8 Supplementary Material for Graph Clustering: Block-models and model free results

Proof of Proposition 2

1. Proof by verification.
2. $LY = Y\hat{\Lambda}Y^TY + (BB^T)\hat{L}(BB^T)Y = Y\hat{\Lambda}$. Since B is the orthogonal complement of Y, it follows that it is a stable subspace as well.
3. This is a well known result; see for example [19].

The celebrated Sinus Theorem is reproduced here for completeness.

Theorem 13 (Sinus Theorem of Davis-Kahan, from [19], Theorem V.3.6) *Let $\hat{L}$ be a Hermitian matrix with spectral resolution given by* (4)*, Y be any $n \times K$ matrix with orthonormal columns, and M any symmetric $K \times K$ matrix with eigenvalues $\mu_{1:K}$. Let $R = \hat{L}Y - YM$ and $\Delta = \min_{\lambda \in \hat{\lambda}_{K+1:n}, \mu \in \mu_{1:K}} |\lambda - \mu| > 0$. Then, for any unitarily invariant norm $||\,||$, $||\operatorname{diag}(\sin\theta_{1:K}(\hat{Y}, Y))|| \leq \frac{||R||}{\Delta}$, where $\theta_{1:K}$ are the canonical angles between $\mathcal{R}(\hat{Y})$ and $\mathcal{R}(Y)$.*

Proof of Proposition 5 This is a corollary of Theorem 3.6 in [19]. If eigenvalues are sorted by their absolute values, then $\hat{\lambda}_{K+1:n} \in [-|\hat{\lambda}_{K+1}|, |\hat{\lambda}_{K+1}|]$ and $\mu_{1:K} \in \mathbb{R} \setminus (-|\hat{\lambda}_{K+1}| - \Delta, |\hat{\lambda}_{K+1}| + \Delta)$. If we set $M = \hat{\Lambda}$, so that $\hat{\lambda}_{1:K} \in \mathbb{R} \setminus (-|\hat{\lambda}_{K+1}| - \Delta, |\hat{\lambda}_{K+1}| + \Delta)$. Now we view Y as a perturbation of $\hat{Y}$, hence

$$R \;=\; \hat{L}Y - Y\hat{\Lambda} \;=\; \hat{L}Y - LY + (LY - Y\hat{\Lambda}) \;=\; (\hat{L} - L)Y \tag{11}$$

$$||R|| \;=\; ||(\hat{L} - L)Y|| \;\leq\; ||\hat{L} - L||\,||Y|| \;\leq\; \varepsilon. \tag{12}$$

From Theorem 13 the result follows. □

Proof of Proposition 6 For 1:

$$\begin{aligned}
||F||_F^2 &= \operatorname{trace} FF^T \;=\; \operatorname{trace} U\Sigma V^TV\Sigma U^T \;=\; \operatorname{trace} U^TU\Sigma V^TV\Sigma \;=\; \operatorname{trace}\Sigma^2 \\
&= 1 + \sum_{k=2}^{K}\cos^2\theta_k \;=\; 1 + \sum_{k=2}^{K}(1 - \sin^2\theta_k) \;=\; K - \sum_{k=2}^{K}\sin^2\theta_k \text{ since } \theta_1 = 0 \quad (13) \\
&\geq K - (K-1)\varepsilon'^2 \quad (14)
\end{aligned}$$

For 2: Denote $\operatorname{trace}\hat{M}^TM = <\hat{M}, M>_F$. Then $||M - \hat{M}||_F^2 = ||M||_F^2 + ||\hat{M}||_F^2 - 2 < \hat{M}, M>_F \leq K + K - 2(K - (K-1)\varepsilon'^2) = 2(K-1)\varepsilon'^2$. □

Proof of Proposition 7 We have that $|<M - \hat{M}, M' - \hat{M}>_F| \leq ||M - \hat{M}||_F||M' - \hat{M}||_F$. From Proposition 6 the r.h.s is no larger than $2(K-1)\varepsilon'^2$.

$$\begin{aligned}
-<M - \hat{M}, M' - \hat{M}>_F &\leq ||M - \hat{M}||_F||M' - \hat{M}||_F \;\leq\; 2(K-1)\varepsilon'^2 \quad (15) \\
-<M, M'>_F &+ <\hat{M}, M>_F + <\hat{M}, M'>_F - ||\hat{M}||_F^2 \;\leq\; 2(K-1)\varepsilon'^2 \quad (16) \\
<M, M'>_F &\geq <\hat{M}, M>_F + <\hat{M}, M'>_F - K - 2(K-1)\varepsilon'^2 \quad (17) \\
&\geq 2K - 2(K-1)\varepsilon'^2 - K - 2(K-1)\varepsilon'^2 \;=\; K - 4(K-1)\varepsilon'^2 \quad (18)
\end{aligned}$$

Now, note that $\operatorname{trace} MM' = \operatorname{trace} YY^TY'(Y')^T = \operatorname{trace}((Y')^TY))(Y^TY') = ||Y^TY'||_F^2$. Moreover, by (7), Y_Z and Y differ by a unitary transformation. Since $||\,||_F$ is unitarily invariant, the result follows.

Proof of Theorem 4 We apply Theorem 9 of [13] with $A_X = Z$, $A_{X'} = Z'$, and $\tilde{A}_X = Y$, $\tilde{A}_{X'} = Y'$. It follows that $p_{XY_{kk'}} = \sum_{i \in k \cap k'} \hat{d}_i / \sum_{i=1}^{n} \hat{d}_i$. Hence, the point weights are proportional to $\hat{d}_{1:n}$. Also, evidently, $p_{min}/p_{max} = \delta_0$, and the result follows.

