Real-Time Decoding of an Integrate and Fire Encoder

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Appendix
We present Theorem 2 in the main text again, and provide the proof here.

Theorem 2. *Real-time reconstruction: Given a signal $f \in \mathcal{L}^{\Omega}_{2,\beta}$ passed through an IAF encoder with known thresholds, and given that the spikes satisfy a certain minimum density $\sup_{i\in\mathbb{Z}}(t_{i+1}-t_i) = \delta$ for some $\delta < \frac{\Omega}{\pi}$, we can construct a causal real-time decoder that reconstructs a function $\tilde{f}_t(t)$ using the recursive algorithm in Equations 11 and 12, s.t.*

$$|f(t) - \tilde{f}_t(t)| \leq \frac{c}{1-\frac{\delta\Omega}{\pi}}\|f\|_{2,\beta}(1+t)^{-\beta} \tag{1}$$

, where c depends only on δ, Ω and β.
Moreover, if we use a finite number of iterations K at every step, we obtain the following error.

$$|f(t) - \tilde{f}^K_t(t)| \leq c\frac{1-\left(\frac{\delta\Omega}{\pi}\right)^{K+1}}{1-\frac{\delta\Omega}{\pi}}\|f\|_{2,\beta}(1+t)^{-\beta} + \left(\frac{\delta\Omega}{\pi}\right)^{K+1}\frac{1+\frac{\delta\Omega}{\pi}}{1-\frac{\delta\Omega}{\pi}}\|f\|_2 \tag{2}$$

Proof. We start with a few preliminaries, and the definition of a constant c that will be used later in the proof. These preliminary notions are also provided in [1]. We define a local maximum function on f:

$$f^{\#}(t) = \sup_{|u|\leq\delta}|f(t+u)| \tag{3}$$

Note the following two properties of $f^{\#}(t)$ for $f \in \mathcal{L}^{\Omega}_{2,\beta}$, with a function $p_\alpha(t)$ such that $\hat{p}_\alpha(\omega) = 1$ for $\omega \in [-\Omega,\Omega]$, and $p_\alpha \in \mathcal{L}_{1,\alpha}$, for some $\alpha \geq \beta$. Here, $*$ denotes the convolution operator.

$$|\sum_{i=1}^{\infty} f(s_i)\mathbb{1}_{[t_i,t_{i+1}]}(t)| \quad \leq \quad f^{\#}(t) \text{ pointwise} \tag{4}$$

$$f^{\#}(t) = (f * p_\alpha)^{\#}(t) \quad \leq \quad (|f| * p^{\#}_\alpha)(t) \tag{5}$$

As a consequence of equation (5), we obtain the following bound on the $\mathcal{L}_{2,\beta}$ norm of the local maximum function (3), using a function $p_\alpha(t)$ as described above.

$$\|f^{\#}\|_{2,\beta} \leq \inf_{p_\alpha}\|p^{\#}_\alpha\|_{1,\alpha}\|f\|_{2,\beta} \tag{6}$$

We denote $\inf_{p_\alpha}\|p^{\#}_\alpha\|_{1,\alpha}$ for some $\alpha \geq \beta$ by c, which depends only on δ, Ω and β.

We now bound the $\mathcal{L}_2$ norm of the error incurred using a decoder acting on all the spikes in a finite time period T, i.e. $\{t_i\}_{i:|t_i|\leq T}$, and show that this error is decaying in T.

