequality for the ℓ_q norm.

To finish the proof of Theorem 4.2, we need to describe what we do in the unlikely event that Algorithm A.1 fails $\lceil \log_2 n \rceil$ times. In this case, we create a new graph on X with edges between pairs of vertices at distance at most $1/n$ from each other and partition it into connected components. We analyze this algorithm in the full version of the paper (see supplemental materials for details).

B Correlation Clustering on Complete Graphs

In this section, we present our algorithm for Correlation Clustering on complete graphs and its analysis. Our algorithm achieves an approximation ratio of 5 and is an improvement over the approximation ratio of 7 by Charikar, Gupta, and Schwartz [2017].

B.1 The Algorithm

Our algorithm is based on rounding an optimal solution to the convex relaxation (P). Recall that for complete graphs, we can get a simpler relaxation by removing the variables z in our convex programming formulation. We start with considering the entire vertex set of unclustered vertices. At each step t of the algorithm, we select a subset of vertices as a cluster C_t and remove it from unclustered vertices. Thus, each vertex is assigned to a cluster exactly once and is never removed from a cluster once it is assigned.

For each vertex $w \in V$, let $\text{Ball}(w, \rho) = \{u \in V : x_{uw} \leq \rho\}$ be the set of vertices within a distance of ρ from w . For $r = 1/5$ the quantity $r - x_{uw}$ where $u \in \text{Ball}(w, r)$ represents the distance from u to the boundary of the ball of radius $1/5$ around w . Let $V_t \subseteq V$ be the set of unclustered vertices at step t , and define

$$L_t(w) = \sum_{u \in \text{Ball}(w, r) \cap V_t} r - x_{uw}.$$

At each step t , we select the vertex w_t that maximizes the quantity $L_t(w)$ over all unclustered vertices $w \in V_t$ and select the set $\text{Ball}(w_t, 2r)$ as a cluster. We repeat this step until all the nodes have been clustered. A pseudo-code for our algorithm is given in Figure B.1.

B.2 Analysis

In this section, we present an analysis of our algorithm.

Theorem B.1. *Algorithm 2 gives a 5-approximation for Correlation Clustering on complete graphs.*

For an edge $(u, v) \in E$, let $LP(u, v)$ be the LP cost of the edge (u, v) : $LP(u, v) = x_{uv}$ if $(u, v) \in E^+$ and $LP(u, v) = 1 - x_{uv}$ if $(u, v) \in E^-$. Let $ALG(u, v) = \mathbb{1}((u, v) \text{ is in disagreement})$.

Define

$$\text{profit}(u) = \sum_{(u,v) \in E} LP(u, v) - r \sum_{(u,v) \in E} ALG(u, v),$$

Input: Optimal solution x to the linear program (P).

Output: Clustering \mathcal{C} .

1. Let $V_0 = V, r = 1/5, t = 0$.
 2. **while** ($V_t \neq \emptyset$)
 - Find $w_t = \arg \max_{w \in V_t} L_t(w)$.
 - Create a cluster $C_t = \text{Ball}(w_t, 2r) \cap V_t$.
 - Set $V_{t+1} = V_t \setminus C_t$ and $t = t + 1$.
 3. Return $\mathcal{C} = (C_0, \dots, C_{t-1})$.
-

Figure B.1: Algorithm for Correlation Clustering on complete graphs.

where $r = 1/5$. We show that for each vertex $u \in V$, we have $\text{profit}(u) \geq 0$ (see Lemma B.2 below) and, therefore, the number of disagreeing edges incident to u is upper bounded by $5y(u)$:

$$ALG(u) = \sum_{v: (u,v) \in E} ALG(u, v) \leq \frac{1}{r} \sum_{v: (u,v) \in E} LP(u, v) = 5y(u).$$

Thus, $\|ALG\|_q \leq 5\|y\|_q$ for any $q \geq 1$. Consequently, the approximation ratio of the algorithm is at most 5 for any norm ℓ_q .

Lemma B.2. *For every $u \in V$, we have $\text{profit}(u) \geq 0$.*

At each step t of the algorithm, we create a new cluster C_t and remove it from the graph. We also remove all edges with at least one endpoint in C_t . Denote this set of edges by

$$\Delta E_t = \{(u, v) : u \in C_t \text{ or } v \in C_t\}.$$

Now let

$$\text{profit}_t(u, v) = \begin{cases} LP(u, v) - rALG(u, v), & \text{if } (u, v) \in \Delta E \\ 0, & \text{otherwise} \end{cases}.$$

$$\text{profit}_t(u) = \sum_{v \in V_t} \text{profit}_t(u, v) = \sum_{(u,v) \in \Delta E_t} LP(u, v) - r \sum_{(u,v) \in \Delta E_t} ALG(u, v). \quad (11)$$

As all sets ΔE_t are disjoint, $\text{profit}(u) = \sum_t \text{profit}_t(u)$. Thus, to prove Lemma B.2, it is sufficient to show that $\text{profit}_t(u) \geq 0$ for all t . Note that we only need to consider $u \in V_t$ as $\text{profit}_t(u) = 0$ for $u \notin V_t$.

Consider a step t of the algorithm and vertex $u \in V_t$. Let $w = w_t$ be the center of the cluster chosen at this step. First, we show that since the diameter of the cluster C_t is $4r$, for all negative edges $(u, v) \in E^-$ with $u, v \in C_t$, we can charge the cost of disagreement to the edge itself, that is, $\text{profit}_t(u, v)$ is nonnegative for $(u, v) \in E^-$ (see Lemma B.3). We then consider two cases: $x_{uw} \in [0, r] \cup [3r, 1]$ and $x_{uw} \in (r, 3r]$.

The former case is fairly simple since disagreeing positive edges $(u, v) \in E^+$ (with $x_{uw} \in [0, r] \cup [3r, 1]$) have a “large” LP cost. In Lemma B.4 and Lemma B.5, we prove that the cost of disagreement can be charged to the edge itself and hence $\text{profit}_t(u) \geq 0$.

We then consider the latter case. For vertices u with $x_{uw} \in (r, 3r]$, $\text{profit}_t(u, v)$ for some disagreeing positive edges (u, v) might be negative. Thus, we split the profit at step t for such vertices u into the profit they get from edges (u, v) with v in $\text{Ball}(w, r) \cap V_t$ and from edges with v in $V_t \setminus \text{Ball}(w, r)$. That is,

$$\text{profit}_t(u) = \underbrace{\sum_{v \in \text{Ball}(w, r)} \text{profit}_t(u, v)}_{P_{\text{high}}(u)} + \underbrace{\sum_{v \in V_t \setminus \text{Ball}(w, r)} \text{profit}_t(u, v)}_{P_{\text{low}}(u)}.$$

Denote the first term by $P_{high}(u)$ and the second term by $P_{low}(u)$. We show that $P_{low}(u) \geq -L_t(u)$ (see Lemma B.9) and $P_{high} \geq L_t(w)$ (see Lemma B.7) and conclude that $\text{profit}_t(u) = P_{high}(u) + P_{low}(u) \geq L_t(w) - L_t(u) \geq 0$ since $L_t(w) = \max_{w' \in V_t} L_t(w') \geq L_t(u)$.

