A Online convex optimization on the simplex

By using a standard reduction, the results of the main body of the paper (for linear optimization on the simplex) can be applied to online convex optimization on the simplex. In this setting, at each step t the forecaster chooses $\widehat{\boldsymbol{p}}_t \in \Delta_d$ and then is given access to a convex loss $\ell_t : \Delta_d \to [0,1]$. Now, using Algorithm 1 with the loss vector $\boldsymbol{\ell}_t \in \partial \ell_t(\widehat{\boldsymbol{p}}_t)$ given by a subgradient of ℓ_t leads to the desired bounds. Indeed, by the convexity of ℓ_t, the regret at each time t with respect to any vector $\boldsymbol{u}_t \in \mathbb{R}_+^d$ with $\|\boldsymbol{u}_t\|_1 > 0$ is then bounded as

$$\|\boldsymbol{u}_t\|_1 \left(\ell_t(\widehat{\boldsymbol{p}}_t) - \ell_t\!\left(\frac{\boldsymbol{u}_t}{\|\boldsymbol{u}_t\|_1} \right) \right) \leqslant \big(\|\boldsymbol{u}_t\|_1 \widehat{\boldsymbol{p}}_t - \boldsymbol{u}_t \big)^\top \boldsymbol{\ell}_t \,.$$

B Proof of Theorem 3; application of the bound to two different updates

Proof. The beginning and the end of the proof are similar to the one of Theorem 2, as they do not depend on the specific weight update. In particular, inequalities (6) and (7) remain the same. The proof is modified after (8), which this time we upper bound using the first condition in (12),

$$\begin{aligned} \sum_{i=1}^d \left(u_{i,t} \ln \frac{1}{\widehat{p}_{i,t}} - u_{i,t-1} \ln \frac{1}{v_{i,t}} \right) = & \sum_{i\,:\,u_{i,t} \geqslant u_{i,t-1}} (u_{i,t} - u_{i,t-1}) \ln \frac{1}{\widehat{p}_{i,t}} + u_{i,t-1} \ln \frac{v_{i,t}}{\widehat{p}_{i,t}} \\ & + \sum_{i\,:\,u_{i,t} < u_{i,t-1}} \underbrace{(u_{i,t} - u_{i,t-1})}_{\leqslant 0} \underbrace{\ln \frac{1}{v_{i,t}}}_{\geqslant \ln(1/w_{i,t})} + u_{i,t} \ln \frac{v_{i,t}}{\widehat{p}_{i,t}} \,. \qquad (14) \end{aligned}$$

By definition of the shared update (11), we have $1/\widehat{p}_{i,t} \leqslant Z_t/(\alpha\, w_{i,t})$ and $v_{i,t}/\widehat{p}_{i,t} \leqslant 1/(1-\alpha)$. We then upper bound the quantity at hand in (14) by

$$\begin{aligned} & \sum_{i\,:\,u_{i,t} \geqslant u_{i,t-1}} (u_{i,t} - u_{i,t-1}) \ln \left(\frac{Z_t}{\alpha\, w_{i,t}} \right) + \left(\sum_{i\,:\,u_{i,t} \geqslant u_{i,t-1}} u_{i,t-1} + \sum_{i\,:\,u_{i,t} < u_{i,t-1}} u_{i,t} \right) \ln \frac{1}{1-\alpha} \\ & + \sum_{i\,:\,u_{i,t} < u_{i,t-1}} (u_{i,t} - u_{i,t-1}) \ln \frac{1}{w_{i,t}} \\ = \; & D_{\mathrm{TV}}(\boldsymbol{u}_t, \boldsymbol{u}_{t-1}) \ln \frac{Z_t}{\alpha} + \big(\|\boldsymbol{u}_t\|_1 - D_{\mathrm{TV}}(\boldsymbol{u}_t, \boldsymbol{u}_{t-1}) \big) \ln \frac{1}{1-\alpha} + \sum_{i=1}^d (u_{i,t} - u_{i,t-1}) \ln \frac{1}{w_{i,t}} \,. \end{aligned}$$

