A KL divergence for the truncated log-normal distribution

We need to introduce some notation to work with the truncated normal distribution. Consider a distribution $\mathrm{t}\mathcal{N}(x \mid \mu, \sigma^2, a, b)$, where μ and σ are the mean and the standard deviation of the corresponding normal distribution before truncation, and a and b are the left and right truncation thresholds respectively. Denote

$$\alpha = \frac{a-\mu}{\sigma} \qquad \beta = \frac{b-\mu}{\sigma} \qquad Z = \Phi(\beta) - \Phi(\alpha)$$

Now we can calculate the KL divergence between a truncated log-normal distribution $q(\theta^i)$ and a log-uniform distribution $p(\theta^i)$ with a bounded support $\theta^i \in [e^a, e^b]$.

$$\begin{aligned}
&\mathrm{KL}(q(\theta^i) \,\|\, p(\theta^i)) = \mathrm{KL}(q(\log\theta^i) \,\|\, p(\log\theta^i)) = \\
&= \mathrm{KL}(\mathrm{t}\mathcal{N}(x \mid \mu, \sigma^2, a, b) \,\|\, \mathcal{U}(x \mid a, b)) = \int_a^b \mathrm{t}\mathcal{N}(x \mid \mu, \sigma^2, a, b) \log \frac{\mathrm{t}\mathcal{N}(x \mid \mu, \sigma^2, a, b)}{\mathcal{U}(x \mid a, b)} dx = \\
&= -\mathcal{H}(\mathrm{t}\mathcal{N}(x \mid \mu, \sigma^2, a, b)) + \log(b-a)
\end{aligned}$$

Entropy for the truncated normal distribution is

$$\begin{aligned}
&\mathcal{H}(\mathrm{t}\mathcal{N}(x \mid \mu, \sigma^2, a, b)) = \log(\sqrt{2\pi e}\sigma Z) + \frac{\alpha\phi(\alpha) - \beta\phi(\beta)}{2Z} \\
&\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \\
&\Phi(x) = \frac{1}{2}(1 + \mathrm{erf}(x/\sqrt{2}))
\end{aligned}$$

Finally, we obtain

$$\begin{gathered}
\mathrm{KL}(q(\theta_i \mid \mu_i, \sigma_i) \,\|\, p(\theta_i)) = \\
= \log(b-a) - \log(\sqrt{2\pi e}\sigma_i) - \log(\Phi(\beta_i) - \Phi(\alpha_i)) - \frac{\alpha_i\phi(\alpha_i) - \beta_i\phi(\beta_i)}{2(\Phi(\beta_i) - \Phi(\alpha_i))}
\end{gathered}$$

B Sampling from the truncated log-normal distribution

To sample θ^i from the truncated log-normal distribution, we first sample $\log\theta^i \sim \mathrm{t}\mathcal{N}(\log\theta^i \mid \mu_i, \sigma_i^2, a, b)$ and then take the exponent. In order to sample from the truncated normal distribution we use the inverse cumulative density function. The CDF for the truncated normal distribution is

$$F(x) = \frac{\Phi(\frac{x-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}{Z} = y$$

Hence inverse CDF can be written as

$$F^{-1}(y) = \mu + \sigma\Phi^{-1}\Big(\Phi(\alpha) + Zy\Big) = x$$

Sampling y from $\mathcal{U}[0,1]$ one can obtain x samples from truncated normal distribution. The final expression for the reparameterization trick for $\theta \sim \log\mathrm{t}\mathcal{N}(\theta \mid \mu, \sigma^2, a, b)$ looks like this:

$$\theta = \exp\left\{\mu + \sigma\Phi^{-1}\Big(\Phi(\alpha) + Zy\Big)\right\}, \text{ where } y \sim \mathcal{U}(y \mid a, b)$$

C Mean of the truncated log-normal ditribution

Let's derive the expected value of θ. In order to this, let's first find the PDF of the truncated log-normal distribution.

$$\begin{aligned}
&\mathrm{t}\mathcal{N}(x \mid \mu, \sigma^2, a, b) = \frac{1}{Z\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \ x \in [a, b], \\
&\mathrm{t}\mathcal{N}(\log x \mid \mu, \sigma^2, a, b) d(\log x) = \mathrm{t}\mathcal{N}(\log x \mid \mu, \sigma^2, a, b)\frac{dx}{x} = \mathrm{LogN}_{[\mathrm{a,b}]}(x \mid \mu, \sigma^2) dx \\
&\mathrm{LogN}_{[\mathrm{a,b}]}(x \mid \mu, \sigma^2) = \frac{1}{Zx\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\log x - \mu)^2\right), \ \log x \in [a, b], \ x > 0
\end{aligned}$$

The obtained PDF is very similar to the log-normal distribution PDF. Hence,

$$\mathbb{E}\Big[x \sim \mathrm{LogN}_{[\mathrm{a,b}]}(x \,|\, \mu, \sigma^2)\Big] = \frac{1}{Z}\int_{e^a}^{e^b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx =$$
$$= \frac{1}{Z}\bigg[\exp(\mu + \sigma^2/2) - \int_0^{e^a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx -$$
$$- \int_{e^b}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx\bigg]$$