In the following claim, we show that we can charge the cost of disagreement of a negative edge to the edge itself.

Claim B.3. *For a negative edge $(u, v) \in E^-$, $\text{profit}_t(u, v)$ is always nonnegative.*

Proof. The only case when (u, v) is in disagreement is when both u and v belong to the new cluster. In this case, they lie in the ball of radius $2r$ around w (and thus $x_{uw}, x_{vw} \leq 2r$). Thus the distance x_{uv} between them is at most $4r$ (because $x_{uv} \leq x_{uw} + x_{vw} \leq 4r$). The LP cost of the edge (u, v) is at least $LP(u, v) = 1 - x_{uv} \geq 1 - 4r = r$. Thus, $\text{profit}_t(u, v) = LP(u, v) - rALG(u, v) = LP(u, v) - r \geq 0$. \square

In Lemma B.4 and Lemma B.5, we consider the case when $x_{uw} \in [0, r] \cup (3r, 1]$.

Lemma B.4. *If $x_{uw} \leq r$, then $\text{profit}_t(u, v) \geq 0$ for all $v \in V_t$.*

Proof. If $x_{uw} \in E^-$, then $\text{profit}_t(u, v) \geq 0$ by Claim B.3. Assume that $x_{uw} \in E^+$. Since $x_{uw} \leq r$, u belongs to the cluster C_t . Thus, (u, v) disagrees only if v does not belong to that cluster. In this case, $x_{vw} \geq 2r$ and by the triangle inequality $x_{uv} \geq x_{vw} - x_{uw} \geq r$. Therefore, $\text{profit}_t(u, v) = x_{u,v} - r \geq 0$. \square

Lemma B.5. *If $x_{uw} \geq 3r$, then $\text{profit}_t(u, v) \geq 0$ for all $v \in V_t$.*

Proof. As in the previous lemma, we can assume that $x_{uw} \in E^+$. If $x_{uw} \geq 3r$, then u does not belong to the new cluster C_t . Thus, (u, v) disagrees only if v belongs to C_t . In this case, $x_{vw} \leq 2r$ and by the triangle inequality $x_{uv} \geq x_{uw} - x_{vw} \geq r$. Therefore, $\text{profit}_t(u, v) = x_{u,v} - r \geq 0$. \square

We next consider u such that $x_{uw} \in (r, 3r]$. First, we show that the profit we obtain from every edge (u, v) with $v \in \text{Ball}(w, r)$ is at least $r - x_{vw}$, regardless of whether the edge is positive or negative.

Claim B.6. *If $x_{uw} \in (r, 3r]$ and $v \in \text{Ball}(w, r) \cap V_t$, then $\text{profit}_t(u, v) \geq r - x_{vw}$.*

Proof. First consider u such that $x_{uw} \in (r, 2r]$. Note that $x_{uv} \geq x_{uw} - x_{vw} \geq r - x_{vw}$. Moreover, $x_{uv} \leq x_{uw} + x_{vw} \leq 2r + x_{vw}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq r - x_{vw}$. Otherwise, $\text{profit}_t(u, v) \geq (1 - 2r - x_{vw}) - r \geq 2r - x_{vw}$.

For $u \in (2r, 3r]$, note that $x_{uv} \geq x_{uw} - x_{vw} \geq 2r - x_{vw}$. Moreover, $x_{uv} \leq x_{uw} + x_{vw} \leq 3r + x_{vw}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq (2r - x_{vw}) - r \geq r - x_{vw}$. Otherwise, $\text{profit}_t(u, v) \geq (1 - 3r - x_{vw}) \geq 2r - x_{vw}$. \square

Using the above claim, we can sum up the profits from all vertices v in $\text{Ball}(w, r)$ and lower bound $P_{high}(u)$ as follows.

Lemma B.7. *If $x_{uw} \in (r, 3r]$, then $P_{high}(u) \geq L_t(w)$.*

Proof. By Claim B.6, we have $\text{profit}_t(u, v) \geq r - x_{vw}$ for all $v \in V_t$. Thus,

$$P_{high}(u) = \sum_{v \in \text{Ball}(w, r) \cap V_t} \text{profit}_t(u, v) \geq \sum_{v \in \text{Ball}(w, r) \cap V_t} r - x_{vw} = L_t(w).$$

\square

We now lower bound $P_{low}(u)$. To this end, we estimate each term $\text{profit}_t(u, v)$ in the definition of P_{low} .

Claim B.8. *If $x_{uw} \in (r, 3r]$ and $v \in V_t \setminus \text{Ball}(w, r)$, then $\text{profit}_t(u, v) \geq \min(x_{uv} - r, 0)$.*

Proof. By Claim B.3, if (u, v) is a negative edge, then $\text{profit}_t(u, v) \geq 0$. The profit is 0 if $x_{uv} \notin \Delta E_t$ (i.e., neither u nor v belong to the new cluster). So let us assume that (u, v) is a positive edge in ΔE_t . Then, the profit obtained from (u, v) is x_{uv} if (u, v) is in agreement and $x_{uv} - r$ if (u, v) is in disagreement. In any case, $\text{profit}_t(u, v) \geq x_{uv} - r \geq \min(x_{uv} - r, 0)$. \square

$$\begin{aligned}
& \text{minimize} && \sum_{(u,v) \in E^+} w_{uv} |x_u - x_v| + \sum_{(u,v) \in E^-} w_{uv} (x_u + x_v - 1)^+ \\
& \text{subject to} && x_z = 1 \\
& && 0 \leq x_u \leq 1 \text{ for all } u \in V
\end{aligned}$$

Here, we use notation $(t)^+ = \max(t, 0)$.

Figure C.1: LP relaxation for covering z with a low cost set S .

Lemma B.9 is an immediate corollary of Claim B.8.

Lemma B.9. *If $x_{uw} \in (r, 3r]$, then $P_{low}(u) \geq -L_t(u)$.*

Proof. By Claim B.8, we have $\text{profit}_t(u, v) \geq \min(x_{uv} - r, 0)$ for all $v \in V_t$. Thus,

$$\begin{aligned}
P_{low}(u) &= \sum_{v \in V_t \setminus \text{Ball}(w, r)} \text{profit}_t(u, v) \\
&\geq \sum_{v \in V_t \setminus \text{Ball}(w, r)} \min(x_{uv} - r, 0) \\
&\stackrel{a}{\geq} \sum_{v \in V_t} \min(x_{uv} - r, 0) \\
&\stackrel{b}{=} \sum_{v \in \text{Ball}(u, r) \cap V_t} x_{uv} - r \\
&= -L(u).
\end{aligned}$$

Here we used that (a) all terms $\min(x_{uv} - r, 0)$ are nonpositive, and (b) $\min(x_{uv} - r, 0) = 0$ if $v \notin \text{Ball}(u, r)$. \square

This finishes the proof of Lemma B.2.

