

# 1 Dynamical Compressed Sensing: Replica Method

We wish to compute the typical properties the Gibbs distribution  $P_G(\mathbf{s}) = \frac{1}{Z} e^{-\beta E(\mathbf{s})}$ , with energy function  $E(\mathbf{s})$  given by

$$E(\mathbf{s}) = \frac{\lambda}{2} \mathbf{u}^T \mathbf{A}^T \mathbf{A} \mathbf{u} + \sum_{i=1}^T |\mathbf{s}_i|, \quad (1)$$

where  $\mathbf{s}^0$  is the true signal history,  $\mathbf{s}$  is a candidate signal reconstruction, and  $\mathbf{u}$  is the residual  $\mathbf{s} - \mathbf{s}^0$ . This energy function depends on the random annealed measurement matrix  $\mathbf{A}$  and we compute the typical properties of  $P_G$  by averaging over  $\mathbf{A}$ . To do so, we must compute the average free energy  $-\beta \bar{F} \equiv \langle \langle \ln Z \rangle \rangle$ , where  $\langle \langle \cdot \rangle \rangle$  is an average over  $\mathbf{A}$ , and  $Z = \int \prod_{i=0}^{\infty} ds_i e^{-\beta E(\mathbf{s})}$ . We use the replica method [1], which relies on the identity  $\ln Z = \lim_{n \rightarrow 0} \frac{Z^n - 1}{n}$ .  $Z^n$  can be written as an integral over  $n$  replicated variables  $s_i^a$ ,  $a = 1, \dots, n$ ,

$$\langle \langle Z^n \rangle \rangle = \left\langle \left\langle \int \prod_{a=1}^n \prod_{i=0}^{\infty} du_i^a e^{\sum_{a=1}^n -\frac{\beta \lambda}{2} \sum_{\mu=1}^N \sum_{ij=0}^{\infty} u_i^a A_{\mu i} A_{\mu j} u_j^a - \beta \sum_{i=0}^{\infty} |u_i^a + s_i^0|} \right\rangle \right\rangle. \quad (2)$$

Here the  $u_i^a \equiv s_i^a - s_i^0$  is the replicated residual. Now (2) depends on  $\mathbf{A}$  only through the variables  $b_\mu^a \equiv \sum_{i=0}^{\infty} A_{\mu i} u_i^a$ , which are jointly gaussian distributed with zero mean and covariance  $\langle \delta b_\mu^a \delta b_\nu^b \rangle = Q_{ab} \delta_{\mu\nu}$ , where  $Q_{ab} \equiv \frac{1}{N} \sum_{i=0}^{\infty} \rho^i u_i^a u_i^b$ . Introducing delta functions

$$\delta \left[ \sum_{i=0}^{\infty} \rho^i u_i^a u_i^b - N Q_{ab} \right] = \int d \hat{Q}_{ab} e^{i \hat{Q}_{ab} (\sum_{i=0}^{\infty} \rho^i u_i^a u_i^b - N Q_{ab})} \quad (3)$$

to decouple the integrals over  $u_i^a$  and  $b_\mu^a$ , and performing the gaussian integral over  $b_\mu^a$  yields,

$$\langle \langle Z^n \rangle \rangle = \int \prod_{a,b=1}^n dQ_{ab} \hat{Q}_{ab} e^{N [i \sum_{ab} \hat{Q}_{ab} Q_{ab} + \frac{1}{N} \sum_{i=0}^{\infty} \ln \mathcal{Z}_i - \frac{1}{2} \text{Tr} \log(I + \beta \lambda Q)]}, \quad (4)$$

where  $\mathcal{Z}_i = \int \prod_{a=1}^n du^a e^{-i \rho^i \sum_{ab} u^a \hat{Q}_{ab} u^b - \beta \sum_a |u^a + s_i^0|}$ . The remaining integrals over  $Q_{ab}$  and  $\hat{Q}_{ab}$  in (4) can be done via the saddle point approximation. We work with a replica symmetric (RS) ansatz for the saddle point:  $Q_{ab} = (Q_1 - Q_0) \delta_{ab} + Q_0$  and  $\hat{Q}_{ab} = (\hat{q}_1 - \hat{q}_0) \delta_{ab} + \hat{q}_0$ . Inserting this ansatz into the above equations and taking the  $n \rightarrow 0$  limit yields

$$-\beta \bar{F} = \text{extr}_{(\hat{q}_1, Q_1, \hat{q}_0, Q_0)} \left\{ i(\hat{q}_1 Q_1 - \hat{q}_0 Q_0) + \langle \langle \ln \zeta_i \rangle \rangle_z - \frac{\alpha}{2} \left( \frac{\beta \lambda}{1 + \beta \lambda \Delta Q} + \ln(1 + \beta \lambda \Delta Q) \right) \right\}, \quad (5)$$

where

$$\zeta_i = \int du e^{-i\rho^i(\hat{q}_1 - \hat{q}_0)u^2 + \sqrt{-2i\hat{q}_0\rho^i}zu - \beta|u + s_i^0|}, \quad (6)$$

and  $\Delta Q \equiv Q_1 - Q_0$ . Here  $z$  is a zero mean unit variance gaussian variable introduced to decouple the replicas under the RS assumption via the identity

$$e^{-i\hat{q}_0\rho^i(\sum_a u^a)^2} = \int \mathcal{D}z e^{\sqrt{-2i\hat{q}_0\rho^i}z\sum_a u^a}. \quad (7)$$

Extremizing (5) with respect to  $Q_0$  and  $Q_1$  yields the saddle point equations

$$-i\hat{q}_0 = \frac{\alpha}{2} \frac{(\beta\lambda)^2 Q_0}{(1 + \beta\lambda\Delta Q)^2}, \quad i(\hat{q}_1 - \hat{q}_0) = \frac{\alpha}{2} \frac{\beta\lambda}{1 + \beta\lambda\Delta Q}, \quad (8)$$

while extremizing with respect to  $\hat{q}_0$  and  $\hat{q}_1$  yields

$$Q_0 = \frac{1}{N} \sum_{i=0}^{\infty} \rho^i \langle\langle \langle u \rangle_{H_i^{MF}}^2 \rangle\rangle_z \quad (9)$$

$$\Delta Q = \frac{1}{N} \sum_{i=0}^{\infty} \rho^i \langle\langle \langle \delta u^2 \rangle_{H_i^{MF}} \rangle\rangle_z, \quad (10)$$

where the effective Hamiltonian

$$H_i^{MF}(s) = \rho^i \frac{\beta\lambda}{2(1 + \beta\lambda\Delta Q)} \left( s - s_k^0 - z\sqrt{Q_0/\rho^k} \right)^2 + \beta|s|, \quad (11)$$

is obtained by substituting (8) into the exponent of (6) and completing the square. According to the replica method [1], the mean field approximation to the marginal distribution of a signal reconstruction component  $s_i$  conditioned on the true signal component  $s_i^0$  is then given by

$$P_i^{MF}(s_i = s) = \int \mathcal{D}z \frac{1}{Z_i^{MF}} \exp(-H_i^{MF}(s)), \quad (12)$$

where  $\mathcal{D}z$  is the standard gaussian measure. In this mean field approximation, the integral over the annealed measurement matrix  $\mathbf{A}$  has been replaced with an integral over a zero mean unit variance gaussian scalar variable  $z$ , which reflects the mean field effect of the measurement constraints in  $P_G$  after averaging over  $\mathbf{A}$ .

## References

- [1] M. Mezard, G. Parisi, and M.A. Virasoro. *Spin glass theory and beyond*. World scientific Singapore, 1987.