Now we need the formula for

$$p(a) := \int_a^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx$$
$$t = \frac{\log x - \mu}{\sigma}, \quad x = e^{t\sigma+\mu}, \quad dx = e^{t\sigma+\mu}\sigma dt$$

$$p(a) = \int_{\frac{\log a - \mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\bigg(-\frac{1}{2}t^2 + t\sigma + \mu\bigg) dt = \int_{\frac{\log a - \mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\bigg(-\frac{1}{2}(t^2 - 2t\sigma + \sigma^2) + \frac{\sigma^2}{2} + \mu\bigg) dt =$$
$$= \exp(\sigma^2/2 + \mu)\int_{\frac{\log a - \mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\bigg(-\frac{1}{2}(t-\sigma)^2\bigg) dt = \exp(\sigma^2/2 + \mu)\int_{\frac{\log a - \mu}{\sigma} - \sigma}^\infty \frac{1}{\sqrt{2\pi}} \exp\bigg(-\frac{1}{2}t^2\bigg) dt$$
$$p(a) = \exp(\sigma^2/2 + \mu)\bigg(1 - \Phi\bigg(\frac{\log a - \mu}{\sigma} - \sigma\bigg)\bigg) = \exp(\sigma^2/2 + \mu)\Phi\bigg(\frac{\sigma^2 + \mu - \log a}{\sigma}\bigg)$$

Using the derived formula, we finally obtain the expectation $\mathbb{E}x$

$$\mathbb{E}\Big[x \sim \mathrm{LogN}_{[\mathrm{a,b}]}(x \,|\, \mu, \sigma^2)\Big] =$$
$$= \frac{1}{Z}\Big[\exp(\mu + \sigma^2/2) - [\exp(\mu + \sigma^2/2) - p(e^a)] - p(e^b)\Big]$$
$$= \frac{\exp(\mu + \sigma^2/2)}{Z}\bigg[\Phi\bigg(\frac{\sigma^2 + \mu - a}{\sigma}\bigg) - \Phi\bigg(\frac{\sigma^2 + \mu - b}{\sigma}\bigg)\bigg]$$

D Signal-to-noise ratio of the truncated log-normal distribution

It is useful to calculate the signal-to-noise ratio $\mathbb{E}x/\sqrt{\mathrm{Var}(x)}$ for the truncated log-normal distribution in order to investigate the sparsity of the resulting layer. We need the variance $\mathrm{Var}(x)$ to calculate it.

$$\mathrm{LogN}_{[\mathrm{a,b}]}(x \,|\, \mu, \sigma^2) = \frac{1}{Zx\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg), \ \log x \in [a, b], \ x > 0$$

$$\mathrm{Var}\Big[x \sim \mathrm{LogN}_{[\mathrm{a,b}]}(x \,|\, \mu, \sigma^2)\Big] = \frac{1}{Z}\int_{e^a}^{e^b} \frac{(x - \mathbb{E}x)^2}{x\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx =$$
$$= \frac{1}{Z}\bigg[\int_{e^a}^\infty \frac{(x - \mathbb{E}x)^2}{x\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx - \int_{e^b}^\infty \frac{(x - \mathbb{E}x)^2}{x\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx\bigg]$$

So we have to calculate

$$p'(a) := \int_a^\infty \frac{(x - \mathbb{E}x)^2}{x\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx$$

$$p'(a) = \int_a^\infty \frac{x}{\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx$$
$$- 2\mathbb{E}x \int_a^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx$$
$$+ (\mathbb{E}x)^2 \int_a^\infty \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\bigg(-\frac{1}{2\sigma^2}(\log x - \mu)^2\bigg) dx$$

We already have the expression for $p(a)$

$$p(a) = \int_a^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(\log x - \mu)^2\Big) dx = \exp(\sigma^2/2 + \mu)\Phi\Big(\sigma - \frac{\log a - \mu}{\sigma}\Big)$$

For the rest two summands we introduce a new variable t.

$$t = \frac{\log x - \mu}{\sigma}, \quad x = e^{t\sigma + \mu}, \quad dx = e^{t\sigma+\mu}\sigma dt$$

Next, we use t in a variable substitution.

$$\int_a^\infty \frac{x}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(\log x - \mu)^2\Big) dx = \int_{\frac{\log a - \mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}t^2 + 2t\sigma + 2\mu\Big) dt =$$
$$= \int_{\frac{\log a - \mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}(t^2 - 4t\sigma + 4\sigma^2) + 2\mu + 2\sigma^2\Big) dt =$$
$$= \exp(2\mu + 2\sigma^2) \int_{\frac{\log a - \mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}(t - 2\sigma)^2\Big) dt = \exp(2\mu + 2\sigma^2) \int_{\frac{\log a - \mu}{\sigma} - 2\sigma}^\infty \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}t^2\Big) dt =$$
$$= \exp(2\mu + 2\sigma^2)\Phi\Big(2\sigma - \frac{\log a - \mu}{\sigma}\Big)$$

$$\int_a^\infty \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(\log x - \mu)^2\Big) dx = \int_{\frac{\log a - \mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}t^2\Big) dt = \Phi\Big(-\frac{\log a - \mu}{\sigma}\Big)$$