C Correlation Clustering with the AKS Objective

Note that, after linearizing the objective function in C.1, we get the LP in C.2.

Claim C.1. *The LP relaxation described in Figure C.2 is a valid relaxation for the subproblem.*

Proof. Let us verify that this is a valid relaxation for the problem. As we discussed above, in the intended integral solution, we have $x_u = 1$ if u is in the set S , and $x_u = 0$, otherwise. That is, x_u is the indicator of the event “ $u \in S$ ”.

Consider a positive edge $(u, v) \in E^+$. In the integral solution, $|x_u - x_v| = 1$ if and only if one of the vertices u or v is in S and the other one is not. In this case, the edge (u, v) is in disagreement with S . Now, consider a negative edge $(u, v) \in E^-$. In the integral solution, $(x_u + x_v - 1)^+ = 1$ if and only if both u and v are in S . Again, in this case, the edge (u, v) is in disagreement with S . Thus, this LP is a relaxation for our problem.

Note that we can linearize the $|\cdot|$ and $(\cdot)^+$ terms in the objective as follows. We can replace terms of the type $|x_u - x_v|$ with variables μ_{uv} and introduce the constraints $\mu_{uv} \geq (x_u - x_v)$ and $\mu_{uv} \geq (x_v - x_u)$. Similarly, we can replace terms of the type $(x_u + x_v - 1)^+$ with variables η_{uv} and introduce the constraints $\eta_{uv} \geq (x_u + x_v - 1)$ and $\eta_{uv} \geq 0$. It is easy to see that the minimum values for the variables μ_{uv} and η_{uv} is attained at $|x_u - x_v|$ and $(x_u + x_v - 1)^+$ respectively. \square

C.1 Half-integrality of Subproblem Polytope

In this subsection, we show that the polytope of the subproblem that we consider is half-integral. First, we linearize the objective as described in the above subsection to obtain the equivalent relaxation (P-Cover- z).

$$\begin{aligned}
& \text{minimize} && \sum_{(u,v) \in E^+} w_{uv} \mu_{uv} + \sum_{(u,v) \in E^-} w_{uv} \eta_{uv} \\
& \text{subject to} && x_z = 1 \\
& && 0 \leq x_u \leq 1 \text{ for all } u \in V \\
& && \mu_{uv} \geq (x_v - x_u) \\
& && \mu_{uv} \geq (x_u - x_v) \\
& && \eta_{uv} \geq 0 \\
& && \eta_{uv} \geq (x_u + x_v - 1)
\end{aligned} \tag{P-Cover-}z$$

Figure C.2: LP relaxation for covering z with a low cost set S .

Lemma C.2. *The polytope (P-Cover- z) is half-integral.*

Proof. We will show that the (P-Cover- z) polytope is half-integral via contradiction, that is, we show that if an extreme point solution (x, μ, η) is not half-integral, then there exist feasible solution pairs (x', μ', η') and (x'', μ'', η'') such that (x, μ, η) can be written as a convex combination of x' and x'' (respectively $(\mu', \mu''), (\eta', \eta'')$).

Towards a contradiction assume that an extreme point solution is not integral. Let $L = \{u : 0 < x_u < 1/2\}$ and $R = \{u : 1/2 < x_u < 1\}$. For each $u \in V$, define,

$$\Delta_1^u = \begin{cases} +\varepsilon, & \text{if } u \in L \\ -\varepsilon, & \text{if } u \in R \\ 0, & \text{otherwise} \end{cases} \quad \Delta_2^u = \begin{cases} -\varepsilon, & \text{if } u \in L \\ +\varepsilon, & \text{if } u \in R \\ 0, & \text{otherwise} \end{cases}$$

Here ε is chosen small enough such that: (1) $x_u \pm \varepsilon \in (0, 1/2)$ if $u \in L$, (2) $x_u \pm \varepsilon \in (1/2, 1)$ if $u \in R$, (3) $(x_u + x_v - 1) \pm 2\varepsilon < 0$ if $(x_u + x_v - 1) < 0$.

For each $u \in V$, we define, $x'_u = x_u + \Delta_1^u$ and $x''_u = x_u + \Delta_2^u$. Thus, from definition, we can see that x' and x'' are feasible, more over, $x = \frac{1}{2}(x' + x'')$.

Consider μ' and μ'' defined by x' and x'' respectively. Notice that if $\mu_{uv} = 0$, then $\mu'_{uv} = \mu''_{uv} = 0$. Otherwise, without loss of generality, let $\mu_{uv} = x_u - x_v > 0$. If both $u, v \in L$ or $u, v \in R$, then $\mu'_{uv} = \mu''_{uv} = \mu_{uv}$. Else, $u \in R$ and $v \in L$, and $\mu'_{uv} = \mu_{uv} + 2\varepsilon$ and $\mu''_{uv} = \mu_{uv} - 2\varepsilon$. Thus, $\mu = \frac{1}{2}(\mu' + \mu'')$.

Consider η' and η'' defined by x' and x'' respectively. Parameter ε was chosen such that if $\eta < 0$, then $\eta', \eta'' < 0$. Moreover, if $(x_u + x_v - 1) = 0$, then either $u \in L$ and $v \in R$ or $u \in R$ and $v \in L$, thus $\eta'_{uv} = \eta''_{uv} = 0$. Finally, if $\eta_{uv} > 0$, then $\eta_{uv} = (x_u + x_v - 1)$. If $u \in R$ and $v \in L$, then $\eta'_{uv} = \eta''_{uv} = \eta_{uv}$. Otherwise, $u, v \in R$ and $\eta'_{uv} = \eta_{uv} - 2\varepsilon$ and $\eta''_{uv} = \eta_{uv} + 2\varepsilon$. Thus, $\eta = \frac{1}{2}(\eta' + \eta'')$.

This contradicts the fact that (x, μ, η) is an extreme point solution. Thus, the polytope (P-Cover- z) is half-integral. \square

C.2 Uncrossing Overlapping Sets

For completeness, we present here a proof of the following lemma from Bansal et al. [2011]. Denote by $\delta(S)$ the set of all positive edges leaving set S in graph G . Then, $\text{cost}^+(S) = w(\delta(S))$.

Lemma C.3 (Uncrossing argument in Bansal et al. [2011]). *There exists a polynomial-time algorithm that given a weighted graph $G = (V, E)$, a family of sets S_1, \dots, S_k that covers all vertices in G , and a parameter $\varepsilon = 1/\text{poly}(n)$, finds disjoint sets P_1, \dots, P_k covering V such that for each i :*

1. $P_i \subset S_i$; and
2. $w(\delta(P_i)) \leq w(\delta(S_i)) + \varepsilon \max_j w(\delta(S_j))$.