Proceeding as in the end of the proof of Theorem 2, we then get the claimed bound, provided that we can show that

$$\sum_{t=2}^T \sum_{i=1}^d (u_{i,t} - u_{i,t-1}) \ln \frac{1}{w_{i,t}} \leqslant n(\boldsymbol{u}_1^T)\,(\ln d + T \ln C) - \|\boldsymbol{u}_1\|_1 \ln d\,,$$

which we do next. Indeed, the left-hand side can be rewritten as

$$\begin{aligned} & \sum_{t=2}^T \sum_{i=1}^d \left(u_{i,t} \ln \frac{1}{w_{i,t}} - u_{i,t} \ln \frac{1}{w_{i,t+1}} \right) + \sum_{t=2}^T \sum_{i=1}^d \left(u_{i,t} \ln \frac{1}{w_{i,t+1}} - u_{i,t-1} \ln \frac{1}{w_{i,t}} \right) \\ \leqslant \; & \left(\sum_{t=2}^T \sum_{i=1}^d u_{i,t} \ln \frac{C\, w_{i,t+1}}{w_{i,t}} \right) + \left(\sum_{i=1}^d u_{i,T} \ln \frac{1}{w_{i,T+1}} - \sum_{i=1}^d u_{i,1} \ln \frac{1}{w_{i,2}} \right) \\ \leqslant \; & \left(\sum_{i=1}^d \left(\max_{t=1,\ldots,T} u_{i,t} \right) \sum_{t=2}^T \ln \frac{C\, w_{i,t+1}}{w_{i,t}} \right) + \left(\sum_{i=1}^d \left(\max_{t=1,\ldots,T} u_{i,t} \right) \ln \frac{1}{w_{i,T+1}} - \sum_{i=1}^d u_{i,1} \ln \frac{1}{w_{i,2}} \right) \\ = \; & \sum_{i=1}^d \left(\max_{t=1,\ldots,T} u_{i,t} \right) \left((T-1) \ln C + \ln \frac{1}{w_{i,2}} \right) - \sum_{i=1}^d u_{i,1} \ln \frac{1}{w_{i,2}} \,, \end{aligned}$$

where we used $C \geqslant 1$ for the first inequality and the second condition in (12) for the second inequality. The proof is concluded by noting that (12) entails $w_{i,2} \geqslant (1/C) w_{i,1} \geqslant (1/C) v_{i,1} = 1/(dC)$ and that the coefficient $\max_{t=1,\ldots,T} u_{i,t} - u_{i,1}$ in front of $\ln(1/w_{i,2})$ is nonnegative. □

The first update uses $w_{j,t} = \max_{s \leqslant t} v_{j,s}$. Then (12) is satisfied with $C = 1$. Moreover, since a sum of maxima of nonnegative elements is smaller than the sum of the sums, $Z_t \leqslant \min\{d, t\} \leqslant T$. This immediately gives the following result.

Corollary 4. *Suppose Algorithm 1 is run with the update* (11) *with* $w_{j,t} = \max_{s \leqslant t} v_{j,s}$*. For all* $T \geqslant 1$*, for all sequences* $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$ *of loss vectors* $\boldsymbol{\ell}_t \in [0,1]^d$*, and for all* $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_T \in \Delta_d$*,*

$$\sum_{t=1}^{T} \widehat{\boldsymbol{p}}_t^\top \boldsymbol{\ell}_t - \sum_{t=1}^{T} \boldsymbol{q}_t^\top \boldsymbol{\ell}_t \leqslant \frac{n(\boldsymbol{q}_1^T) \ln d}{\eta} + \frac{\eta}{8} T + \frac{m(\boldsymbol{q}_1^T)}{\eta} \ln \frac{T}{\alpha} + \frac{T - m(\boldsymbol{q}_1^T) - 1}{\eta} \ln \frac{1}{1-\alpha} \,.$$

The second update we discuss uses $w_{j,t} = \max_{s \leqslant t} e^{\gamma(s-t)} v_{j,s}$ in (11) for some $\gamma > 0$. Both conditions in (12) are satisfied with $C = e^\gamma$. One also has that

$$Z_t \leqslant d \qquad \text{and} \qquad Z_t \leqslant \sum_{\tau \geqslant 0} e^{-\gamma \tau} = \frac{1}{1 - e^{-\gamma}} \leqslant \frac{1}{\gamma}$$

as $e^x \geqslant 1 + x$ for all real x. The bound of Theorem 3 then instantiates as

$$\frac{n(\boldsymbol{q}_1^T) \ln d}{\eta} + \frac{n(\boldsymbol{q}_1^T)\, T\gamma}{\eta} + \frac{\eta}{8} T + \frac{m(\boldsymbol{q}_1^T)}{\eta} \ln \frac{\min\{d,\, 1/\gamma\}}{\alpha} + \frac{T - m(\boldsymbol{q}_1^T) - 1}{\eta} \ln \frac{1}{1-\alpha}$$