Note that we use the fact that both PFM's have degrees equal to $\hat{d}_{1:n}$ to obtain this proof. □

Proposition 14 *Assumptions 3 and 4, imply $||\operatorname{diag}(\sin\theta_{1:K}(\hat{Y}, Y))|| \leq \varepsilon / |\hat{\lambda}_K^A| = \varepsilon'$, where $\hat{\lambda}_K^A$ is the K-th eigenvalue of $\hat{A}$.*

Proof of Proposition 14 We consider $\hat{A}$ a perturbation of A, its eigenvectors $\hat{Y}$ as the perturbed eigenvectors of A and $M = \hat{\Lambda}$. Then, $R = A\hat{Y} - \hat{Y}\hat{\Lambda}$

$$
\begin{aligned}
||R|| &= ||A\hat{Y} - \hat{Y}\hat{\Lambda}|| && (19)\\
&= ||(A\hat{Y} - \hat{A}\hat{Y}) + (\hat{A}\hat{Y} - \hat{Y}\hat{\Lambda})|| && (20)\\
&\leq ||(A - \hat{A})\hat{Y}|| && (21)\\
&\leq ||A - \hat{A}||||\hat{Y}|| \;\leq\; \varepsilon. && (22)
\end{aligned}
$$

The separation between $\hat{\Lambda}$ and the residual spectrum of A is $|\hat{\lambda}_K|$. From the main Davis-Kahan theorem 13 the result follows. □

Proof of Proposition 8 The proofs of 1 and 2 are straightforward. To show 3, note that $A = ZC^{-1}Z^T\hat{A}ZC^{-1}Z^T = Y_Z C^{1/2}BC^{1/2}Y_Z^T = Y_Z U\Lambda U^T Y_Z^T = Y\Lambda Y^T$. The definition of B above shows that this is the Maximum Likelihood estimator of B given the clustering $\mathcal{C}$.

$$\Leftrightarrow \quad B_{kl} = \frac{\#\text{edges from cluster } k \text{ to cluster } l}{n_k n_l} \tag{23}$$

Proof of Theorem 9 We now follow the steps outlined in section 3 with ε' from Proposition 14 to obtain our main stability result.

Proof of Proposition 10 In the Proof of Proposition 7, we replace the bounds corresponding to $< \hat{M}, M >_F, ||\hat{M} - M||_F$ by the actual values computed from $M, \hat{M}$. We obtain

$$< M, M' >_F \geq < \hat{M}, M >_F - (K-1)(\varepsilon')^2 - 2\sqrt{2(K-1)}\varepsilon' ||\hat{M} - M||_F. \tag{24}$$

Proof of Proposition 3

From the Proof of this theorem, we have that $||L^* - \hat{L}|| = o(1)$, $||(D^*)^{1/2} - \hat{D}^{1/2}|| = o(1)$, $||\lambda^* - \hat{\Lambda}|| = o(1)$, and $||\hat{Y} - Y^*|| = o(1)$. Let Z be the indicator matrix of $\mathcal{C}^*$. The principal eigenvectors of L^* are $Y^* = (D^*)^{1/2}Z(C^*)^{-1/2}$. It follows then that $||Z^T\hat{D}Z - Z^T D^* Z|| = o(1)$, and since $C = Z^T\hat{D}Z, Y_Z = \hat{D}^{1/2}ZC^{-1/2}$ we have that $||Y_Z - Y^*|| = o(1)$, $||F^* - F|| = o(1)$ where $F^* = Y^T Y^*$. Moreover, since $||\hat{Y} - Y^*|| = o(1)$, $||F - I|| = o(1)$ Hence $||UV^T - I|| = o(1)$. Since the choice of B depends only on $\mathcal{R}(Y_Z)$, it follows immediately that $||BB^T\hat{L}B^TB - B^*(B^*)^T L^*(B^*)^T B^*|| = o(1)$. Now, $L = Y_Z UV^T\hat{\Lambda}VU^T Y_Z^T + BB^T\hat{L}B^TB$, and $L^* = Y^*\Lambda^*(Y^*)^T + B^*(B^*)^T L^*(B^*)^T B^*$, which completes the proof. □

perturbation of the PFM model To obtain a noisy PFM model A, we calculate the first K piecewise constant [14] eigenvectors V of the transition matrix $P = D^{-1}A$, from which we obtain V^* by perturbing each entry in V with a noise $\epsilon \sim unif(0, 10^{-4})$. The perturbed similarity matrix A is then obtained as $A = D^{1/2}(D^{1/2}V^*\hat{\Lambda}V^{*T}D^{1/2} + \hat{Y}_{low}\hat{\Lambda}_{low}\hat{Y}_{low}^T)D^{1/2}$. An adjacency matrix $\hat{A}$ is generated from A. In figure 2, we show the perturbed graphs A and $\hat{A}$.

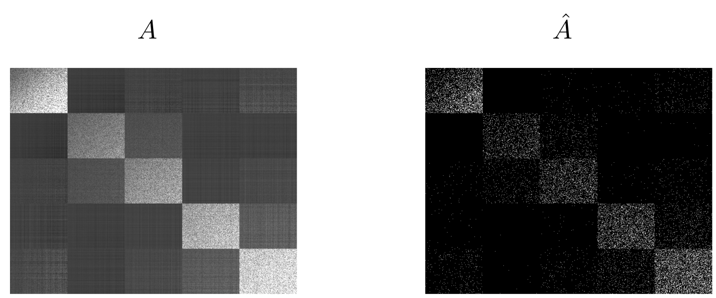

Figure 2: Left: the visualization of the perturbed A. Right: the visualization of the perturbed $\hat{A}$