Proof. Let us first describe the uncrossing algorithm from the paper Bansal et al. [2011]. Initially, the algorithm sets $P_i^0 = S_i \setminus \cup_{j < i} S_j$ for each $i \in \{1, \dots, k\}$. Then, at every step t , it finds a set P_i^t

violating the desired bound

$$w(\delta(P_i^t)) \leq w(\delta(S_i)) + \varepsilon \max_j w(\delta(S_j)) \quad (12)$$

and updates all sets as follows: $P_i^{t+1} = S_i$; and $P_j^{t+1} = P_j^t \setminus S_i$. The algorithm terminates and outputs sets P_i^t when bound (12) holds for all sets P_i^t .

It easy to see that the following loop invariants hold at every step of the algorithm: (1) each P_i^t is a subset of S_i ; (2) sets P_i^t are disjoint; (4) sets P_i^t cover all vertices in V . It is also immediate that when or if the algorithm terminates sets P_i^t satisfy (12). We only need to check that the algorithm stops in polynomial time.

Let $B = \max_j w(\delta(S_j))$. Define a potential function $\varphi(t) = \sum_{i=1}^k w(\delta(P_i^t))$. Observe that initially $\varphi(0) \leq 2 \sum_i w(\delta(S_i))$, since every edge cut by the partition (P_1, \dots, P_k) belongs to some S_i . Since, $w(\delta(S_i)) \leq B$ for all i , we have $\varphi(0) \leq 2kB$. We will show that at every step of the algorithm $\varphi(t)$ decreases by at least $2\varepsilon B$ and thus the algorithm terminates in at most k/ε steps.

Consider step t of the algorithm. Suppose that at this step of the algorithm, set P_i^t violated the constraint and thus it was replaced by S_i . Write,

$$\begin{aligned} \varphi(t+1) - \varphi(t) &= \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) + \sum_{j \neq i} (w(\delta(P_j^{t+1})) - w(\delta(P_j^t))) \\ &= \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) + \sum_{j \neq i} (w(\delta(P_j^t \setminus S_i)) - w(\delta(P_j^t))). \end{aligned}$$

Observe that for every two subsets of vertices P and S we have the following inequality:

$$\begin{aligned} w(\delta(P \setminus S)) - w(\delta(P)) &= \left(w(E(P \setminus S, V \setminus P)) + w(E(P \setminus S, P \cap S)) \right) \\ &\quad - \left(w(E(P \setminus S, V \setminus P)) + w(E(P \cap S, V \setminus P)) \right) \\ &= w(E(P \cap S, P \setminus S)) - w(E(P \cap S, V \setminus P)) \\ &\leq w(E(P \cap S, P \setminus S)) - w(E(P \cap S, S \setminus P)) \\ &= \left(w(E(P \cap S, P \setminus S)) + w(E(S \setminus P, P \setminus S)) \right) \\ &\quad - \left(w(E(P \cap S, S \setminus P)) + w(E(P \setminus S, S \setminus P)) \right) \\ &= w(E(S, P \setminus S)) - w(E(P, S \setminus P)). \end{aligned}$$

Also, note that $P_i^t \subset S_i \setminus P_j^t$ (since $P_i^t \subset S_i$ and all P_j^t are disjoint). Consequently, $w(E(P_i^t, P_j^t)) \leq w(E(S_i \setminus P_j^t, P_j^t))$. Therefore,

$$\begin{aligned} \varphi(t+1) - \varphi(t) &= \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) + \sum_{j \neq i} w(E(S_i, P_j^t \setminus S_i)) - w(E(P_j^t, S_i \setminus P_j^t)) \\ &\leq \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) + \sum_{j \neq i} w(E(S_i, P_j^t \setminus S_i)) - w(E(P_j^t, P_i^t)). \end{aligned}$$

Using again that the sets P_j^t partition V into disjoint pieces, we get

$$\begin{aligned} \varphi(t+1) - \varphi(t) &\leq \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) + \sum_{j \neq i} w(E(S_i, P_j^t \setminus S_i)) - w(E(P_j^t, P_i^t)) \\ &= \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) + w(E(S_i, \cup_{j \neq i} P_j^t \setminus S_i)) - w(E(\cup_{j \neq i} P_j^t, P_i^t)) \\ &= \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) + \underbrace{w(E(S_i, V \setminus S_i))}_{= \delta(S_i)} - \underbrace{w(E(V \setminus P_i^t, P_i^t))}_{= \delta(P_i^t)} \\ &= 2 \left(w(\delta(S_i)) - w(\delta(P_i^t)) \right) \leq -2\varepsilon B. \end{aligned}$$

This concludes the proof. \square

C.3 Hardness

We show that AKS problem is at least as hard as the vertex cover problem unless $P = NP$. This can be seen as follows.

Theorem C.4. *The AKS problem Min-Max objective is NP-Hard to approximate within a factor of $2 - \varepsilon$.*

Proof. We prove the above theorem by reducing the AKS problem to vertex cover.

Reduction: Let $G = (V, E)$ be the vertex cover instance. We reduce it to an instance of the AKS problem $G' = (V', E^+, E^-)$ as follows. We define the new vertex set V' as $V' = V \cup \{z\}$ and the set of positive edges as the ones that connect z to the vertices in V . That is, $E^+ = \{(z, u) : u \in V\}$. We assign each positive edge a weight of 1. Finally, the set of negative edges are the edges E , that is, $E^- = E$. We assign each negative edge a weight of ∞ .

$OPT_{AKS} \leq OPT_{VC}$ Let S be an optimal vertex cover in G , that is $OPT_{VC} = |S|$. We will use S to construct a partitioning \mathcal{P} of the vertex set of G' , such that the cost of \mathcal{P} under the AKS objective is $|S|$. We define the partitioning \mathcal{P} as follows: For every vertex $v \in S$, we form a singleton cluster $\{v\}$. We also form a cluster $\{z\} \cup (V \setminus S)$ consisting of all the vertices in $V \setminus S$ and z . For every singleton cluster $\{v\}$, the only edge in disagreement is the similar edge connecting v to z of weight 1. Thus, the cost of every singleton cluster is 1.

Consider the cluster $\{z\} \cup (V \setminus S)$. There are no negative edges (u, v) within this cluster, since for every edge $(u, v) \in E^- = E$, one of its endpoints must belong to the vertex cover S . Thus, the cost of the set $\{z\} \cup (V \setminus S)$ equals the number of positive edges which are cut which are exactly the edges connecting z to S . Thus, $OPT_{AKS} \leq cost(\mathcal{P}) = |S| = OPT_{VC}$.