when sequences $\boldsymbol{u}_t = \boldsymbol{q}_t \in \Delta_d$ are considered. This bound is best understood when γ is tuned optimally based on T and on two bounds m_0 and n_0 over the quantities $m(\boldsymbol{q}_1^T)$ and $n(\boldsymbol{q}_1^T)$. Indeed, by optimizing $n_0 T\gamma + m_0 \ln(1/\gamma)$, i.e., by choosing $\gamma = m_0/(n_0\, T)$, one gets a bound that improves on the one of the previous corollary:

Corollary 5. *Let* $m_0, n_0 > 0$*. Suppose Algorithm 1 is run with the update* $w_{j,t} = \max_{s \leqslant t} e^{\gamma(s-t)} v_{j,s}$ *where* $\gamma = m_0/(n_0\, T)$*. For all* $T \geqslant 1$*, for all sequences* $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$ *of loss vectors* $\boldsymbol{\ell}_t \in [0,1]^d$*, and for all* $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_T \in \Delta_d$ *such that* $m(\boldsymbol{q}_1^T) \leqslant m_0$ *and* $n(\boldsymbol{q}_1^T) \leqslant n_0$*, we have*

$$\begin{aligned} \sum_{t=1}^{T} \widehat{\boldsymbol{p}}_t^\top \boldsymbol{\ell}_t - \sum_{t=1}^{T} \boldsymbol{q}_t^\top \boldsymbol{\ell}_t \quad &\leqslant \quad \frac{n_0 \ln d}{\eta} + \frac{m_0}{\eta} \left(1 + \ln \, \min\left\{ d,\, \frac{n_0\, T}{m_0} \right\} \right) \\ &\quad + \frac{\eta}{8} T + \frac{m_0}{\eta} \ln \frac{1}{\alpha} + \frac{T - m_0 - 1}{\eta} \ln \frac{1}{1-\alpha} \,. \end{aligned}$$

As the factors $e^{-\gamma t}$ cancel out in the numerator and denominator of the ratio in (11), there is a straightforward implementation of the algorithm (not requiring the knowledge of T) that needs to maintain only d weights.

In contrast, the corresponding algorithm of [6], using the updates $\widehat{p}_{j,t} = (1-\alpha) v_{j,t} + \alpha S_t^{-1} \sum_{s \leqslant t-1} (s-t)^{-1} v_{j,s}$ or $\widehat{p}_{j,t} = (1-\alpha) v_{j,t} + \alpha S_t^{-1} \max_{s \leqslant t-1} (s-t)^{-1} v_{j,s}$, where S_t denote normalization factors, needs to maintain $O(dT)$ weights with a naive implementation, and $O(d \ln T)$ weights with a more sophisticated one. In addition, the obtained bounds are slightly worse than the one stated above in Corollary 5 as an additional factor of $m_0 \ln(1 + \ln T)$ is present in [6, Corollary 9].

C Proof of Theorem 4; illustration of the obtained bound

We first adapt Lemma 1.

Lemma 3. *The forecaster based on the loss and shared updates* (13) *satisfies, for all* $t \geqslant 1$ *and for all* $\boldsymbol{q}_t \in \Delta_d$*,*

$$\left(\widehat{\boldsymbol{p}}_t - \boldsymbol{q}_t \right)^\top \boldsymbol{\ell}_t \leqslant \sum_{i=1}^{d} q_{i,t} \left(\frac{1}{\eta_{t-1}} \ln \frac{1}{\widehat{p}_{i,t}} - \frac{1}{\eta_t} \ln \frac{1}{v_{i,t+1}} \right) + \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \ln d + \frac{\eta_{t-1}}{8} \,,$$

whenever $\eta_t \leqslant \eta_{t-1}$*.*

Proof. By Hoeffding's inequality,

$$\sum_{j=1}^{d} \widehat{p}_{j,t}\, \ell_{j,t} \leqslant -\frac{1}{\eta_{t-1}} \ln \left(\sum_{j=1}^{d} \widehat{p}_{j,t}\, e^{-\eta_{t-1}\, \ell_{j,t}} \right) + \frac{\eta_{t-1}}{8} \,.$$