Using the expression for $\mathbb{E}x$ we obtain

$$p'(a) = \exp(2\mu + 2\sigma^2)\Phi\Big(2\sigma - \frac{\log a - \mu}{\sigma}\Big) - 2\exp(\sigma^2/2 + \mu)\Phi\Big(\sigma - \frac{\log a - \mu}{\sigma}\Big)\mathbb{E}x + (\mathbb{E}x)^2\Phi\Big(-\frac{\log a - \mu}{\sigma}\Big)$$

$$\begin{aligned}
\mathrm{Var}\Big[x \sim \mathrm{LogN}_{[a,b]}(x\,|\,\mu, \sigma^2)\Big] &= \frac{1}{Z}\int_{e^a}^{e^b} \frac{(x - \mathbb{E}x)^2}{x\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(\log x - \mu)^2\Big) dx = \\
&= \frac{\exp(2\mu + \sigma^2)}{Z}\Big[\exp(\sigma^2)(\Phi(2\sigma - \alpha) - \Phi(2\sigma - \beta)) \\
&- \frac{2}{Z}(\Phi(\sigma - \alpha) - \Phi(\sigma - \beta))^2 + \frac{1}{Z^2}(\Phi(\sigma - \alpha) - \Phi(\sigma - \beta))^2(\Phi(-\alpha) - \Phi(-\beta))\Big] \\
&= \frac{\exp(2\mu + \sigma^2)}{Z}\Big[\exp(\sigma^2)(\Phi(2\sigma - \alpha) - \Phi(2\sigma - \beta)) - \frac{1}{Z}(\Phi(\sigma - \alpha) - \Phi(\sigma - \beta))^2\Big]
\end{aligned}$$

Finally, we obtain the signal-to-noise ratio

$$\mathrm{SNR}(x) = \frac{\mathbb{E}x}{\sqrt{\mathrm{Var}(x)}} = \frac{\Big[\Phi(\sigma - \alpha) - \Phi(\sigma - \beta)\Big]}{\sqrt{Z\exp(\sigma^2)(\Phi(2\sigma - \alpha) - \Phi(2\sigma - \beta)) - (\Phi(\sigma - \alpha) - \Phi(\sigma - \beta))^2}}$$

E Stable Computation of Statistics

Straightforward computation of the SNR and the mean of the truncated log-normal distribution can lead to indeterminate values like $0 \cdot \infty$ when the values of σ are high. In order to make our calculations stable we use the scaled complementary error function $\mathrm{erfcx}(x) = \exp(x^2)\mathrm{erfc}(x)$. Given the equation

$$\Phi(a) - \Phi(b) = \frac{1}{2}\Big[\mathrm{erf}\Big(\frac{a}{\sqrt{2}}\Big) - \mathrm{erf}\Big(\frac{b}{\sqrt{2}}\Big)\Big] = \frac{1}{2}\Big[\mathrm{erfcx}\Big(\frac{b}{\sqrt{2}}\Big)\exp\Big(-\frac{b^2}{2}\Big) - \mathrm{erfcx}\Big(\frac{a}{\sqrt{2}}\Big)\exp\Big(-\frac{a^2}{2}\Big)\Big],$$

we can rewrite equations (19), (20) in the following form

$$\mathbb{E}\theta = \exp(\mu)\frac{1}{2Z}\left[\operatorname{erfcx}\left(\frac{\sigma-\beta}{\sqrt{2}}\right)\exp\left(b-\mu-\frac{\beta^2}{2}\right)-\operatorname{erfcx}\left(\frac{\sigma-\alpha}{\sqrt{2}}\right)\exp\left(a-\mu-\frac{\alpha^2}{2}\right)\right]$$

$$\begin{aligned}
\mathrm{SNR}(\theta) =& \frac{1}{\sqrt{D}}\left[\operatorname{erfcx}\left(\frac{\sigma-\beta}{\sqrt{2}}\right)\exp\left(b-\mu-\frac{\beta^2}{2}\right)-\operatorname{erfcx}\left(\frac{\sigma-\alpha}{\sqrt{2}}\right)\exp\left(a-\mu-\frac{\alpha^2}{2}\right)\right],\\
D =& 2Z\left[\operatorname{erfcx}\left(\frac{2\sigma-\beta}{\sqrt{2}}\right)\exp\left(2(b-\mu)-\frac{\beta^2}{2}\right)-\operatorname{erfcx}\left(\frac{2\sigma-\alpha}{\sqrt{2}}\right)\exp\left(2(a-\mu)-\frac{\alpha^2}{2}\right)\right]\\
&-\left[\operatorname{erfcx}\left(\frac{\sigma-\beta}{\sqrt{2}}\right)\exp\left(b-\mu-\frac{\beta^2}{2}\right)-\operatorname{erfcx}\left(\frac{\sigma-\alpha}{\sqrt{2}}\right)\exp\left(a-\mu-\frac{\alpha^2}{2}\right)\right]
\end{aligned}$$