$OPT_{VC} \leq OPT_{AKS}$ Let \mathcal{P} be an optimal partitioning of V' of cost OPT_{AKS} . Consider the cluster $P \in \mathcal{P}$ containing the vertex z . Since \mathcal{P} is an optimal partitioning, there can be no negative edges present within P . Thus the cost of the cluster P equals the number of positive edges which are cut, which is exactly $|V'| - |P| = |V| + 1 - |P|$. We define the vertex cover for the graph G to be $T = V \setminus (P \setminus \{z\})$. The set T is a feasible set cover because every edge $(u, v) \in E$ is a negative edge of infinite weight and thus both vertices u and v cannot belong to the cluster P . Therefore, at least one of them – u or v – must belong to T . The cost of T equals $|T| = |V| - (|P| - 1)$. Hence, $OPT_{VC} \leq cost(P) \leq OPT_{AKS}$. \square

D Integrality gap

In this section, we present an integrality gap example for the convex program (P). We describe an instance of the ℓ_q $s - t$ cut problem on $\Theta(n)$ vertices that has an integrality gap of $\Omega(n^{\frac{1}{2} - \frac{1}{2q}})$. In our integrality gap example, we describe a layered graph with $\Theta(n^{\frac{1}{2}})$ layers, with each layer consisting of a complete bipartite graph on $\Theta(n^{\frac{1}{2}})$ vertices. Between each layer i and $i + 1$, there is a terminal s_i which connects these two layers. Finally, the terminals s and t are located at opposite ends of this layered graph. We will observe that for any integral cut separating s and t , there will be at least one vertex such that a large fraction of the edges incident to it are cut. We will show that there is a corresponding fractional solution that is cheaper compared to any integral cut as the fractional solution can “spread” the cut equally across the layers, thus not penalizing any individual layer too harshly. In doing so, we will prove the following theorem,

Theorem D.1. *The integrality gap for the convex relaxation (P) is $\Omega(n^{\frac{1}{2} - \frac{1}{2q}})$.*

Proof. We now give a more formal description of the layered graph discussed above. The construction has two parameters a and b , so we will call such a graph $G_{a,b}$. The graph consists of b layers with each layer consisting of the complete bipartite graph $K_{a,a}$. We refer to layer i of the graph as $G_{a,b}^i$ and refer to the left and right hand of the bipartition as $L(G_{a,b}^i)$ and $R(G_{a,b}^i)$ respectively. In addition to these layers, the graph consists of $b + 1$ terminals $\{s, t, s_1, \dots, s_{b-1}\}$ (we will refer to s as s_0 and t as s_b interchangeably). For each $i \in \{1, \dots, b - 1\}$, the vertex s_i is connected to all the vertices in $R(G_{a,b}^i)$ and $L(G_{a,b}^{i+1})$. Finally, s is connected to all the vertices in $L(G_{a,b}^1)$ and t is connected to all the vertices in $R(G_{a,b}^b)$.

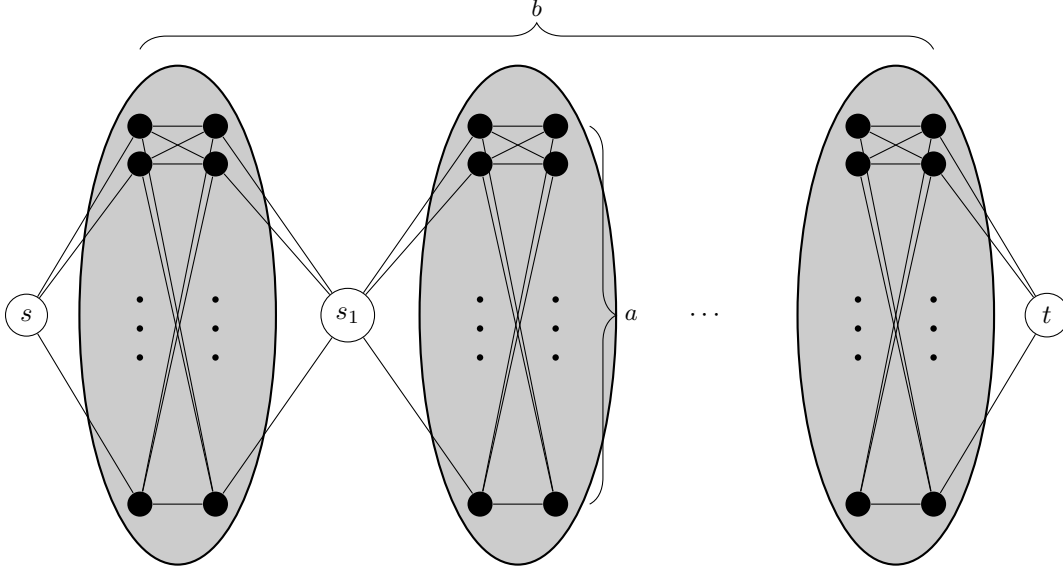


Figure D.1: Integrality gap example.

Consider any integral cut separating s and t in the graph $G_{a,b}$. Any such cut must disconnect at least one pair of consecutive terminals (if all pairs of consecutive terminals are connected, then s is still connected to t). Thus let $j \in \{0, 1, \dots, b\}$ be such that s_{j-1} is disconnected from s_j and consider the subgraph induced on $\{s_{j-1} \cup s_j \cup G_{a,b}^j\}$. We will show that this induced subgraph contains a vertex such that $\Omega(a^{\frac{1}{2}})$ of its incident edges are cut. Intuitively, since s_{j-1} is separated from s_j , if the majority of the edges incident to s_{j-1} and s_j are not cut, then s_{j-1} and s_j have many neighbors in $L(G_{a,b}^j)$ and $R(G_{a,b}^j)$ respectively. As $G_{a,b}^j$ is highly connected, in order for s_{j-1} to be separated from s_j , there must be a vertex in $G_{a,b}^j$ with many incident edges which are cut. If $\text{cut}(s_{j-1})$ or $\text{cut}(s_j)$ is at least $a/2$, then we are done. Otherwise, s_j is connected to at least $a/2$ vertices in $R(G_{a,b}^j)$, so every u adjacent to s_{j-1} must have at least $a/2$ incident edges which are cut. Therefore, $\text{OPT}^q \geq \Omega(a^q)$.

We now present a fractional cut separating s and t . If an edge e connects s_i to a vertex in $R(G_{a,b}^i)$ for some $i \in \{1, \dots, b\}$, set the length of the edge to be $1/b$; otherwise set the edge length to be 0. We let x_{uv} be the shortest path metric in this graph. It is easy to see that such a solution is feasible. We now analyze the quality of this solution. For each $i \in \{1, \dots, b\}$, we have $y_{s_i} = a/b$ and for each $u \in R(G_{a,b}^i)$, we have $y_u = 1/b$. Thus

$$LP^q = ab\left(\frac{1}{b}\right)^q + b\left(\frac{a}{b}\right)^q.$$

If $b > a$, then

$$LP^q \leq ab\left(\frac{1}{b}\right) + b\left(\frac{a}{b}\right) = 2a$$

and if $b < a$, then

$$LP^q \leq ab\left(\frac{1}{b}\right) + b\left(\frac{a}{b}\right)^q \leq a^q \left(a^{-(q-1)} + b^{-(q-1)} \right).$$

Setting $a = b = \Omega(n^{\frac{1}{2}})$ gives

$$\frac{\text{OPT}^q}{LP^q} = \Omega\left(n^{\frac{q}{2} - \frac{1}{2}}\right),$$

so the integrality gap is $\frac{\text{OPT}}{LP} = \Omega(n^{\frac{1}{2} - \frac{1}{2q}})$. \square

E Hardness of approximation

In this section, we prove the following hardness result.