By Jensen's inequality, since $\eta_t \leqslant \eta_{t-1}$ and thus $x \mapsto x^{\frac{\eta_{t-1}}{\eta_t}}$ is convex,

$$\frac{1}{d} \sum_{j=1}^{d} \widehat{p}_{j,t}\, e^{-\eta_{t-1}\ell_{j,t}} = \frac{1}{d} \sum_{j=1}^{d} \left(\widehat{p}_{j,t}^{\,\frac{\eta_t}{\eta_{t-1}}}\, e^{-\eta_t \ell_{j,t}} \right)^{\frac{\eta_{t-1}}{\eta_t}} \geqslant \left(\frac{1}{d} \sum_{j=1}^{d} \widehat{p}_{j,t}^{\,\frac{\eta_t}{\eta_{t-1}}}\, e^{-\eta_t \ell_{j,t}} \right)^{\frac{\eta_{t-1}}{\eta_t}} .$$

Substituting in Hoeffding's bound we get

$$\widehat{\boldsymbol{p}}_t^{\top} \boldsymbol{\ell}_t \leqslant -\frac{1}{\eta_t} \ln \left(\sum_{j=1}^{d} \widehat{p}_{j,t}^{\,\frac{\eta_t}{\eta_{t-1}}}\, e^{-\eta_t \ell_{j,t}} \right) + \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \ln d + \frac{\eta_{t-1}}{8} \,.$$

Now, by definition of the loss update in (13), for all $i \in \{1, \ldots, d\}$,

$$\sum_{j=1}^{d} \widehat{p}_{j,t}^{\,\frac{\eta_t}{\eta_{t-1}}}\, e^{-\eta_t \ell_{j,t}} = \frac{1}{v_{i,t+1}}\, \widehat{p}_{i,t}^{\,\frac{\eta_t}{\eta_{t-1}}}\, e^{-\eta_t \ell_{i,t}} \,,$$

which, after substitution in the previous bound leads to the inequality

$$\widehat{\boldsymbol{p}}_t^{\top} \boldsymbol{\ell}_t \leqslant \ell_{i,t} + \frac{1}{\eta_{t-1}} \ln \frac{1}{\widehat{p}_{i,t}} - \frac{1}{\eta_t} \ln \frac{1}{v_{i,t+1}} + \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \ln d + \frac{\eta_{t-1}}{8} \,,$$

valid for all $i \in \{1, \ldots, d\}$. The proof is concluded by taking a convex aggregation over i with respect to $\boldsymbol{q}_t$. $\square$

The proof of Theorem 4 follows the steps of the one of Theorem 2; we sketch it below.

Proof of Theorem 4. Applying Lemma 3 with $\boldsymbol{q}_t = \boldsymbol{u}_t / \|\boldsymbol{u}_t\|_1$, and multiplying by $\|\boldsymbol{u}_t\|_1$, we get for all $t \geqslant 1$ and $\boldsymbol{u}_t \in \mathbb{R}_+^d$,

$$\begin{aligned} \|\boldsymbol{u}_t\|_1 \, \widehat{\boldsymbol{p}}_t^{\top} \boldsymbol{\ell}_t - \boldsymbol{u}_t^{\top} \boldsymbol{\ell}_t \leqslant \frac{1}{\eta_{t-1}} \sum_{i=1}^{d} u_{i,t} \ln \frac{1}{\widehat{p}_{i,t}} - \frac{1}{\eta_t} \sum_{i=1}^{d} u_{i,t} \ln \frac{1}{v_{i,t+1}} & \\ + \|\boldsymbol{u}_t\|_1 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \ln d + \frac{\eta_{t-1}}{8} \, \|\boldsymbol{u}_t\|_1 \,. & \qquad (15) \end{aligned}$$

We will sum these bounds over $t \geqslant 1$ to get the desired result but need to perform first some additional boundings for $t \geqslant 2$; in particular, we examine

$$\begin{aligned} & \frac{1}{\eta_{t-1}} \sum_{i=1}^{d} u_{i,t} \ln \frac{1}{\widehat{p}_{i,t}} - \frac{1}{\eta_t} \sum_{i=1}^{d} u_{i,t} \ln \frac{1}{v_{i,t+1}} \\ & \quad = \frac{1}{\eta_{t-1}} \sum_{i=1}^{d} \left(u_{i,t} \ln \frac{1}{\widehat{p}_{i,t}} - u_{i,t-1} \ln \frac{1}{v_{i,t}} \right) + \sum_{i=1}^{d} \left(\frac{u_{i,t-1}}{\eta_{t-1}} \ln \frac{1}{v_{i,t}} - \frac{u_{i,t}}{\eta_t} \ln \frac{1}{v_{i,t+1}} \right) , \qquad (16) \end{aligned}$$