We consider that after the first approximation $\mathcal{A}_T f$ with a finite number of spikes, we can construct $\{t_i\}_{i:|t_i|>T}$ such that $\sup_{i:|t_i|>T}(t_{i+1}-t_i)$ is less than the required δ. Thus we can construct an operator $\mathcal{A}$ as long as it is not acting directly on f, where $\mathcal{A}$ is defined as in Theorem 1 in the main text. The adjoint operators of $\mathcal{A}$ and $\mathcal{A}_T$ for $f \in \mathcal{L}_2^{\Omega}$ are $\mathcal{A}^*$ and $\mathcal{A}_T^*$ respectively.

$$\mathcal{A}^* f = \sum_{i \in \mathbb{Z}} f(s_i)(\mathrm{sinc}_{\Omega} * \mathbb{1}_{[t_i, t_{i+1}]}) \tag{7}$$

$$\mathcal{A}_T^* f = \sum_{i:|t_i| \leq T} f(s_i)(\mathrm{sinc}_{\Omega} * \mathbb{1}_{[t_i, t_{i+1}]}) \tag{8}$$

Equation 7 is shown in [2], and Equation 8 follows similarly.
We first define $\tilde{f}_T$ as the result of the following iteration, i.e. $\tilde{f}_T = \lim_{k \to \infty} \tilde{f}_T^k$.

$$\tilde{f}_T^0 = \mathcal{A}_T f \tag{9}$$

$$\tilde{f}_T^1 = (I - \mathcal{A})\tilde{f}_T^0 + \mathcal{A}_T f = (I - \mathcal{A})\mathcal{A}_T f + \mathcal{A}_T f \tag{10}$$

$$\tilde{f}_T^k = (I - \mathcal{A})\tilde{f}_T^{k-1} + \mathcal{A}_T f = \sum_{n=0}^{k} (I - \mathcal{A})^n \mathcal{A}_T f \tag{11}$$

To derive $\|f - \tilde{f}_T\|_2$ for $f \in \mathcal{L}_2^{\Omega}$, we first note that the error incurred using a finite number of spikes is the same as the error in the adjoint space, i.e. $\|f - \tilde{f}_T\|_2 = \left\|f - \sum_{n=0}^{\infty}(I - \mathcal{A})^n \mathcal{A}_T f\right\|_2 = \left\|f - \sum_{n=0}^{\infty}(I - \mathcal{A}^*)^n \mathcal{A}_T^* f\right\|_2$

We can thus work exclusively with the adjoint operators $\mathcal{A}^* f$ and $\mathcal{A}_T^* f$ in order to derive $\|f - \tilde{f}_T\|_2$ [3].

$$\begin{aligned}
\|f - \tilde{f}_T\|_2 &= \left\| \sum_{n=0}^{\infty} (I - \mathcal{A}^*)^n (\mathcal{A}^* f - \mathcal{A}_T^* f) \right\|_2 && (12)\\
&\leq \sum_{n=0}^{\infty} \left\|(I - \mathcal{A}^*)^n (\mathcal{A}^* f - \mathcal{A}_T^* f)\right\|_2 && (13)\\
&\leq \sum_{n=0}^{\infty} \left(\frac{\delta\Omega}{\pi}\right)^{n+1} \left\|\mathcal{A}^* f - \mathcal{A}_T^* f\right\|_2 && (14)\\
&= \frac{1}{1 - \frac{\delta\Omega}{\pi}} \left\| \sum_{i:|t_i| > T} f(s_i) \mathbb{1}_{[t_i, t_{i+1}]} * \mathrm{sinc}_{\Omega} \right\|_2 && (15)\\
&\leq \frac{1}{1 - \frac{\delta\Omega}{\pi}} \left\| \sum_{i:|t_i| > T} f(s_i) \mathbb{1}_{[t_i, t_{i+1}]} \right\|_2 && (16)\\
&\leq \frac{1}{1 - \frac{\delta\Omega}{\pi}} \left\| f^{\#} \mathbb{1}_{\mathbb{R} \setminus [-T, T]} \right\|_2 \quad (using\ (5)) && (17)\\
&= \frac{1}{1 - \frac{\delta\Omega}{\pi}} \left\| f^{\#} \mathbb{1}_{\mathbb{R} \setminus [-T, T]} (1 + |t|)^{\beta} (1 + |t|)^{-\beta} \right\|_2 && (18)\\
&\leq \frac{1}{1 - \frac{\delta\Omega}{\pi}} \left\| f^{\#} \right\|_{2,\beta} \sup_{t \in \mathbb{R} \setminus [-T, T]} (1 + |t|)^{-\beta} && (19)\\
&\leq \frac{c}{1 - \frac{\delta\Omega}{\pi}} \|f\|_{2,\beta} (1 + T)^{-\beta} \quad (using\ (6)) && (20)
\end{aligned}$$

, where c depends only on δ, Ω and β.