Theorem E.1. *It is NP-hard to approximate the min ℓ_∞ s-t cut problem within a factor of $2 - \varepsilon$ for every positive ε .*

Proof. The proof follows a reduction from 3SAT. We will describe a procedure that reduces every instance of a 3CNF formula ϕ to a graph G_ϕ such that the minimum ℓ_∞ s-t cut for G_ϕ has a certain value if and only if the formula ϕ is satisfiable.

Reduction from 3SAT: Given a 3CNF instance ϕ with n literals and m clauses, we describe a graph G_ϕ with $(2 + 4n + 5m)$ vertices and $(6n + 8m)$ edges. We refer to the vertex and edge set of G_ϕ as $V(G_\phi)$ and $E(G_\phi)$. For every literal $x_i, i \in \{1, \dots, n\}$, we have four nodes, $x_i^T, x_i^F, x_i^\dagger$ and \bar{x}_i^\dagger . Additionally, we have a “False” and a “True” node. For every $i \in \{1, \dots, n\}$, we connect “False” with x_i^F and “True” with x_i^T using an infinite weight edge. Both x_i^F and x_i^T are connected to x_i^\dagger and \bar{x}_i^\dagger using edges of weight 1.

For every clause C in ϕ , we will create a gadget in G_ϕ consisting of five nodes. We will refer to the subgraph induced by these nodes as $G_\phi[C]$. Let the clause $C = (y_1 \vee y_2 \vee y_3)$. We have a node in the gadget for each $y_i, i \in \{1, 2, 3\}$, and two additional nodes C_a and C_b . We connect y_2 and y_3 to C_b , and y_1 and C_b to C_a , all using unit weight edges.

We connect the gadget $G_\phi[C]$ for clause $C = (y_1 \vee y_2 \vee y_3)$ to the main graph as follows. For each $i \in \{1, 2, 3\}$, connect the vertex for the literal y_i to the vertex y_i^\dagger with a unit weight edge. Finally connect the node C_a to the “True” vertex using an infinite weight edge. An example of a 3CNF formula ϕ and the corresponding G_ϕ is given in Figure E.1.

Fact 1. Consider the gadget $G_\phi[C]$ for the clause $C = (y_1 \vee y_2 \vee y_3)$. If all three nodes y_1, y_2 , and y_3 need to be disconnected from C_a , then either $|\text{cut}_{C_a}| = 2$ or $\text{cut}_{C_b} = 2$. If at most two of the three nodes y_1, y_2 and y_3 need to be disconnected from C_a , then there is a cut that separates those nodes from C_a such that both cut_{C_a} and cut_{C_b} are at most 1.

Lemma E.2. *Given a 3CNF formula ϕ , consider the graph G_ϕ constructed according to the reduction described above. The formula ϕ is satisfiable if and only if the minimum ℓ_∞ True-False cut \mathcal{P} for the graph G_ϕ has value 1, that is, $\|\text{cut}_{\mathcal{P}}\|_\infty = 1$.*

Proof. **3SAT \Rightarrow minimum ℓ_∞ True-False cut:** If the 3CNF formula ϕ is satisfiable, then the graph G_ϕ has a minimum ℓ_∞ s-t cut of value exactly 1. This can be seen as follows. Given a satisfying assignment x^* , we will construct a cut $E_{\mathcal{P}}$ (and corresponding partition \mathcal{P}) such that for every vertex $u \in V(G_\phi)$, $\text{cut}_{\mathcal{P}}(u) \leq 1$. For every $i \in \{1, \dots, n\}$, if x_i^* is True, then include (x_i^\dagger, x_i^F) and $(\bar{x}_i^\dagger, x_i^T)$ as part of the cut $E_{\mathcal{P}}$, else include (x_i^\dagger, x_i^T) and $(\bar{x}_i^\dagger, x_i^F)$ as part of the cut $E_{\mathcal{P}}$. Note that this cuts exactly one edge incident to each vertex $x_i^\dagger, x_i^F, \bar{x}_i^\dagger$ and x_i^T for $i \in \{1, \dots, n\}$. Since ϕ has a satisfiable assignment, each clause C has at least one literal which is True, and hence the node corresponding to this literal is not connected to the vertex False in $G_\phi - E_{\mathcal{P}}$. Thus, each clause C has at most two literals that are False, and thus there are at most two False-True paths that go through this gadget. From Fact 1, we can know that we can include edges from $E(G_\phi[C])$ in $E_{\mathcal{P}}$ such that both $\text{cut}_{\mathcal{P}}(C_a)$ and $\text{cut}_{\mathcal{P}}(C_b)$ are at most 1 and the False-True paths through this gadget are disconnected. Thus, cut $E_{\mathcal{P}}$ disconnects True from False such that $\|\text{cut}_{\mathcal{P}}(G_\phi)\|_\infty = 1$.

minimum ℓ_∞ True-False cut \Rightarrow 3SAT: Let G_ϕ be the graph constructed for the 3CNF formula ϕ such that there is a cut $E_{\mathcal{P}} \subseteq E(G_\phi)$ (and corresponding partition \mathcal{P}) such that \mathcal{P} separates True from False and $\|\text{cut}_{\mathcal{P}}(G_\phi)\|_\infty = 1$. We will construct a satisfying assignment x^* from the formula ϕ . Since $\text{cut}_{\mathcal{P}}(u) \leq 1$ for every $u \in V(G_\phi)$, none of the $(\text{True}, x_i^T), (x_i^F, \text{False})$ edges are part of the cut \mathcal{P} for $i \in \{1, \dots, n\}$. In order for True to be separated from False, either the edges (x_i^\dagger, x_i^F) and $(\bar{x}_i^\dagger, x_i^T)$ are part of the cut $E_{\mathcal{P}}$, or the edges (x_i^\dagger, x_i^T) and $(\bar{x}_i^\dagger, x_i^F)$ are part of the cut $E_{\mathcal{P}}$. This gives us our assignment; for each $i \in \{1, \dots, n\}$, if $(x_i^T, x_i^\dagger) \in E \setminus E_{\mathcal{P}}$, then assign x_i^* as True and \bar{x}_i^* as False. Otherwise $(x_i^F, x_i^\dagger) \in E \setminus E_{\mathcal{P}}$, so assign x_i^* as False and \bar{x}_i^* as True. Now, we show that x^* is a satisfiable assignment for ϕ . To see this, note that for each clause C , there exists at least one