where the first difference in the right-hand side can be bounded as in (8) by

$$\begin{aligned} & \sum_{i=1}^{d} \left(u_{i,t} \ln \frac{1}{\widehat{p}_{i,t}} - u_{i,t-1} \ln \frac{1}{v_{i,t}} \right) \\ & \quad \leqslant \sum_{i \,:\, u_{i,t} \geqslant u_{i,t-1}} \left((u_{i,t} - u_{i,t-1}) \ln \frac{1}{\widehat{p}_{i,t}} + u_{i,t-1} \ln \frac{v_{i,t}}{\widehat{p}_{i,t}} \right) + \sum_{i \,:\, u_{i,t} < u_{i,t-1}} u_{i,t} \ln \frac{v_{i,t}}{\widehat{p}_{i,t}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \quad D_{TV}(\boldsymbol{u}_t,\boldsymbol{u}_{t-1}) \ln \frac{d}{\alpha_t} + \big(\|\boldsymbol{u}_t\|_1 - D_{TV}(\boldsymbol{u}_t,\boldsymbol{u}_{t-1})\big) \ln \frac{1}{1-\alpha_t} \\
&\leqslant \quad D_{TV}(\boldsymbol{u}_t,\boldsymbol{u}_{t-1}) \ln \frac{d(1-\alpha_T)}{\alpha_T} + \|\boldsymbol{u}_t\|_1 \ln \frac{1}{1-\alpha_t}\,, \qquad (17)
\end{aligned}
$$

where we used for the second inequality that the shared update in (13) is such that $1/\widehat{p}_{i,t} \leqslant d/\alpha_t$ and $v_{i,t}/\widehat{p}_{i,t} \leqslant 1/(1-\alpha_t)$, and for the third inequality, that $\alpha_t \geqslant \alpha_T$ and $x \mapsto (1-x)/x$ is increasing on $(0,1]$. Summing (16) over $t = 2,\dots,T$ using (17) and the fact that $\eta_t \geqslant \eta_T$, we get

$$
\begin{aligned}
&\sum_{t=2}^{T} \left(\frac{1}{\eta_{t-1}} \sum_{i=1}^{d} u_{i,t} \ln \frac{1}{\widehat{p}_{i,t}} - \frac{1}{\eta_t} \sum_{i=1}^{d} u_{i,t} \ln \frac{1}{v_{i,t+1}} \right) \\
&\quad \leqslant \quad \frac{m(\boldsymbol{u}_1^T)}{\eta_T} \ln \frac{d(1-\alpha_T)}{\alpha_T} + \sum_{t=2}^{T} \frac{\|\boldsymbol{u}_t\|_1}{\eta_{t-1}} \ln \frac{1}{1-\alpha_t} + \sum_{i=1}^{d} \bigg(\frac{u_{i,1}}{\eta_1} \ln \frac{1}{v_{i,2}} - \underbrace{\frac{u_{i,T}}{\eta_T} \ln \frac{1}{v_{i,T+1}}}_{\geqslant 0} \bigg).
\end{aligned}
$$

An application of (15) —including for $t = 1$, for which we recall that $\widehat{p}_{i,1} = 1/d$ and $\eta_1 = \eta_0$ by convention— concludes the proof. $\square$

We now instantiate the obtained bound to the case of, e.g., T–adaptive regret guarantees, when T is unknown and/or can increase without bounds.

Corollary 6. *The forecaster based on the updates discussed above with $\eta_t = \sqrt{\big(\ln(dt)\big)/t}$ for $t \geqslant 3$ and $\eta_0 = \eta_1 = \eta_2 = \eta_3$ on the one hand, $\alpha_t = 1/t$ on the other hand, is such that for all $T \geqslant 3$ and for all sequences $\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_T$ of loss vectors $\boldsymbol{\ell}_t \in [0,1]^d$,*

$$
\max_{[r,s]\subset[1,T]} \left\{ \sum_{t=r}^{s} \widehat{\boldsymbol{p}}_t^\top \boldsymbol{\ell}_t - \min_{\boldsymbol{q}\in\Delta_d} \sum_{t=r}^{s} \boldsymbol{q}^\top \boldsymbol{\ell}_t \right\} \leqslant \sqrt{2T\ln(dT)} + \sqrt{3\ln(3d)}\,.
$$

