Optimal Response Initiation: Why Recent Experience Matters

SUPPLEMENTARY MATERIAL

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NIPS 2008

This supplement provides the mathematical derivations of the predictions of our multiresponse diffusion model. In this model, evidence for each response in a decision task accrues according to a diffusion process. The drift rates for all incorrect responses are known to be zero, and there is a prior distribution over the drift rate for the correct response (derived from the drifts on recent trials). This information, together with the actual values of the accrued evidence, is used to derive a posterior distribution over the identity of the correct response, which then drives the model's decision.

The first section derives standard results for Bayesian inference of the parameters of a diffusion process. For the drift rate, the total change of the process over the period of observation is a sufficient statistic; observation at arbitrarily fine timescales does not provide additional information. For the diffusion rate, the likelihood function becomes increasingly peaked with more frequent measurement, so that the true value can be estimated with perfect precision in the limit of continuous observation. (The difference in inferential properties of these two parameters is due to the continuous, undifferentiable property of Brownian motion.) Therefore there can be no sequential learning effects involving the diffusion rate, and we focus subsequent attention on the drift rate. The likelihood function for the drift rate is derived, which corresponds to Equation 2 in the main paper.

The second section derives the posterior probability over the correct response in a many-alternative forced-choice task. The result corresponds to Equation 3 in the main paper.

1. Inferring the parameters of a diffusion process

Let $x(t)_{t\in[0,T]}$ be a diffusion process with drift and diffusion rates μ and σ. For any times *i* and *j*, the change in *x* over the intervening period is distributed as

$$x(j)-x(i) \sim N\left((j-i)\mu, (j-i)\sigma^2\right).$$

For any positive step size $\tau = T/n$, consider the discrete set of points $x(i\tau)_{0\le i\le \frac{T}{\tau}}$. The likelihood of these observations is given by

$$P\left[x(i\tau)_{0\le i\le \frac{T}{\tau}} \,|\, \mu,\sigma\right] = \prod_{i=1}^{\frac{T}{\tau}}\left[\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{\left(x(i\tau)-x((i-1)\tau)-\tau\mu\right)^2}{2\tau\sigma^2}}\right].$$

Given two parameter vectors, (μ,σ) and (μ',σ'), their log-likelihood ratio is

$$\ln \frac{P\left[x(i\tau)_{0\le i\le \frac{T}{\tau}} \mid \mu,\sigma\right]}{P\left[x(i\tau)_{0\le i\le \frac{T}{\tau}} \mid \mu',\sigma'\right]}$$

$$=\sum_{i=1}^{\frac{T}{\tau}}\left[\ln\frac{\sigma'}{\sigma}+\frac{\tau}{2}\left(\frac{\mu'^2}{\sigma'^2}-\frac{\mu^2}{\sigma^2}\right)+\left(\frac{\mu}{\sigma^2}-\frac{\mu'}{\sigma'^2}\right)\left(x(i\tau)-x((i-1)\tau)\right)+\frac{1}{2\tau}\left(\frac{1}{\sigma'^2}-\frac{1}{\sigma^2}\right)\left(x(i\tau)-x((i-1)\tau)\right)^2\right]$$

$$=\frac{T}{\tau}\ln\frac{\sigma'}{\sigma}+\frac{T}{2}\left(\frac{\mu'^2}{\sigma'^2}-\frac{\mu^2}{\sigma^2}\right)+\left(\frac{\mu}{\sigma^2}-\frac{\mu'}{\sigma'^2}\right)\left(x(T)-x(0)\right)+\frac{1}{2\tau}\left(\frac{1}{\sigma'^2}-\frac{1}{\sigma^2}\right)\sum_{i=1}^{\frac{T}{\tau}}\left(x(i\tau)-x((i-1)\tau)\right)^2.$$

If (μ,σ) are the true parameters, then the expected value of this expression is

$$\frac{T}{\tau}\ln\frac{\sigma'}{\sigma}+\frac{T}{2}\left(\frac{\mu'^2}{\sigma'^2}-\frac{\mu^2}{\sigma^2}\right)+\left(\frac{\mu}{\sigma^2}-\frac{\mu'}{\sigma'^2}\right)\mu T+\frac{1}{2\tau}\left(\frac{1}{\sigma'^2}-\frac{1}{\sigma^2}\right)\sum_{i=1}^{\left\lfloor\left(\frac{T}{\tau}\right)\right\rfloor}\left(\tau\sigma^2+\tau^2\mu^2\right)$$

$$=T\left[\frac{1}{2}\left(\frac{\mu'^2}{\sigma'^2}-\frac{\mu^2}{\sigma^2}\right)+\mu\left(\frac{\mu}{\sigma^2}-\frac{\mu'}{\sigma'^2}\right)\right]+\frac{T\mu^2}{2}\left(\frac{1}{\sigma'^2}-\frac{1}{\sigma^2}\right)+\frac{T}{2\tau}\left[\frac{\sigma^2}{\sigma'^2}-1-\ln\frac{\sigma^2}{\sigma'^2}\right].$$