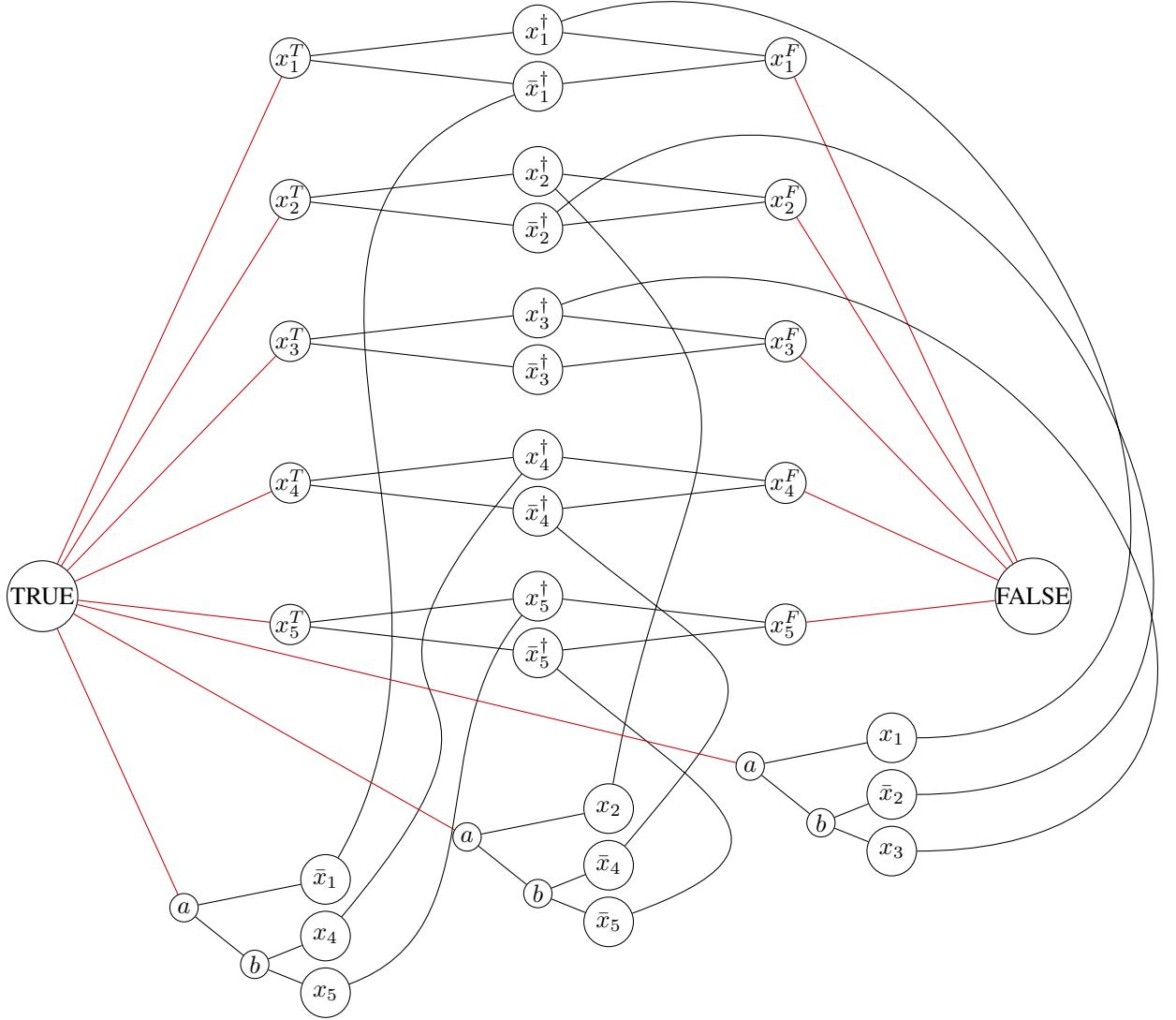


Figure E.1: G_ϕ for the 3CNF formula $\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee \bar{x}_4 \vee \bar{x}_5) \wedge (\bar{x}_1 \vee x_4 \vee x_5)$.

literal y_i such that the node corresponding to y_i is still connected to C_a . As the cut $E_{\mathcal{P}}$ separates True and False, $(y_i^+, y_i^T) \in E \setminus E(G_\phi)$ and hence $y_i^* = \text{True}$. Thus, the assignment x^* is satisfiable for ϕ . \square

Thus, we can conclude Theorem 5.1 from the reduction procedure provided and Lemma 5.2. \square

F Correlation Clustering on Complete Bipartite Graphs

Let $(V = L \cup R, E)$ be a complete bipartite graph with L and R being the bipartition of the vertex set. In this section, we provide and analyze an algorithm for correlation clustering on complete graphs

Input: Optimal solution x to the linear program (P).

Output: Clustering \mathcal{C} .

1. Let $V_0 = V, r = 1/5, t = 0$.
 2. **while** ($V_t \cap L \neq \emptyset$)
 - Find $w_t = \arg \max_{w \in L} L_t^R(w)$.
 - Create a cluster $C_t = \text{Ball}_{L \cup R}(w_t, 2r) \cap V_t$.
 - Set $V_{t+1} = V_t \setminus C_t$ and $t = t + 1$.
 3. Let $\mathcal{C}_L = (C_0, \dots, C_{t-1})$.
 4. **if** ($R \cap V_t \neq \emptyset$)
 - Let $C_R = R \cap V_t$.
 5. Return $\mathcal{C} = \mathcal{C}_L \cup \{C_R\}$.
-

Figure F.1: Algorithm for Correlation Clustering on complete bipartite graphs.

with an approximation guarantee of 5 for minimizing the mistakes on one side of the bipartition (which without loss of generality will be L). The algorithm and analysis for complete bipartite graphs is very similar to the algorithm and analysis for complete graphs. At each step t of our algorithm, we select a cluster center $w_t \in L$ and a cluster $C_t \subseteq (L \cup R)$ and remove it from the graph. This clustering step is repeated until all vertices in L are part of some cluster. If there are any remaining vertices in R which are unclustered, we put them in a single cluster.

Similar to the definition of $\text{Ball}(w, \rho)$ in Section B, we define $\text{Ball}_S(w, \rho) = \{u \in S : x_{uw} \leq \rho\}$. We select the cluster centers w_t in step t as follows. Let $V_t \subseteq V$ be the set of unclustered vertices at the start of step t . We redefine $L_t^S(w) = \sum_{u \in \text{Ball}_{V_t \cap S}(w, r)} r - x_{uw}$. We select w_t as the vertex $w \in L$ that maximizes $L_t(w)$. We then select $\text{Ball}_{L \cup R}(w, 2r)$ as our cluster and repeat. A pseudocode for the above algorithm is provided in Figure F.1.

F.1 Analysis

In this section, we present an analysis of our algorithm.

Theorem F.1. *Algorithm 3 gives a 5-approximation for Correlation Clustering on complete bipartite graphs where disagreements are measured on only one side of the bipartition.*

The proof of this theorem is almost identical to the proof of Theorem B.1. We define $LP(u, v)$, $ALG(u, v)$, $\text{profit}_t(u, v)$ for every edge (u, v) and $\text{profit}(u)$, $\text{profit}_t(u)$ for every vertex u as in Section B. We then show that for each vertex $u \in L$, we have $\text{profit}(u) \geq 0$ and, therefore, the number of disagreeing edges incident to u is upper bounded by $5y(u)$:

$$ALG(u) = \sum_{v: (u, v) \in E} ALG(u, v) \leq \frac{1}{r} \sum_{v: (u, v) \in E} LP(u, v) = 5y(u).$$

Thus, $\|ALG\|_q \leq 5\|y\|_q$ for any $q \geq 1$. Consequently, the approximation ratio of the algorithm is at most 5 for any norm ℓ_q .