Proof. The sequence $n \mapsto \ln(n)/n$ is only non-increasing after round $n \geqslant 3$, so that the defined sequences of (α_t) and (η_t) are non-increasing, as desired. For a given pair (r,s) and a given $\boldsymbol{q} \in \Delta_d$, we consider the sequence ν_1^T defined in the proof of Corollary 2; it satisfies that $m(\boldsymbol{u}_1^T) \leqslant 1$ and $\|\boldsymbol{u}_t\|_1 \leqslant 1$ for all $t \geqslant 1$. Therefore, Theorem 4 ensures that

$$
\sum_{t=r}^{s} \widehat{\boldsymbol{p}}_t^\top \boldsymbol{\ell}_t - \min_{\boldsymbol{q}\in\Delta_d} \sum_{t=r}^{s} \boldsymbol{q}^\top \boldsymbol{\ell}_t \leqslant \frac{\ln d}{\eta_T} + \frac{1}{\eta_T} \ln \underbrace{\frac{d(1-\alpha_T)}{\alpha_T}}_{\leqslant dT} + \underbrace{\sum_{t=2}^{T} \frac{1}{\eta_{t-1}} \ln \frac{1}{1-\alpha_t}}_{\leqslant (1/\eta_T)\sum_{t=2}^{T} \ln(t/(t-1)) = (\ln T)/\eta_T} + \sum_{t=1}^{T} \frac{\eta_{t-1}}{8}\,.
$$

It only remains to substitute the proposed values of η_t and to note that

$$
\sum_{t=1}^{T} \eta_{t-1} \leqslant 3\eta_3 + \sum_{t=3}^{T-1} \frac{1}{\sqrt{t}} \sqrt{\ln(dT)} \leqslant 3\sqrt{\frac{\ln(3d)}{3}} + 2\sqrt{T}\,\sqrt{\ln(dT)}\,. \qquad \square
$$

D Proof of Theorem 1

We recall that the forecaster at hand is the one described in Algorithm 1, with the shared update $\widehat{\boldsymbol{p}}_{t+1} = \psi_{t+1}\big(\underline{V}_{t+1}\big)$ for

$$
\psi_{t+1}\big(\underline{V}_{t+1}\big) \in \operatorname*{argmin}_{\boldsymbol{x}\in\Delta_d^\alpha} \mathcal{K}(\boldsymbol{x},\boldsymbol{v}_{t+1})\,, \qquad \text{where} \quad \mathcal{K}(\boldsymbol{x},\boldsymbol{v}_{t+1}) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{v_{i,t+1}} \qquad (18)
$$

is the Kullback-Leibler divergence and $\Delta_d^\alpha = [\alpha/d,1]^d \cap \Delta_d$ is the simplex of convex vectors with the constraint that each component be larger than α/d.

The proof of the performance bound starts with an extension of Lemma 1.

Lemma 4. *For all $t \geqslant 1$ and for all $\boldsymbol{q}_t \in \Delta_d^\alpha$, the generalized forecaster with the shared update* (18) *satisfies*

$$
(\widehat{\boldsymbol{p}}_t - \boldsymbol{q}_t)^\top \boldsymbol{\ell}_t \leqslant \frac{1}{\eta} \sum_{i=1}^{d} q_{i,t} \ln \frac{\widehat{p}_{i,t+1}}{\widehat{p}_{i,t}} + \frac{\eta}{8}\,.
$$

Proof. We rewrite the bound of Lemma 1 in terms of Kullback-Leibler divergences,

$$
\begin{aligned}
(\widehat{\boldsymbol{p}}_t - \boldsymbol{q}_t)^\top \boldsymbol{\ell}_t &\leqslant \frac{1}{\eta} \sum_{i=1}^{d} q_{i,t} \ln \frac{v_{i,t+1}}{p_{i,t}} + \frac{\eta}{8} = \frac{\mathcal{K}(\boldsymbol{q}_t, \widehat{\boldsymbol{p}}_t) - \mathcal{K}(\boldsymbol{q}_t, \boldsymbol{v}_{t+1})}{\eta} + \frac{\eta}{8} \\
&\leqslant \frac{\mathcal{K}(\boldsymbol{q}_t, \widehat{\boldsymbol{p}}_t) - \mathcal{K}(\boldsymbol{q}_t, \widehat{\boldsymbol{p}}_{t+1})}{\eta} + \frac{\eta}{8} = \frac{1}{\eta} \sum_{i=1}^{d} q_{i,t} \ln \frac{\widehat{p}_{i,t+1}}{\widehat{p}_{i,t}} + \frac{\eta}{8} \,,
\end{aligned}
$$

where the last inequality holds by applying a generalized Pythagorean theorem for Bregman divergences (here, the Kullback-Leibler divergence) —see, e.g., [3, Lemma 11.3]. □