The function $z-1-\ln z$ is nonnegative and is strictly positive iff $z\neq 1$. Therefore if $\sigma\neq\sigma'$ then the third term above goes to infinity as $\tau\to 0$. This implies that observation of a *single* trajectory from a diffusion process allows knowledge of the diffusion rate to arbitrary precision (depending in practice on the temporal resolution with which x is measured).

Assume, then, that σ is known but μ is unknown. The log-likelihood ratio above reduces to

$$\ln \frac{P\left[x(i\tau)_{0\le i\le \frac{T}{\tau}} \mid \mu\right]}{P\left[x(i\tau)_{0\le i\le \frac{T}{\tau}} \mid \mu'\right]}=\frac{1}{\sigma^2}\left[\frac{T}{2}\left(\mu'^2-\mu^2\right)+\left(x(T)-x(0)\right)\left(\mu-\mu'\right)\right],$$

which is independent of τ and, more generally, of the set of time-points sampled. The total change in x, which we write as $\Delta x(T)=x(T)-x(0)$, is thus a sufficient statistic for estimating μ. Therefore we have the following expression for the likelihood function:

$$\ln P[x\mid\mu]=\frac{1}{\sigma^2}\left[-\frac{T}{2}\mu^2+\Delta x(T)\mu\right]+C,$$

where C is a normalization constant. This corresponds to Equation 2 of the main text. As a check, it is easy to see that this likelihood attains a maximum at $\mu = \Delta x(T)/T$.

2. Application to nAFC

Assume N trajectories, $\{x_i\}$, are observed. It is known that one of them, x_s, has drift rate μ_s and the rest have drift rate 0, but s (which represents the identity of the correct response) is unknown with uniform prior. We also have a prior on μ_s, given by

$$\mu_s \sim N(a, b^2).$$

For any j, the posterior probability that $s = j$ is

$$P[s = j \mid (x_i)] \propto \prod_{i \neq j} \exp\left(\frac{1}{\sigma^2}\left[-\frac{T}{2}\cdot 0^2 + \Delta x_i(T)\cdot 0\right]\right)\cdot \int dP(\mu_s)\exp\left(\frac{1}{\sigma^2}\left[-\frac{T}{2}\mu_s^2 + \Delta x_j(T)\mu_s\right]\right)$$

$$= \int d\mu_s \frac{1}{b\sqrt{2\pi}}\exp\left(-\frac{(\mu_s - a)^2}{2b^2}\right)\cdot \exp\left(\frac{1}{\sigma^2}\left[-\frac{T}{2}\mu_s^2 + \Delta x_j(T)\mu_s\right]\right)$$

$$\propto \int d\mu_s \exp\left(-\left(\frac{1}{2b^2} + \frac{T}{2\sigma^2}\right)\mu_s^2 + \left(\frac{a}{b^2} + \frac{\Delta x_j(T)}{\sigma^2}\right)\mu_s - \frac{a^2}{2b^2}\right)$$

$$= \exp\left(\frac{\left(a\sigma^2 + \Delta x_j(T)b^2\right)^2}{2b^2\sigma^2(\sigma^2 + Tb^2)} - \frac{a^2}{2b^2}\right)\cdot \int d\mu_s \exp\left(-\frac{\left(\mu_s - \left(\frac{a}{b^2} + \frac{\Delta x_j(T)}{\sigma^2}\right)\frac{b^2\sigma^2}{\sigma^2 + Tb^2}\right)^2}{\frac{2b^2\sigma^2}{\sigma^2 + Tb^2}}\right)$$

$$\propto \exp\left(\frac{\left(a\sigma^2 + \Delta x_j(T)b^2\right)^2}{2b^2\sigma^2(\sigma^2 + Tb^2)} - \frac{a^2}{2b^2}\right)$$

$$\propto \exp\left(\frac{b^2\Delta x_j(T)^2 + 2a\sigma^2\Delta x_j(T)}{2\sigma^2(\sigma^2 + Tb^2)}\right).$$

This posterior probability corresponds to that given in Equation 3 of the main text.