Lemma F.2. *For every $u \in L$, we have $\text{profit}(u) \geq 0$.*

As in Lemma 4.4, we need to show that $\text{profit}_t(u) \geq 0$ for all t . Note that we only need to consider $u \in V_t \cap L$ as $\text{profit}_t(u) = 0$ for $u \notin V_t$.

Consider a step t of the algorithm and vertex $u \in V_t \cap L$. Let $w = w_t$ be the center of the cluster chosen at this step. First, we show that since the diameter of the cluster C_t is $4r$, for all negative edges $(u, v) \in E^-$ with $u, v \in C_t$, we can charge the cost of disagreement to the edge itself, that

is, $\text{profit}_t(u, v)$ is nonnegative for $(u, v) \in E^-$ (see Lemma B.3). We then consider two cases: $x_{uw} \in [0, r] \cup [3r, 1]$ and $x_{uw} \in (r, 3r]$.

The former case is fairly simple since disagreeing positive edges $(u, v) \in E^+$ (with $x_{uw} \in [0, r] \cup [3r, 1]$) have a “large” LP cost. In Lemma B.4 and Lemma B.5, we prove that the cost of disagreement can be charged to the edge itself and hence $\text{profit}_t(u) \geq 0$.

We then consider the latter case. Similarly to Lemma 4.4, we split the profit at step t for vertices u with $x_{uw} \in (r, 3r]$ into the profit they get from edges (u, v) with v in $\text{Ball}_R(w, r) \cap V_t$ and from edges with v in $V_t \setminus \text{Ball}_R(w, r)$. That is,

$$\begin{aligned} \text{profit}_t(u) &= \\ &= \underbrace{\sum_{v \in \text{Ball}_R(w, r) \cap V_t} \text{profit}_t(u, v)}_{P_{\text{high}}(u)} + \underbrace{\sum_{v \in V_t \setminus \text{Ball}_R(w, r)} \text{profit}_t(u, v)}_{P_{\text{low}}(u)}. \end{aligned}$$

Denote the first term by $P_{\text{high}}(u)$ and the second term by $P_{\text{low}}(u)$. We show that $P_{\text{low}}(u) \geq -L_t^R(u)$ (see Lemma F.6) and $P_{\text{high}} \geq L_t^R(w) = \sum_{v \in \text{Ball}_R(w, r) \cap V_t} r - x_{vw}$ (see Lemma F.4) and conclude that $\text{profit}_t(u) = P_{\text{high}}(u) + P_{\text{low}}(u) \geq L_t^R(w) - L_t^R(u) \geq 0$ since $L_t^R(w) = \max_{w' \in V_t} L_t^R(w') \geq L_t^R(u)$.

Consider u such that $x_{uw} \in (r, 3r]$. First, we show that the profit we obtain from every edge (u, v) with $v \in \text{Ball}_R(w, r)$ is at least $r - x_{vw}$, regardless of whether the edge is positive or negative.

Claim F.3. *If $x_{uw} \in (r, 3r]$ and $v \in \text{Ball}_R(w, r) \cap V_t$, then $\text{profit}_t(u, v) \geq r - x_{vw}$.*

Proof. First consider u such that $x_{uw} \in (r, 2r]$. Note that $x_{uv} \geq x_{uw} - x_{vw} \geq r - x_{vw}$. Moreover, $x_{uv} \leq x_{uw} + x_{vw} \leq 2r + x_{vw}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq r - x_{vw}$. Otherwise, $\text{profit}_t(u, v) \geq (1 - 2r - x_{vw}) - r \geq 2r - x_{vw}$.

For $u \in (2r, 3r]$, note that $x_{uv} \geq x_{uw} - x_{vw} \geq 2r - x_{vw}$. Moreover, $x_{uv} \leq x_{uw} + x_{vw} \leq 3r + x_{vw}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq (2r - x_{vw}) - r \geq r - x_{vw}$. Otherwise, $\text{profit}_t(u, v) \geq (1 - 3r - x_{vw}) \geq 2r - x_{vw}$. \square

Using the above claim, we can sum up the profits from all vertices v in $\text{Ball}_R(w, r)$ and lower bound $P_{\text{high}}(u)$ as follows.

Lemma F.4. *If $x_{uw} \in (r, 3r]$, then $P_{\text{high}}(u) \geq L_t^R(w)$.*

Proof. By Claim , we have $\text{profit}_t(u, v) \geq r - x_{vw}$ for all $v \in R \cap V_t$. Thus,

$$\begin{aligned} P_{\text{high}}(u) &= \sum_{v \in \text{Ball}_R(w, r) \cap V_t} \text{profit}_t(u, v) \\ &\geq \sum_{v \in \text{Ball}_R(w, r) \cap V_t} r - x_{vw} = L_t^R(w). \end{aligned}$$

\square

We now lower bound $P_{\text{low}}(u)$. To this end, we estimate each term $\text{profit}_t(u, v)$ in the definition of P_{low} .

Claim F.5. *If $x_{uw} \in (r, 3r]$ and $v \in V_t \setminus \text{Ball}_R(w, r)$, then $\text{profit}_t(u, v) \geq \min(x_{uv} - r, 0)$.*

Proof. By Claim B.3, if (u, v) is a negative edge, then $\text{profit}_t(u, v) \geq 0$. The profit is 0 if $x_{uv} \notin \Delta E_t$ (i.e., neither u nor v belong to the new cluster). So let us assume that (u, v) is a positive edge in ΔE_t . Then, the profit obtained from (u, v) is x_{uv} if (u, v) is in agreement and $x_{uv} - r$ if (u, v) is in disagreement. In any case, $\text{profit}_t(u, v) \geq x_{uv} - r \geq \min(x_{uv} - r, 0)$. \square

Lemma F.6 is an immediate corollary of Claim F.5.

Lemma F.6. *If $x_{uw} \in (r, 3r]$, then $P_{\text{low}}(u) \geq -L_t^R(u)$.*

Proof. By Claim B.8, we have $\text{profit}_t(u, v) \geq \min(x_{uv} - r, 0)$ for all $v \in V_t$. Thus,

$$\begin{aligned}
P_{low}(u) &= \sum_{v \in V_t \setminus \text{Ball}_R(w, r)} \text{profit}_t(u, v) \\
&\geq \sum_{v \in V_t \setminus \text{Ball}_R(w, r)} \min(x_{uv} - r, 0) \\
&\stackrel{a}{\geq} \sum_{v \in V_t} \min(x_{uv} - r, 0) \\
&\stackrel{b}{=} \sum_{v \in \text{Ball}_R(u, r) \cap V_t} x_{uv} - r \\
&= -L_t^R(u).
\end{aligned}$$

Here we used that (a) all terms $\min(x_{uv} - r, 0)$ are nonpositive, and (b) $\min(x_{uv} - r, 0) = 0$ if $v \notin \text{Ball}(u, r)$. \square