Now, only using a finite number of iterations, we have the following error bound.

$$\begin{aligned}
\|f - \tilde{f}_T^K\|_2 &= \left\| \sum_{n=0}^{\infty} (I - \mathcal{A}^*)^n \mathcal{A}^* f - \sum_{n=0}^{K} (I - \mathcal{A}^*)^n \mathcal{A}_T^* f \right\|_2 && (21)\\
&\leq \left\| \sum_{n=0}^{K} (I - \mathcal{A}^*)^n (\mathcal{A}^* f - \mathcal{A}_T^* f) \right\|_2 + \left\| \sum_{n=K+1}^{\infty} (I - \mathcal{A}^*)^n \mathcal{A}^* f \right\|_2 && (22)\\
&\leq c \frac{1 - \left(\frac{\delta\Omega}{\pi}\right)^{K+1}}{1 - \frac{\delta\Omega}{\pi}} \|f\|_{2,\beta} (1 + T)^{-\beta} + \left(\frac{\delta\Omega}{\pi}\right)^{K+1} \frac{1 + \frac{\delta\Omega}{\pi}}{1 - \frac{\delta\Omega}{\pi}} \|f\|_2 && (23)
\end{aligned}$$

We now construct $\tilde{f}_t(t)$ using spikes $\{t_i\}_{i:|t_i|\leq t}$ at every time t. Thus, at every time t, we have a causal decoder that uses all spikes that have already occurred. We bound the error at every time t as the following.

$$
\begin{aligned}
|f(t) - \tilde{f}_t(t)| &\leq \sup_{\tau\in\mathbb{R}} |f(\tau) - \tilde{f}_t(\tau)| && (24)\\
&\leq \|f - \tilde{f}_t\|_2 && (25)\\
&\leq \frac{c}{1 - \frac{\delta\Omega}{\pi}} \|f\|_{2,\beta} (1+t)^{-\beta} && (26)
\end{aligned}
$$

Here, we used the fact that $\|x\|_\infty \leq \|x\|_2 \; \forall x \in \mathcal{L}_2$ for the inequality in Equation 25, and Equation 20 for the inequality in Equation 26. The proof for Equation 2 follows similarly from Equation 23. We note that as a new spike t_{i+1} arrives, we can calculate the new estimate as a function of the old estimate due to the following.

$$
\tilde{f}^0_{t_{i+1}} = \mathcal{A}_{t_{i+1}} f = \mathcal{A}_{t_i} f + \left(\int_{t_i}^{t_{i+1}} f(\tau) d\tau \right) \operatorname{sinc}(t - s_i) = \tilde{f}^0_{t_i} + g^0_{t_{i+1}} \qquad (27)
$$

We can carry the term $g^0_{t_{i+1}}$ forward, and track its effect on future iterations to calculate $g^k_{t_{i+1}}$ as a function of $g^{k-1}_{t_{i+1}}$, to obtain Equation 12 in the main text. $\square$

References

[1] H. G. Feichtinger and K. Gröchenig, “Irregular sampling theorems and series expansions of band-limited functions,” *Journal of mathematical analysis and applications*, vol. 167, no. 2, pp. 530–556, 1992.

[2] ——, “Theory and practice of irregular sampling,” *Wavelets: mathematics and applications*, vol. 1994, pp. 305–363, 1994.

[3] P. Butzer, H. G. Feichtinger, and K. Grōchenig, “Error analysis in regular and irregular sampling theory,” *Applicable Analysis*, vol. 50, no. 3-4, pp. 167–189, 1993.