Proof. Let $\boldsymbol{q}_t = \dfrac{\alpha}{d} + (1-\alpha) \dfrac{\boldsymbol{u}_t}{\|\boldsymbol{u}_t\|_1} \in \Delta_d^\alpha$. We have by rearranging the terms for all t,

$$
\begin{aligned}
\big(\|\boldsymbol{u}_t\|_1 \widehat{\boldsymbol{p}}_t - \boldsymbol{u}_t\big)^\top \boldsymbol{\ell}_t &= \|\boldsymbol{u}_t\|_1 (\widehat{\boldsymbol{p}}_t - \boldsymbol{q}_t)^\top \boldsymbol{\ell}_t + \Big(\frac{\alpha}{d} \|\boldsymbol{u}_t\|_1 - \alpha \boldsymbol{u}_t\Big)^\top \boldsymbol{\ell}_t \\
&\leqslant \|\boldsymbol{u}_t\|_1 (\widehat{\boldsymbol{p}}_t - \boldsymbol{q}_t)^\top \boldsymbol{\ell}_t + \alpha \|\boldsymbol{u}_t\|_1 \;.
\end{aligned}
$$

Therefore, by applying Lemma 4 with $\boldsymbol{q}_t \in \Delta_d^\alpha$, we further upper bound the quantity of interest as

$$
\big(\|\boldsymbol{u}_t\|_1 \widehat{\boldsymbol{p}}_t - \boldsymbol{u}_t\big)^\top \boldsymbol{\ell}_t \leqslant \frac{\|\boldsymbol{u}_t\|_1}{\eta} \sum_{i=1}^{d} q_{i,t} \ln \frac{\widehat{p}_{i,t+1}}{\widehat{p}_{i,t}} + \frac{\eta}{8} \|\boldsymbol{u}_t\|_1 + \alpha \|\boldsymbol{u}_t\|_1 \;.
$$

The upper bound is rewritten by summing over t and applying an Abel transform to its first term,

$$
\begin{aligned}
&\sum_{t=1}^{T} \frac{\|\boldsymbol{u}_t\|_1}{\eta} \sum_{i=1}^{d} q_{i,t} \ln \frac{\widehat{p}_{i,t+1}}{\widehat{p}_{i,t}} + \frac{\eta}{8} \|\boldsymbol{u}_t\|_1 + \alpha \|\boldsymbol{u}_t\|_1 \\
= \;& \frac{\|\boldsymbol{u}_1\|_1 \ln d}{\eta} + \frac{\|\boldsymbol{u}_T\|_1}{\eta} \underbrace{\sum_{i=1}^{d} q_{i,T} \ln \widehat{p}_{i,T+1}}_{\leqslant 0} + \frac{1}{\eta} \sum_{t=2}^{T} \sum_{i=1}^{d} \underbrace{\big(\|\boldsymbol{u}_t\|_1 q_{i,t} - \|\boldsymbol{u}_{t-1}\| q_{i,t-1}\big)}_{=(1-\alpha)(u_{i,t} - u_{i,t-1})} \underbrace{\ln \frac{1}{\widehat{p}_{i,t}}}_{0 \leqslant \,\cdot\, \leqslant \ln \frac{d}{\alpha}} \\
&+ \Big(\frac{\eta}{8} + \alpha\Big) \sum_{t=1}^{T} \|\boldsymbol{u}_t\|_1 \\
\leqslant \;& \frac{\|\boldsymbol{u}_1\|_1 \ln d}{\eta} + \frac{1-\alpha}{\eta} \left(\sum_{t=2}^{T} D_{\mathrm{TV}}(\boldsymbol{u}_t, \boldsymbol{u}_{t-1}) \right) \ln \frac{d}{\alpha} + \Big(\frac{\eta}{8} + \alpha\Big) \sum_{t=1}^{T} \|\boldsymbol{u}_t\|_1 \;.
\end{aligned}
$$

□